June 1, 2021

A bi-Hamiltonian nature of the Gaudin algebras

Oksana Yakimova

arxiv:2105.01020

Institut für Mathematik

Fakultät für Mathematik und Informatik

FRIEDRICH-SCHILLER-
UNIVERSITÄT
JENA

Algebra Seminar, Universität zu Köln

1. Poisson brackets and Poisson-commutative subalgebras

Let $\mathfrak{q}$ be a non-Abelian Lie algebra over a field $\Bbbk$ ($\mathsf{char}\,\Bbbk = 0$). The symmetric algebra $\mathcal{S}(\mathfrak{q})$ carries the standard Lie–Poisson structure:

- ◇ $\{\xi, \eta\} = [\xi, \eta]$ for all $\xi, \eta \in \mathfrak{q}$, extends further by the Leibniz rule (**algebra**);
- ◇ $\{F_1, F_2\}(\gamma) = \gamma([d_\gamma F_1, d_\gamma F_2])$ for all $F_1, F_2 \in \mathcal{S}(\mathfrak{q}), \gamma \in \mathfrak{q}^*$ (**geometrie**);
- ◇ $\{\boldsymbol{f} + \mathcal{U}_a(\mathfrak{q}), \boldsymbol{h} + \mathcal{U}_b(\mathfrak{q})\} = [\boldsymbol{f}, \boldsymbol{h}] + \mathcal{U}_{a+b}(\mathfrak{q})$ for $\boldsymbol{f} \in \mathcal{U}_{a+1}(\mathfrak{q}), \boldsymbol{h} \in \mathcal{U}_{b+1}(\mathfrak{q})$.

The third definition uses the fact that $\mathcal{S}(\mathfrak{q}) \cong \mathsf{gr}\,\mathcal{U}(\mathfrak{q})$ and one may sat that it belongs to **representation theory**.

If $\mathfrak{q}$ is finite-dimensional, then $\mathcal{S}(\mathfrak{q}) = \Bbbk[\mathfrak{q}^*]$ (this belongs to **geometrie**).

Definition 1. A subalgebra $A \subset \mathcal{S}(\mathfrak{q})$ is *Poisson-commutative* if $\{A, A\} = 0$.

If $C \subset \mathcal{U}(\mathfrak{q})$ is a commutative algebra, then $\mathsf{gr}(C) \subset \mathcal{S}(\mathfrak{q})$ is Poisson-commutative.

Quantisation problem: given a Poisson-commutative $A \subset \mathcal{S}(\mathfrak{q})$, find a commutative subalgebra $\tilde{A} \subset \mathcal{U}(\mathfrak{q})$ such that $A = \mathsf{gr}(\tilde{A})$.

Some notation:

- ◇ For $\gamma \in \mathfrak{q}^*$, set $\hat{\gamma}(\xi, \eta) = \gamma([\xi, \eta])$ if $\xi, \eta \in \mathfrak{q}$.
- ◇ For $A \subset \mathcal{S}(\mathfrak{q})$, $d_\gamma A := \langle d_\gamma F \mid F \in A \rangle_{\Bbbk}$.
- ◇ Let $\mathfrak{q}_\gamma = \ker \hat{\gamma}$ be the stabiliser of γ, then

$$\operatorname{ind} \mathfrak{q} := \min_{\gamma \in \mathfrak{q}^*} \dim \mathfrak{q}_\gamma \quad \text{and} \quad \boldsymbol{b}(\mathfrak{q}) := \tfrac{1}{2}(\dim \mathfrak{q} + \operatorname{ind} \mathfrak{q}).$$

Suppose that $A \subset \mathcal{S}(\mathfrak{q})$ and $\{A, A\} = 0$. Then $\hat{\gamma}(d_\gamma A, d_\gamma A) = 0$ and therefore

$$\dim d_\gamma A \leqslant \frac{1}{2} \dim(\dim \mathfrak{q} - \dim \mathfrak{q}_\gamma) + \dim \mathfrak{q}_\gamma.$$

Hence $\operatorname{tr.deg} A \leqslant \boldsymbol{b}(\mathfrak{q})$. More generally, if $\mathfrak{l} \subset \mathfrak{q}$ is a Lie subalgebra and

$$\mathcal{S}(\mathfrak{q})^{\mathfrak{l}} = \{F \in \mathcal{S}(\mathfrak{q}) \mid \{\xi, F\} = 0 \ \forall \xi \in \mathfrak{l}\},$$

then

$$\operatorname{tr.deg} A \leqslant \boldsymbol{b}(\mathfrak{q}) - \boldsymbol{b}(\mathfrak{l}) + \operatorname{ind} \mathfrak{l} =: \boldsymbol{b}^{\mathfrak{l}}(\mathfrak{q}), \tag{1}$$

for any Poisson-commutative subalgebra $A \subset \mathcal{S}(\mathfrak{q})^{\mathfrak{l}}$ [Molev–Y. (2019)].

Remark. We have also $\operatorname{tr.deg} \mathcal{A} \leqslant \boldsymbol{b}(\mathfrak{q})$ for any commutative subalgebra $\mathcal{A} \subset \mathcal{U}(\mathfrak{q})$ and $\operatorname{tr.deg} \mathcal{C} \leqslant \boldsymbol{b}^{\mathfrak{l}}(\mathfrak{q})$ for any commutative subalgebra $\mathcal{C} \subset \mathcal{U}(\mathfrak{q})^{\mathfrak{l}}$.

2. The Lenard–Magri scheme (compatible Poisson brackets)

Two Poisson brackets are *compatible* if their sum (and hence any linear combination of them) is again a Poisson bracket. Roughly speaking, a *bi-Hamiltonian system* is a pair of compatible Poisson structures $\{\ ,\ \}', \{\ ,\ \}''$, or rather a pencil

$$\{a\{\ ,\ \}' + b\{\ ,\ \}'' \mid a, b \in \Bbbk\}$$

spanned by them.

Let π', π'' be the Poisson tensors of $\{\ ,\ \}', \{\ ,\ \}''$. Then $\pi_{a,b} = a\pi' + b\pi''$ is the Poisson tensors of $a\{\ ,\ \}' + b\{\ ,\ \}''$. For almost all $(a, b) \in \Bbbk^2$, $\mathsf{rk}(a\pi' + b\pi'')$ has one and the same (maximal) value, let it be $\boldsymbol{r}$, and we say that $a\{\ ,\ \}' + b\{\ ,\ \}''$ is *regular* (or that (a, b) is a *regular* point) if $\mathsf{rk}(a\pi' + b\pi'') = \boldsymbol{r}$. The Poisson centres $\mathcal{Z}_{a,b}$ of regular structures in the pencil generate a subalgebra $\mathcal{Z}(\{\ ,\ \}', \{\ ,\ \}'')$, which is Poisson-commutative w.r.t. all Poisson brackets in the pencil.

The Poisson tensor (bivector) π of the Lie–Poisson bracket $\{\ ,\ \}$ of $\mathcal{S}(\mathfrak{q})$ is defined by the formula $\pi(dH \wedge dF) = \{H, F\}$ for $H, F \in \mathcal{S}(\mathfrak{q})$. We have $\widehat{\gamma} = \pi(\gamma)$ and in this terms,

$$\mathsf{ind}\,\mathfrak{q} = \mathsf{dim}\,\mathfrak{q} - \mathsf{rk}\,\pi,$$

where $\mathsf{rk}\,\pi = \mathsf{max}_{\gamma\in\mathfrak{q}^*}\,\mathsf{rk}\,\pi(\gamma)$.

The Poisson centre of $(\mathcal{S}(\mathfrak{q}), \{\ ,\ \})$ is $\mathcal{Z}(\mathcal{S}(\mathfrak{q}), \{\ ,\ \}) = \mathcal{Z}(\mathfrak{q}) = \mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$.

There is a well-developed geometric machinery for dealing with algebras $\mathcal{Z}(\{\ ,\ \}', \{\ ,\ \}'') = \mathsf{alg}\left\langle \mathcal{Z}_{a,b} \mid \mathsf{rk}(a\pi'+b\pi'') = \boldsymbol{r} \right\rangle$.

3. Gaudin models

Suppose $\mathfrak{q} = \mathfrak{g}$ is semisimple. A Gaudin model related to $\mathfrak{h} = \mathfrak{g}^{\oplus n}$ consists of n quadratic Hamiltonians depending on $\vec{z} = (z_1, \ldots, z_n) \in \Bbbk^n$.

Let $\{x_i \mid 1 \leqslant i \leqslant \dim \mathfrak{g}\}$ be a basis of $\mathfrak{g}$ that is orthonormal w.r.t. the Killing form κ. Let $x_i^{(k)} \in \mathfrak{h}$ be a copy of x_i belonging to the k-th copy of $\mathfrak{g}$. Assume that $z_j \neq z_k$ for $j \neq k$ and set

$$\mathcal{H}_k = \sum_{j \neq k} \frac{\sum_{i=1}^{\dim \mathfrak{g}} x_i^{(k)} x_i^{(j)}}{z_k - z_j}, \quad 1 \leqslant k \leqslant n. \tag{2}$$

The Gaudin Hamiltonians $\mathcal{H}_k$ can be regarded as elements of either

$$\mathcal{U}(\mathfrak{g})^{\otimes n} \cong \mathcal{U}(\mathfrak{h}) \quad \text{or} \quad \mathcal{S}(\mathfrak{h}).$$

They commute in $\mathcal{U}(\mathfrak{h})$ and hence Poisson-commutate in $\mathcal{S}(\mathfrak{h})$.

Note that $\sum_{k=1}^{n} \mathcal{H}_k = 0$.

By the construction, each $\mathcal{H}_k$ is an invariant of the diagonal copy of $\mathfrak{g}$, i.e., of $\Delta\mathfrak{g} \subset \mathfrak{h}$.

4. Gaudin algebras

In 1994, B. Feigin, E .Frenkel, and N. Reshetikhin constructed *a large commutative algebra* $\mathcal{C}(\vec{z}) \subset \mathcal{U}(\mathfrak{h})^{\Delta\mathfrak{g}}$ *that contains all* $\mathcal{H}_k$.

The enveloping algebra $\mathcal{U}(\mathfrak{g}[t^{-1}])$ contains a large commutative subalgebra, the *Feigin–Frenkel centre* $\mathfrak{z}(\widehat{\mathfrak{g}}, t^{-1})$. Let $\Delta\mathcal{U}(\mathfrak{g}[t^{-1}]) \cong \mathcal{U}(\mathfrak{g}[t^{-1}])$ be the diagonal of $\mathcal{U}(\mathfrak{g}[t^{-1}])^{\otimes n}$. Suppose that $\vec{z} \in (\Bbbk^{\times})^n$. Then $\vec{z}$ defines a natural homomorphism $\rho_{\vec{z}}\colon \Delta\mathcal{U}(\mathfrak{g}[t^{-1}]) \to \mathcal{U}(\mathfrak{g})^{\otimes n}$, where

$$\rho_{\vec{a}}(xt^k) = z_1^k x^{(1)} + z_2^k x^{(2)} + \ldots + z_n^k x^{(n)} \in \mathfrak{g} \oplus \mathfrak{g} \oplus \ldots \oplus \mathfrak{g} \ \text{ for } \ x \in \mathfrak{g}.$$

Let $\mathcal{C}(\vec{z})$ be the image of $\mathfrak{z}(\widehat{\mathfrak{g}}, t^{-1})$ under $\rho_{\vec{z}}$. If $z_j \neq z_k$ for $j \neq j$, then $\mathcal{C}(\vec{z})$ contains the Hamiltonians $\mathcal{H}_k$ associated with $\vec{z}$.

According to [Chervov, Falqui, and Rybnikov (2010)],

- $\text{tr.deg}\, \mathcal{C}(\vec{z}) = \frac{n-1}{2}(\dim\mathfrak{g} + \text{rk}\,\mathfrak{g}) + \text{rk}\,\mathfrak{g} = \boldsymbol{b}^{\Delta\mathfrak{g}}(\mathfrak{h})$,
- $\mathcal{C}(\vec{z})$ is a polynomial algebra (with $\boldsymbol{b}^{\Delta\mathfrak{g}}(\mathfrak{h})$ generators).

In the literature, one finds often the following (wrong) statement:

- $\mathcal{C}(\vec{z})$ *is a maximal commutative subalgebra of* $\mathcal{U}(\mathfrak{h})$.

The correct one is

◇ $\mathcal{C}(\vec{z})$ is a maximal commutative subalgebra of $\mathcal{U}(\mathfrak{h})^{\Delta\mathfrak{g}}$.

The proof in [CFR] uses some limit-constructions and a connection with *Mishchenko–Fomenko subalgebars*.

The associated graded algebra $\mathrm{gr}(\mathcal{C}(\vec{z})) \subset \mathcal{S}(\mathfrak{h})$ is Poisson-commutative.

Question. Is there a pair of compatible Poisson structures on $\mathfrak{h}^*$ that produces $\mathrm{gr}(\mathcal{C}(\vec{z}))$ by the Lenard–Magri scheme?

Not claiming this to be a general remedy, but nevertheless:

If you do not see a solution, let the problem stand on its head.

5. Quotients of the current algebra

Let $p \in \Bbbk[t]$ be a normalised polynomial of degree $n \geqslant 1$. Then the quotient $\mathfrak{q}[t]/(p) \cong \mathfrak{q} \otimes (\Bbbk[t]/(p))$ is a Lie algebra and as a vector space it is isomorphic to

$$\mathbb{W} = \mathbb{W}(\mathfrak{q}, n) = \mathfrak{q}{\cdot}1 \oplus \mathfrak{q}\bar{t} \oplus \ldots \oplus \mathfrak{q}\bar{t}^{n-1},$$

where $\bar{t}$ identifies with $t + (p)$. Let $[\ ,\ ]_p$ be the Lie bracket on $\mathbb{W}$ given by p, i.e., $\mathfrak{q}[t]/(p) \cong (\mathbb{W}, [\ ,\ ]_p)$ as a Lie algebra. We identify $\mathfrak{q}$ with $\mathfrak{q}{\cdot}1 \subset \mathbb{W}$. In a particular case $p = t^n$, set $\mathfrak{q}\langle n\rangle = \mathfrak{q}[t]/(t^n)$. The Lie algebra $\mathfrak{q}\langle n\rangle$ is known as a (generalised) *Takiff algebra* modelled on $\mathfrak{q}$. Note that $\mathfrak{q}\langle 1\rangle \cong \mathfrak{q}$. If $\dim \mathfrak{q} < \infty$, then by [Raïs–Tauvel (1992)], we have

$$\operatorname{ind} \mathfrak{q}\langle n\rangle = n{\cdot}\operatorname{ind} \mathfrak{q}. \qquad (3)$$

From now on, assume that $\Bbbk = \bar{\Bbbk}$.

Proposition 2. *Suppose* $p = \prod_{i=1}^{u}(t - a_i)^{m_i}$, *where* $a_i \neq a_j$ *for* $i \neq j$, *we have* $m_i \geqslant 1$ *for each* $i \leqslant u$, *and* $\sum_{i=1}^{u} m_i = n$. *Then* $\mathfrak{q}[t]/(p) \cong \bigoplus_{i=1}^{u} \mathfrak{q}\langle m_i\rangle$.

In a finite-dimensional case, Proposition 2 implies: $\operatorname{ind}(\mathbb{W}, [\ ,\ ]_p) = n{\cdot}\operatorname{ind} \mathfrak{q}$.

Example 3. Suppose $m_i = 1$ for each i. Set $r_i = \dfrac{p}{(t-a_i)} \prod\limits_{j\neq i}(a_i - a_j)^{-1}$. Then $r_i^2 \equiv r_i \pmod{p}$. This is an explicit application of the Chinese remainder theorem. Each subspace $\mathfrak{q}\bar{r}_i$ is a Lie subalgebra of $\mathfrak{q}[t]/(p)$, isomorphic to $\mathfrak{q}$, and

$$\mathfrak{q}[t]/(p) = \mathfrak{q}\bar{r}_1 \oplus \mathfrak{q}\bar{r}_2 \oplus \ldots \oplus \mathfrak{q}\bar{r}_n. \tag{4}$$

In particular, $\mathfrak{g}[t]/(p) \cong \mathfrak{g}^{\oplus n}$ is semisimple if $\mathfrak{g}$ is semisimple.

6. Compatible brackets $\{\ ,\ \}_{p_1}$ and $\{\ ,\ \}_{p_2}$ on $\mathcal{S}(\mathbb{W})$

Proposition 4. *Let $p_1, p_2 \in \Bbbk[t]$ be distinct normalised polynomials of degree n. If we have $\deg(p_1 - p_2) \leqslant 1$, then the Lie–Poisson brackets $\{\ ,\ \}_{p_1}$ and $\{\ ,\ \}_{p_2}$ are compatible. More explicitly, $a\{\ ,\ \}_{p_1} + (1-a)\{\ ,\ \}_{p_2} = \{\ ,\ \}_{ap_1+(1-a)p_2}$.*

The pencil $L(p_1,p_2) := \langle \{\ ,\ \}_{p_1}, \{\ ,\ \}_{p_2} \rangle$ contains the unique singular line $\Bbbk\ell$ with $\ell = \{\ ,\ \}_{p_1} - \{\ ,\ \}_{p_2}$ and always $\mathrm{ind}(\mathbb{W},\ell) = \dim\mathfrak{q} + (n-1)\mathrm{ind}\,\mathfrak{q}$.

The bracket $[\mathfrak{q}\cdot 1, \mathbb{W}]_p$ is independent of p, thus $\mathcal{Z}_p = \mathcal{Z}(\mathcal{S}(\mathbb{W}), \{\ ,\ \}_p) \subset \mathcal{S}(\mathbb{W})^{\mathfrak{q}}$ and hence $\mathcal{Z}(p_1,p_2) = \mathrm{alg}\langle \mathcal{Z}_p \mid p = ap_1 + (1-a)p_2 \rangle \subset \mathcal{S}(\mathbb{W})^{\mathfrak{q}}$.

Example 5. Set $p = p_1 = t^n - 1$, $\tilde{p} = p_2 = t^n$. Then

$$L(p, \tilde{p}) = \Big\{ \Bbbk\{\ ,\ \}_{t^n+\alpha},\ \ \Bbbk\ell \mid \alpha \in \Bbbk, \ell = \{\ ,\ \}_{t^n-1} - \{\ ,\ \}_{t^n} \Big\}.$$

Here $(\mathbb{W}, [\ ,\ ]_{t^n+\alpha}) \cong \mathfrak{q}^{\oplus n}$ if $\alpha \neq 0$;

$(\mathbb{W}, [\ ,\ ]_{t^n})$ is the Takiff algebra $\mathfrak{q}\langle n\rangle$;

and $\ell(x\bar{t}^a, y\bar{t}^b) = \begin{cases} 0, & \text{if } a+b<n; \\ [x,y]\bar{t}^{a+b-n}, & \text{if } a+b \geqslant n, \end{cases}$ for $x, y \in \mathfrak{q}$.

The Lie algebra $(\mathbb{W}, \ell)$ is an $\mathbb{N}$-graded: $\mathbb{W} = \mathfrak{q}\bar{t}^{n-1} \oplus \mathfrak{q}\bar{t}^{n-2} \oplus \ldots \oplus \mathfrak{q}\bar{t} \oplus \mathfrak{q}{\cdot}1$, it is isomorphic to $(\tilde{t}\mathfrak{q}[\tilde{t}])/(\tilde{t}^{n+1})$ and to the nilpotent radical of $\mathfrak{q}\langle n{+}1\rangle$.

The bracket $\{\ ,\ \}_{t^n}$ is a *contraction* of $\{\ ,\ \}_{t^n-1}$ related to a cyclic permutation of the summands. In case $\mathfrak{q} = \mathfrak{g}$ is reductive, $\mathcal{Z}(t^n - 1, t^n)$ was already studied [PANYUSHEV–Y. (2021)].

Example 6. Suppose $n \geqslant 2$. Set $p = t^n - t$, $\tilde{p} = t^n$, $\ell = \{\ ,\ \}_{t^n-t} - \{\ ,\ \}_{t^n}$. Then

$$\ell(x\bar{t}^a, y\bar{t}^b) = \begin{cases} 0, & \text{if } a+b<n; \\ [x,y]\bar{t}^{a+b+1-n}, & \text{if } a+b \geqslant n, \end{cases} \quad \text{for } x, y \in \mathfrak{q}$$

and $(\mathbb{W}, \ell) \cong \mathfrak{q}\langle n{-}1\rangle \oplus \mathfrak{q}^{\mathsf{ab}}$.

Theorem (The case $\mathfrak{q}=\mathfrak{g}$). (i) *If* $p=\prod_{i=1}^{n}(x-a_i)$, *where* $a_i\neq a_j$ *for* $i\neq j$ *and* $p(0)\neq 0$, *then we have* $\mathcal{Z}(p,p+t)=\mathrm{gr}(\mathcal{C}(\vec{z}))$ *for* $\vec{z}=(a_1^{-1},\ldots,a_n^{-1})$.

(ii) *Under the same assumptions on* p, *each* $\tilde{\mathcal{H}}_k=\sum\limits_{j\neq k}\frac{\sum_{i=1}^{\dim\mathfrak{g}}x_i^{(k)}x_i^{(j)}}{a_k-a_j}\in\mathcal{S}(\mathfrak{h})$ *with* $1\leqslant k\leqslant n$ *is an element of* $\mathcal{Z}(p,p+1)$.

7. Some explanations

If $p(0)\neq 0$, then the quotient map $\psi_p\colon\Bbbk[t]\to\Bbbk[t]/(p)$ extends to $\mathfrak{q}[t,t^{-1}]$ and to $\mathcal{U}(\mathfrak{q}[t^{-1}])$. If $\mathfrak{q}=\mathfrak{g}$ and the roots of p are distinct, then we identify $\mathfrak{h}=\mathfrak{g}^{\oplus n}$ with $\mathfrak{q}[t]/(p)$ and write also $\mathfrak{h}=\psi_p(\mathfrak{q}[t^{-1}])$. As can be easily seen,

$$\mathcal{C}(\vec{a})=\psi_p(\mathfrak{z}(\widehat{\mathfrak{g}},t^{-1}))\quad\text{if}\quad p=\textstyle\prod_i(x-a_i).$$

Next $\mathrm{gr}(\mathcal{C}(\vec{a}))=\psi_p(\mathcal{Z}(\widehat{\mathfrak{g}},t^{-1}))$, where $\mathcal{Z}(\widehat{\mathfrak{g}},t^{-1})=\mathrm{gr}(\mathfrak{z}(\widehat{\mathfrak{g}},t^{-1}))$.

For any $\mathfrak{q}$, the algebra $\mathcal{Z}(\widehat{\mathfrak{q}},t^{-1})$ is defined as

◇ $\mathcal{Z}(\widehat{\mathfrak{q}},t^{-1})=\mathcal{S}(t^{-1}\mathfrak{q}[t^{-1}])^{\mathfrak{q}[t]}$, where $\mathcal{S}(t^{-1}\mathfrak{q}[t^{-1}])$ is regarded as the quotient of $\mathcal{S}(\mathfrak{q}[t,t^{-1}])$ by the ideal $(\mathfrak{q}[t])$, i.e., $\mathcal{Z}(\widehat{\mathfrak{q}},t^{-1})$ consists of the elements $Y\in\mathcal{S}(t^{-1}\mathfrak{q}[t^{-1}])$ such that $\{xt^k,Y\}\in\mathfrak{q}[t]\mathcal{S}(\mathfrak{q}[t,t^{-1}])$ for all $x\in\mathfrak{q}$ and $k\geqslant 0$.

It is more convenient to switch the variable: $t^{-1} \mapsto t$ in $\mathfrak{q}[t, t^{-1}]$,

i.e., $\mathcal{Z}(\widehat{\mathfrak{q}}, t^{-1}) \mapsto \mathcal{Z}(\widehat{\mathfrak{q}}, t)$.

Then part (i) of the theorem reads: $\mathcal{Z}(p, p+t) = \psi_p(\mathcal{Z}(\widehat{\mathfrak{g}}, t))$.

Conjecture 7. *For any finite-dimensional Lie algebra $\mathfrak{q}$ and any normalised $p \in \Bbbk[t]$ of degree n such that $p(0) \neq 0$, we have $\psi_p(\mathcal{Z}(\widehat{\mathfrak{q}}, t)) = \mathcal{Z}(p, p+t)$.*

8. Non-reductive and reductive-like Lie algebras

◇ For any $\mathfrak{q}$, we have $\{\mathcal{Z}(\widehat{\mathfrak{q}}, t), \mathcal{Z}(\widehat{\mathfrak{q}}, t)\} = 0$.

◇ The existence of $\mathfrak{z}(\widehat{\mathfrak{q}}, t^{-1})$ (i.e., of a quantisation for $\mathcal{Z}(\widehat{\mathfrak{q}}, t^{-1})$) is not well-documented. Probably one has to assume that $\mathfrak{q}$ is quadratic. For the centralisers $\mathfrak{g}_\gamma$ with $\gamma \in \mathfrak{g}$, the problem is settled, affirmatively, [Arakawa–Premet (2017)].

Reductive-like properties of a Lie algebra:

Set $\mathfrak{q}^*_{\mathsf{sing}} = \{\eta \in \mathfrak{q}^* \mid \dim \mathfrak{q}_\eta > \operatorname{ind} \mathfrak{q}\}$.

$(\Diamond_1)$ $\operatorname{tr.deg} \mathcal{Z}(\mathfrak{q}) = \operatorname{ind} \mathfrak{q}$ (enough symmetric invariants).

$(\Diamond_k)_{k=2,3}$ $\dim \mathfrak{q}^*_{\mathsf{sing}} \leqslant \dim \mathfrak{q} - k$ (*codim–k* property).

$(\Diamond_4)$ $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}} = \Bbbk[F_1, \dots, F_m]$ is a polynomial ring in $m = \operatorname{ind} \mathfrak{q}$ variables and $\Omega_{\mathfrak{q}^*} = \{\xi \in \mathfrak{q}^* \mid (d_\xi F_1) \wedge \ldots \wedge (d_\xi F_m) \neq 0\}$ is a big open subset of $\mathfrak{q}^*$ (i.e., the complement of $\Omega_{\mathfrak{q}^*}$ does not contain divisors).

Results:

$\diamond$ If $\mathfrak{q}$ satisfies $(\Diamond_1)$ and $(\Diamond_2)$, then $\operatorname{tr.deg} \mathcal{Z}(p, p+l) = \frac{n-1}{2}(\dim \mathfrak{q} + \operatorname{ind} \mathfrak{q}) + \operatorname{ind} \mathfrak{q}$ for any p and any l with $0 \leqslant \deg l \leqslant 1$.

$\diamond$ If $\mathfrak{q}$ satisfies $(\Diamond_2)$ and $(\Diamond_4)$, then $\mathcal{Z}(p, p+l)$ is a polynomial ring (for any p and l as above).

$\diamond$ If $\mathfrak{q}$ satisfies $(\Diamond_3)$ and $(\Diamond_4)$, then $\mathcal{Z}(p, p+l)$ is a maximal (w.r.t. inclusion) Poisson-commutative subalgebra of $(\mathcal{S}(\mathbb{W}), \{\ ,\ \}_p)^{\mathfrak{q}}$.

$\diamond$ If $\mathfrak{q}$ satisfies $(\Diamond_4)$ and $p(0) \neq 0$, then $\mathcal{Z}(p, p+t) = \psi_p(\mathcal{Z}(\widehat{\mathfrak{q}}, t))$ (Conj. 7 holds).

9. Reasons and ideas

Definition 8 (Polarisation)**.** For $\vec{k} = (k_1, \ldots, k_d) \in \mathbb{Z}$ such that $0 \leqslant k_j < n$ for each j, the $\vec{k}$*-polarisation* of $Y = \prod_i y_i \in \mathcal{S}^d(\mathfrak{q})$ is

$$Y[\vec{k}] := (d!)^{-1} |\mathsf{S}_d{\cdot}\vec{k}| \sum_{\theta \in \mathsf{S}_d} y_1 \bar{t}^{\theta(k_1)} \ldots y_d \bar{t}^{\theta(k_d)} \in \mathcal{S}(\mathbb{W}) \,.$$

The notion extends to all $F \in \mathcal{S}^d(\mathfrak{q})$ by linearity; $\mathsf{Pol}(F) := \left\langle F[\vec{k}] \mid \vec{k} \text{ as above} \right\rangle$.

Theorem 9 (Raïs–Tauvel, Arakawa–Premet, Panyushev–Y.)**.** *Suppose that* $\mathfrak{q}$ *satisfies* $(\diamondsuit_4)$*. Then, for any* $n \geqslant 1$*, the Takiff algebra* $\mathfrak{q}\langle n\rangle$ *has the same properties as* $\mathfrak{q}$*. In particular,* $\mathcal{Z}(\mathfrak{q}\langle n\rangle)$ *is a graded polynomial ring of Krull dimension* $\mathsf{ind}\,\mathfrak{q}\langle n\rangle = nm$ *and algebraically independent generators of* $\mathcal{Z}(\mathfrak{q}\langle n\rangle)$ *are polarisations of the polynomials* F_j *with* $1 \leqslant j \leqslant m$*.*

The evaluation at $t = 1$ defines an isomorphism $\mathsf{Ev}_1 : \mathcal{S}(\mathfrak{q}t) \to \mathcal{S}(\mathfrak{q})$ of $\mathfrak{q}$-modules. For $F \in \mathcal{S}(\mathfrak{q})$, set $F[t] := \mathsf{Ev}_1^{-1}(F) \in \mathcal{S}(\mathfrak{q}t)$. If $F \in \mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$, then $F[t] \in \mathcal{Z}(\widehat{\mathfrak{q}}, t)$. Set $\tau = t^2\partial_t$.

Corollary 10. *If* $\mathfrak{q}$ *satisfies* $(\diamondsuit_4)$*, then* $\mathcal{Z}(\widehat{\mathfrak{q}}, t)$ *is a polynomial ring generated by* $\tau^k(F_j[t])$ *with* $k \geqslant 0$ *and* $1 \leqslant j \leqslant m$*.*

Assume that $\mathfrak{q}$ satisfies ($\lozenge_4$). On the one hand, each $\mathcal{Z}_p$, and hence also each $\mathcal{Z}(p, p+l)$, is generated by polarisations of the invariants F_j; on the other hand, each $\psi_p(\tau^k(F_j[t])) \in \mathsf{Pol}(F_j)$ for any j and k.

Suppose that $\mathfrak{q}$ is quadratic and let $\boldsymbol{h} \in \mathcal{S}^2(\mathfrak{q})^{\mathfrak{q}}$ be a non-degenerate scalar product. (In case $\mathfrak{q} = \mathfrak{g}$ is semisimple let $\boldsymbol{h}$ be the dual of κ.) Then $\boldsymbol{h}[t] \in \mathcal{Z}(\widehat{\mathfrak{q}}, t)$ and also $\boldsymbol{h}[t] \in \mathfrak{z}(\widehat{\mathfrak{q}}, t)$. By [RYBNIKOV (2008)], $\mathfrak{z}(\widehat{\mathfrak{g}}, t)$ is the centraliser of $\boldsymbol{h}[t]$ in $\mathcal{U}(t\mathfrak{g}[t])$ and $\mathcal{Z}(\widehat{\mathfrak{g}}, t)$ is the Poisson centraliser of $\boldsymbol{h}[t]$ in $\mathcal{S}(t\mathfrak{g}[t])$.

Suppose $n \geqslant 2$. Clearly $\boldsymbol{h}[(1,1)] \in \psi_p(\mathcal{Z}(\widehat{\mathfrak{q}}, t))$ for any p. By a small calculation, $\boldsymbol{h}[(1,1)] \in \mathcal{Z}(p, p+t)$ for any p.

Proposition 11. *Suppose that* $\mathfrak{g} = \mathfrak{sl}_d$ *and* $F \in \mathcal{S}^d(\mathfrak{g})$ *is such that* $F(\xi) = \mathsf{det}(\xi)$ *for* $\xi \in \mathfrak{g}^* \cong \mathfrak{sl}_d$. *Then for any* p *of degree* n, *we have*

$$\mathsf{dim}\{f \in \mathsf{Pol}(F) \mid \{f, \boldsymbol{h}[(1,1)]\}_p = 0\} \leqslant (n-1)d + 1.$$

From this inequality one can deduce: $\mathsf{dim}(\mathsf{Pol}(F) \cap \mathcal{Z}(p, p+l)) = (n-1)d + 1$ and $\mathsf{Pol}(F) \cap \mathcal{Z}(p, p+l) = \mathsf{Pol}(F) \cap \psi_p(\mathcal{Z}(\widehat{\mathfrak{q}}, t))$, whenever $F \in \mathcal{S}^d(\mathfrak{q})^{\mathfrak{q}}$ is nonzero and $p(0) \neq 0$.

Having a non-degenerate invariant scalar product, we may choose an orthonormal basis $\{x_i\} \subset \mathfrak{q}$ and defined the (generalised) Gaudin Hamiltonian by the same formulas as in (2). Set $p = \prod_i (t - z_i)$.

Example 12. Return to Example 3 and polynomials $\bar{r}_i$ defined there.

Let $\boldsymbol{h}[\bar{r}_i] \in \mathcal{S}^2(\mathfrak{q}\bar{r}_i)^{\mathfrak{q}}$ be the image of $\boldsymbol{h}$ under the canonical isomorphism extended from the map $x \mapsto x\bar{r}_i$ (here $x \in \mathfrak{q}$). Then

$$\boldsymbol{h}[(1,1)] = \sum_{i=1}^{\dim \mathfrak{q}} (x_i \bar{t})^2 = -2 \left(\sum_k z_k \mathcal{H}_k \right) + \sum_k z_k^2 \boldsymbol{h}[\bar{r}_k].$$

Furthermore $\dim\{f \in \mathrm{Pol}(\boldsymbol{h}) \mid \{f, \boldsymbol{h}[(1,1)]\}_p = 0\} = 2n - 1$ and this subspace has a basis $\{\mathcal{H}_k,\ \boldsymbol{h}[\bar{r}_j] \mid 1 \leqslant k < n,\ 1 \leqslant j \leqslant n\}$. If p has nonzero distinct roots, then this another basis:

$$\{\boldsymbol{h}[(1,1)],\ \psi_p \circ \tau(\boldsymbol{h}[t]), \ldots, \psi_p \circ \tau^{n-2}(\boldsymbol{h}[t]),\ \boldsymbol{h}[\bar{r}_j] \mid 1 \leqslant j \leqslant n\}.$$

One may say that the generalised Gaudin model $(\mathfrak{q}^{\oplus n}, \mathcal{H}_1, \ldots, \mathcal{H}_n)$ is equivalent to $(\mathbb{W}, [\ ,\ ]_p, \boldsymbol{h}[(1,1)], \psi_p \circ \tau(\boldsymbol{h}[t]), \ldots, \psi_p \circ \tau^{n-2}(\boldsymbol{h}[t]))$. The elements $\psi_p \circ \tau^k(\boldsymbol{h}[t])$ with $k \leqslant n-2$ do not depend on p, we have

$$\psi_p \circ \tau^k(\boldsymbol{h}[t]) = k! \sum_{1 \leqslant a,b;\ a+b=k+2} \left(\textstyle\sum_{i=1}^{\dim \mathfrak{q}} x_i \bar{t}^a x_i \bar{t}^b \right).$$