CENTRES OF CENTRALISERS IN THE CLASSICAL LIE ALGEBRAS

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Let $\mathfrak{g}$ be a simple classical Lie algebra defined over a field $\Bbbk$ such that $\operatorname{char}\Bbbk \neq 2$. Suppose that $e \in \mathfrak{g}$ is a nilpotent element. The goal of this note is to describe the centre $\mathfrak{z}$ of the centraliser $\mathfrak{g}_e \subset \mathfrak{g}$. Proposition 3.5 of [2], which addresses the same problem, is not correct.

Let V be a vector space of the defining representation of $\mathfrak{g}$, i.e., $\mathfrak{g} = \mathfrak{sl}(V)$, $\mathfrak{sp}(V)$, or $\mathfrak{so}(V)$. Consider e as a matrix in $\mathfrak{gl}(V)$. Then the powers of e (as a matrix) are also elements of $\mathfrak{gl}(V)$. Let d be the minimal number such that $e^d = 0$. Set $\mathfrak{E} := \mathfrak{g} \cap \langle e, e^2, \ldots, e^{d-1}\rangle_{\Bbbk}$. If $\mathfrak{g}$ is of the type A or C, then $\mathfrak{z} = \mathfrak{E}$; if $\mathfrak{g}$ is of the type B or D, then there are nilpotent elements such that $\mathfrak{z} = \mathfrak{E} \oplus \Bbbk\xi$ for some $\xi \in \mathfrak{g}_e$. The following notation is needed to give a precise formulation and proofs.

Suppose that e is given by the partition $(d_1+1, \ldots, d_m+1)$, where $m = \dim \operatorname{Ker}(e)$ and $d_i \geqslant d_j$ for $i < j$. Then, by the theory of Jordan normal form, there are vectors $w_1, \ldots, w_m \in V$ such that $e^{d_i+1}{\cdot}w_i = 0$ and $\{e^s{\cdot}w_i \mid 1 \leqslant i \leqslant m,\ 0 \leqslant s \leqslant d_i\}$ is a basis of V. The spaces $V_i := \langle w_i, e{\cdot}w_i, \ldots, e^{d_i}{\cdot}w_i\rangle$ are called the Jordan (or *cyclic*) spaces of the nilpotent element e.

Let $\hat{\mathfrak{g}}_e$ be the centraliser of e in $\mathfrak{gl}(V)$. Suppose $\varphi \in \hat{\mathfrak{g}}_e$. Because $\varphi(e^s{\cdot}w_i) = e^s{\cdot}\varphi(w_i)$, the linear map φ is determined by its values on W. In other words, if

$$\varphi(w_i) = \sum_{j,s} c_i^{j,s}(e^s{\cdot}w_j), \text{ where } c_i^{j,s} \in \Bbbk\ ,$$

then φ is determined by the coefficients $c_i^{j,s} = c_i^{j,s}(\varphi)$. In what follows, we will only indicate the values of φ on the cyclic vectors $\{w_i\}$.

A basis of $\hat{\mathfrak{g}}_e$ consists of the maps $\{\xi_i^{j,s}\}$ given by

$$\xi_i^{j,s} : \left\{ \begin{array}{lcll} w_i & \mapsto & e^s{\cdot}w_j & \\ w_t & \mapsto & 0 & \text{if } t \neq i \end{array} \right. , \quad \text{where} \quad 1 \leqslant i, j \leqslant m \ \text{ and } \ \max\{d_j - d_i, 0\} \leqslant s \leqslant d_j\ .$$

The following result is well-known, but we will need some of these ideas later.

Theorem 1. *If $\mathfrak{g} = \mathfrak{sl}(V)$, then $\mathfrak{z} = \mathfrak{E}$.*

Proof. We have $e^k = \sum\limits_{i=1}^{m} \xi_i^{i,k}$ and $e^k \in \mathfrak{g}$ if and only if $i \geqslant 1$. Suppose $\eta \in \mathfrak{z}$. Then η commutes with the maximal torus $\mathfrak{t} := \langle \xi_i^{i,0}\rangle_{\Bbbk} \subset \hat{\mathfrak{g}}_e$. We have

$$[\xi_i^{i,0}, \xi_j^{t,s}] = \left\{ \begin{array}{rl} -\xi_i^{t,s} & \text{if } i = j, i \neq t; \\ \xi_j^{i,s} & \text{if } i = t, i \neq j; \\ 0 & \text{otherwise.} \end{array} \right. ,$$

Therefore $\eta \in \langle \xi_i^{i,s} \rangle_{\Bbbk}$. Adding an element of $\mathfrak{E}$ we may assume that $c_1^{1,s}(\eta) = 0$ for all s. If $\eta \notin \mathfrak{E}$, then there is some $c_i^{i,s}(\eta)$, which is not zero. Now take $\xi_1^{i,0} \in \mathfrak{g}_e$ and compute that

$$[\eta, \xi_1^{i,0}] = c_i^{i,0}(\eta)\xi_1^{i,0} + c_i^{i,1}(\eta)\xi_1^{i,1} + \ldots + c_i^{i,d_i}(\eta)\xi_1^{i,d_i} \neq 0.$$

A contradiction! Thus $\mathfrak{z} = \mathfrak{E}$. □

Recall that both $\mathfrak{sp}(V)$ and $\mathfrak{so}(V)$ are symmetric subalgebras of $\mathfrak{gl}(V)$. Let $(\ ,\ )$ be a non-degenerate symmetric or skew-symmetric form on V. In other words, $(v, w) = \varepsilon(w, v)$, where $v, w \in V$ and $\varepsilon = +1$ or -1. Let $\mathfrak{gl}(V) = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be the corresponding $\mathbb{Z}_2$-grading, i.e., $\mathfrak{g}_0$ consists of the linear transformations such that $(vA, w) = -(v, Aw)$ for all $v, w \in V$. Then $\mathfrak{g}_0 = \mathfrak{sp}(V)$ if $\varepsilon = -1$; and $\mathfrak{g}_0 = \mathfrak{so}(V)$ if $\varepsilon = 1$.

Lemma 1. *In the above setting, suppose that $e \in \mathfrak{g}_0$ is a nilpotent element. Then the cyclic vectors $\{w_i\}$ and thereby the spaces $\{V_i\}$ can be chosen such that there is an involution $i \mapsto i^*$ on the set $\{1, \ldots, m\}$ satisfying the following conditions:*

- $d_i = d_{i^*}$;
- $(V_i, V_j) = 0$ *if* $i \neq j^*$;
- $i = i^*$ *if and only if* $(-1)^{d_i}\varepsilon = 1$.

Proof. This is a standard property of the nilpotent orbits in $\mathfrak{sp}(V)$ and $\mathfrak{so}(V)$, see, for example, [1, Sect. 5.1] or [3, Sect. 1]. □

Let $\{w_i\}$ be a set of cyclic vectors chosen according to Lemma 1. Consider the restriction of the $\mathfrak{g}$-invariant form $(\ ,\ )$ to $V_i + V_{i^*}$. Since $(w, e^s{\cdot}v) = (-1)^s(e^s{\cdot}w, v)$, a vector $e^{d_i}{\cdot}w_i$ is orthogonal to all vectors $e^s{\cdot}w_{i^*}$ with $s > 0$. Therefore $(w_{i^*}, e^{d_i}{\cdot}w_i) = (-1)^{d_i}(e^{d_i}{\cdot}w_{i^*}, w_i) \neq 0$. Let us normalise the cyclic vectors such that $(w_i, e^{d_i}{\cdot}w_{i^*}) = \pm 1$ and $(w_i, e^{d_i}{\cdot}w_{i^*}) > 0$ if $i \leqslant i^*$. Then $\mathfrak{g}_e$ is generated (as a vector space) by the vectors $\xi_i^{j,d_j-s} + \varepsilon(i,j,s)\xi_{j^*}^{i^*,d_i-s}$, where $\varepsilon(i,j,s) = \pm 1$ depending on i, j and s in the following way

$$(e^{d_j-s}{\cdot}w_j, e^s{\cdot}w_{j^*}) = -\varepsilon(i,j,s)(w_i, e^{d_i}{\cdot}w_{i^*}).$$

Enumerate the Jordan blocks such that $i^* \in \{i, i+1, i-1\}$ and $d_i \geqslant d_j$ for $i < j$.

Theorem 2. *If $\mathfrak{g} = \mathfrak{so}(V)$ and e is given by a partition $(d_1+1, d_2+1, d_3+1, \ldots, d_m+1)$ with $m \geqslant 2$, where $d_1 \geqslant d_2 > d_3 \geqslant d_4 \geqslant \ldots \geqslant d_m$ and both d_1 and d_2 are even, then $\mathfrak{z} = \mathfrak{E} \oplus \Bbbk(\xi_1^{2,d_2} - \xi_2^{1,d_1})$. For all other simple $\mathfrak{g}$ and nilpotent elements $e \in \mathfrak{g}$ we have $\mathfrak{z} = \mathfrak{E}$.*

Proof. First we show that indeed in the special case indicated in the theorem we have an additional central element $x := \xi_1^{2,d_2} - \xi_2^{1,d_1}$. Note that $\xi_1^{2,d_2}, \xi_2^{1,d_1}$ do not commute only with the elements $\xi_1^{1,0}$, $\xi_2^{2,0}$, ξ_2^{1,d_1-d_2}, and $\xi_1^{2,0}$. Since $\mathfrak{g}_e$ contains no elements of the form $a\xi_1^{1,0} + b\xi_2^{2,0}$, we only have to check that $[x, \xi_1^{2,0} + \varepsilon(1,2,d_2)\xi_2^{1,d_1-d_2}] = 0$. Here d_1 and d_2 are even, therefore $\varepsilon(1,2,d_2) = -1$. We get

$$[x, \xi_1^{2,0} - \xi_2^{1,d_1-d_2}] = -\xi_1^{1,d_1} - \xi_2^{2,d_1} + \xi_1^{1,d_1} + \xi_2^{2,d_1} = 0.$$

Let us prove that $\mathfrak{z}$ contains no other elements. The case $\mathfrak{g} = \mathfrak{sl}(V)$ was treated above. Thus assume that $\mathfrak{g}$ is either $\mathfrak{sp}(V)$ or $\mathfrak{so}(V)$. Then $\mathfrak{E}$ is a vector space generated by all odd powers of e. Note that whenever $i \neq i^*$ there is an $\mathfrak{sl}_2$-triple $\mathfrak{q}_i = \langle \xi_i^{i,0} - \xi_{i^*}^{i^*,0}, \xi_i^{i^*,0}, \xi_{i^*}^{i,1} \rangle_{\Bbbk}$, which is a subalgebra of $\mathfrak{g}_e$. Hence, if $i \ne i^*$ or $j \ne j^*$ and $i \ne j$, then $c_i^{j,s}(\eta) = 0$. Also for $i \ne i^*$ we have $c_i^{i,s}(\eta) = c_{i^*}^{i^*,s}(\eta)$.

Suppose that $\eta \in \mathfrak{z}$. If η preserve the cyclic spaces, then $\eta \in \mathfrak{E}$. It can be shown exactly in the same way as in the $\mathfrak{sl}(V)$ case.

Assume that $\eta \notin \mathfrak{E}$. Take the minimal i such that there is a non-zero $c_i^{j,s}(\eta)$ with $j \ne i$. Fix this i and take the minimal j, and then the minimal s, with this property. We have just seen that necessary $i^* = i$ and $j^* = j$. Since $c_j^{i,d_j-d_i+s}(\eta) \ne 0$, we have also $j > i$ and therefore $j > 1, 1^*$. There is an element $\xi := \xi_j^{1,d_1-s} + \varepsilon(j,1,s)\xi_{1^*}^{j,d_j-s} \in \mathfrak{g}_e$. Consider the commutator $[\xi,\eta] = \xi\eta - \eta\xi$. We are interested in the coefficient $a_i := c_i^{1,d_1}([\xi,\eta])$. Since all coefficients $c_i^{1^*,r}(\eta)$ are zeros and $j \ne i$, we get

$$a_i = c_i^{j,s}(\eta) - \delta_{i,1}\varepsilon(j,1,s)c_j^{1,d_1-d_j+s}(\eta).$$

In particular, if $i \ne 1$, then η is not a central element. Therefore $i = 1$.

In the symplectic case d_1 and d_j are odd, hence $d_j - s$ and s have different parity and $\varepsilon(j,1,s)\varepsilon(1,j,d_j-s) = -1$. Thus $a_i = 2c_i^{j,s} \ne 0$. We get a contradiction.

The orthogonal case is more complicated. If $j > 2$, then also $j > 2^*$ and

$$c_1^{2,d_2}([\xi_j^{2,d_2-s} + \varepsilon(j,2,s)\xi_{2^*}^{j,d_j-s}, \eta]) = c_1^{j,s}(\eta) \ne 0.$$

Since $\eta \in \mathfrak{z}$, we get $j = 2$. If $d_3 = d_2$, then $3^* = 3$ and there is a semisimple element $\xi_2^{3,0} - \xi_3^{2,0} \in \mathfrak{g}_e$, which does not commute with η.

Thus we have to consider only the special case $d_2 > d_3$. There is no harm in replacing η by $\eta - c_1^{2,d_2}(\eta)x$. In other word, we may assume that $c_1^{2,d_2}(\eta) = 0$ and thereby $s < d_2$. It is not difficult to see that η does not commute either with $\xi_1^{2,1} + \xi_2^{1,d_1-d_2+1}$ or $\xi_1^{2,0} - \xi_2^{1,d_1-d_2}$. Thus if $\eta \notin \mathfrak{E} \oplus \Bbbk(\xi_1^{2,d_2} - \xi_2^{1,d_1})$, then η is not a central element. This completes the proof. □

References

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