Symmetrisation and the Feigin–Frenkel centre

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Let $\mathfrak{g} = \mathrm{Lie}\, G$ be reductive. Assume that $G = G^\circ$ is connected. Let $\mathcal{Z}(\mathfrak{g}) \subset \mathcal{U}(\mathfrak{g})$ be the centre of the enveloping algebra. Then

$$\mathbb{C}[\mathfrak{g}^*]^G = \mathcal{S}(\mathfrak{g})^{\mathfrak{g}} \cong \mathcal{Z}(\mathfrak{g}).$$

We call $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ the algebra *of symmetric invariants of* $\mathfrak{g}$.

As is well-known, $\mathfrak{g}^* \cong \mathfrak{g}$ as a G-module, $\varphi\colon \mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[\mathfrak{t}]^W$ ist an isomorphism (Chevalley) for a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$ and the Weyl group $W = N_G(\mathfrak{t})/Z_G(\mathfrak{t})$ is a finite reflection group. Hence $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}} = \mathbb{C}[H_1, \ldots, H_\ell]$ is a polynomial ring in $\ell = \mathrm{rk}\,\mathfrak{g} = \dim \mathfrak{t}$ variables.

For a vector space V, let $\varpi\colon \mathcal{S}^k(V) \to V^{\otimes k}$ be the canonical symmetrisation map. If V is a G-module, then ϖ is a homomorphism of G-modules. For a Lie algebra $\mathfrak{q}$, we let ϖ stand also for the symmetrisation map from $\mathcal{S}(\mathfrak{q})$ to $\mathcal{U}(\mathfrak{q})$. Then $\varpi\colon \mathcal{S}(\mathfrak{q})^{\mathfrak{q}} \to \mathcal{Z}(\mathfrak{q})$ is an isomorphism of vector spaces.

Current and Takiff algebras

Let $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$ be the *current algebra* associated with $\mathfrak{g}$. The *truncated* current algebra

$$\mathfrak{g}\langle n\rangle := \mathfrak{g} \otimes \mathbb{C}[t]/(t^n) = \mathfrak{g}[t]/(t^n),$$

is also known as a (generalised) *Takiff algebra* modelled on $\mathfrak{g}$. If $n > 1$, then $\mathfrak{g}\langle n\rangle$ is no longer reductive. Nevertheless, $\mathcal{S}(\mathfrak{g}\langle n\rangle)^{\mathfrak{g}\langle n\rangle}$ is a polynomial ring in $n{\cdot}\mathrm{rk}\,\mathfrak{g}$ variables by a theorem of Raïs and Tauvel.

The current algebra $\mathfrak{g}[t]$ acts on $\mathfrak{g}\langle n\rangle^* = \mathfrak{g}^* \oplus (\mathfrak{g}\bar{t})^* \oplus (\mathfrak{g}\bar{t}^2)^* \oplus \ldots \oplus (\mathfrak{g}\bar{t}^{n-1})^*$ in the same way as it acts on $\mathbb{W}_n := \mathfrak{g}^* t^{-n} \oplus \mathfrak{g}^* t^{-n+1} \oplus \ldots \oplus \mathfrak{g}^* t^{-1} \subset \mathfrak{g}^*[t, t^{-1}]/\mathfrak{g}^*[t]$.

Set $\widehat{\mathfrak{g}}^- = t^{-1}\mathfrak{g}[t^{-1}]$ and identify $\mathcal{S}(\widehat{\mathfrak{g}}^-)$ with $\mathcal{S}(\mathfrak{g}[t,t^{-1}])/(\mathfrak{g}[t])$. Then $\mathcal{S}(\widehat{\mathfrak{g}}^-)^{\mathfrak{g}[t]} = \varinjlim \mathcal{S}(\mathbb{W}_n)^{\mathfrak{g}[t]}$ is a polynomial ring in infinitely many variables.

Two features of $\bar{\mathfrak{z}}(\widehat{\mathfrak{g}}) = \mathcal{S}(\widehat{\mathfrak{g}}^-)^{\mathfrak{g}[t]}$ are

- algebraically independent generators can be described very explicitly due to a construction of Raïs–Tauvel;
- it is a *Poisson-commutative subalgebra* of $\mathcal{S}(\widehat{\mathfrak{g}}^-)$.

Poisson bracket on $\mathcal{S}(\mathfrak{q})$

Let $\mathfrak{q}$ be a Lie algebra. The symmetric algebra $\mathcal{S}(\mathfrak{q})$ carries the standard Lie–Poisson structure:

- ◇ $\{\xi,\eta\} = [\xi,\eta]$ for all $\xi,\eta \in \mathfrak{q}$, extends further by the Leibniz rule;
- ◇ $\{F_1,F_2\}(\gamma) = \gamma([d_\gamma F_1, d_\gamma F_2])$ for all $F_1, F_2 \in \mathcal{S}(\mathfrak{q})$, $\gamma \in \mathfrak{q}^*$;
- ◇ $\{\mathbf{f} + \mathcal{U}_a(\mathfrak{q}), \mathbf{h} + \mathcal{U}_b(\mathfrak{q})\} = [\mathbf{f},\mathbf{h}] + \mathcal{U}_{a+b}(\mathfrak{q})$ for $\mathbf{f} \in \mathcal{U}_{a+1}(\mathfrak{q})$, $\mathbf{h} \in \mathcal{U}_{b+1}(\mathfrak{q})$.

The third definition uses the fact that $\mathcal{S}(\mathfrak{q}) \cong \mathrm{gr}\,\mathcal{U}(\mathfrak{q})$. In these terms, $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}} = \{F \in \mathcal{S}(\mathfrak{q}) \mid \{\xi, F\} = 0\ \forall \xi \in \mathfrak{q}\}$.

Definition

A subalgebra $A \subset \mathcal{S}(\mathfrak{q})$ is *Poisson-commutative* if $\{A, A\} = 0$.

Quantisation problem: given a Poisson-commutative $A \subset \mathcal{S}(\mathfrak{q})$, find a commutative subalgebra $\widetilde{A} \subset \mathcal{U}(\mathfrak{q})$ such that $A = \mathrm{gr}(\widetilde{A})$.

The Feigin–Frenkel centre

There is a commutative subalgebra $\mathfrak{z}(\widehat{\mathfrak{g}}) \subset \mathcal{U}(\widehat{\mathfrak{g}}^-)^{\mathfrak{g}}$ s.t. $\mathrm{gr}(\mathfrak{z}(\widehat{\mathfrak{g}})) = \bar{\mathfrak{z}}(\widehat{\mathfrak{g}})$.

The fist proof of this result is given by B. Feigin and E. Frenkel in 1992. Roughly, $\mathfrak{z}(\widehat{\mathfrak{g}}) = \mathcal{U}(\widehat{\mathfrak{g}}^-)^{\mathfrak{g}[t]}$. Here $\mathcal{U}(\widehat{\mathfrak{g}}^-)$ has to be considered as a quotient $\mathcal{U}(\widehat{\mathfrak{g}})/J$ *at the critical level* for $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$.

Suppose $\mathfrak{g}$ is simple and $\mathrm{h}^\vee$ is the dual Coxeter number of $\mathfrak{g}$,

$$[xt^r, yt^m] = [x, y]t^{r+m} + r\delta_{r,-m}\frac{\mathrm{tr}(\mathrm{ad}(x)\mathrm{ad}(y))}{2\mathrm{h}^\vee}K \ \text{ for } x, y \in \mathfrak{g},$$

then take as J the left ideal generated by $\mathfrak{g}[t]$ and $K + \mathrm{h}^\vee$ (the reason is that $\mathcal{U}(\widehat{\mathfrak{g}})/J$ is a *vertex* algebra and $\mathfrak{z}(\widehat{\mathfrak{g}})$ is its centre). Some other features:

- the centre of the completed enveloping algebra $\widetilde{\mathcal{U}}_{-\mathrm{h}^\vee}(\widehat{\mathfrak{g}})$ can be obtained from $\mathfrak{z}(\widehat{\mathfrak{g}})$ by employing the vertex algebra structure;
- the image of $\mathfrak{z}(\widehat{\mathfrak{g}})$ in any quotient of $\mathcal{U}(\widehat{\mathfrak{g}}^-)$, by a two-sided ideal, is commutative, several quantisation problems are solved in this way (e.g. in the Gaudin model or for Mishchenko–Fomenko subalgebras).

On the structure of $\mathfrak{z}(\widehat{\mathfrak{g}})$

Set $\mathfrak{g}[a] = \mathfrak{g}t^a$, $x[a] = xt^a$ for $x \in \mathfrak{g}$.

By Feigin–Frenkel (1992), $\mathfrak{z}(\widehat{\mathfrak{g}}) = \mathbb{C}[\partial_t^r S_k \mid 1 \leqslant k \leqslant \ell, r \geqslant 0]$, where the symbols $\mathrm{gr}(S_k)$ generate $\mathcal{S}(\mathfrak{g}[-1])^{\mathfrak{g}}$. Such a set $\{S_k\}$ is said to be a *complete set of Segal–Sugawara vectors*.

The evaluation at $t = 1$ defines an isomorphism $\mathrm{Ev}_1 : \mathcal{S}(\mathfrak{g}[-1]) \to \mathcal{S}(\mathfrak{g})$ of $\mathfrak{g}$-modules. For $F \in \mathcal{S}(\mathfrak{g})$, let $F[-1]$ stand for $\mathrm{Ev}_1^{-1}(F) \in \mathcal{S}(\mathfrak{g}[-1])$. If $H \in \mathcal{S}^d(\mathfrak{g})^G$, then there is $S \in \mathfrak{z}(\widehat{\mathfrak{g}})$ such that

$$S = \varpi(H[-1]) + (\text{something mysterious in } \mathcal{U}_{<d}(\widehat{\mathfrak{g}}^-)^{\mathfrak{g}}).$$

Let $\{x_i\}$ be a basis of $\mathfrak{g}$ orthonormal w.r.t. a non-degenerate $\mathfrak{g}$-invariant scalar product. Then $\mathcal{H}[-1] = \sum_{i=1}^{\dim \mathfrak{g}} x_i[-1]x_i[-1] \in \mathfrak{z}(\widehat{\mathfrak{g}})$.

Theorem (L. Rybnikov, 2008)

We have $\mathfrak{z}(\widehat{\mathfrak{g}}) = \{X \in \mathcal{U}(\widehat{\mathfrak{g}}^-) \mid [X, \mathcal{H}[-1]] = 0\}$.

Explicit formulas in type A

In case $\mathfrak{g} = \mathfrak{gl}_n$, there are several explicit formulas for S_k by Chervov–Talalaev (2006) and Chervov–Molev (2009).

For $\gamma \in \mathfrak{g}^*$, write $\chi_\gamma(\lambda) = \det(\lambda I_n - \gamma)$ as

$$\lambda^n - \Delta_1(\gamma)\lambda^{n-1} + \ldots + (-1)^k \Delta_k(\gamma)\lambda^{n-k} + \ldots + (-1)^n \Delta_n(\gamma)$$

with $\Delta_k \in \mathcal{S}^k(\mathfrak{gl}_n)$. Then $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}} = \mathbb{C}[\Delta_1, \ldots, \Delta_n]$.

Set $\tau = -\partial_t$ and assume the conventions that

$$\tau x[a] - x[a]\tau = [\tau, x[a]] = \tau(x[a]) = -ax[a-1]$$

and $\tau \cdot 1 = 0$. For example, this leads to $\tau x[-1] \cdot 1 = x[-2]$.

Form the matrix $\boldsymbol{E}[-1] + \tau = (E_{ij}[-1]) + \tau I_n$ with $E_{ij} \in \mathfrak{gl}_n$ and calculate its column- and symmetrised determinants. Due to the fact that this matrix is *Manin*, the results are the same.

Explicit formulas in type A, continuation

The elements S_k are coefficients of τ^{n-k} in

$$\det_{\mathrm{sym}}(\boldsymbol{E}[-1]+\tau)=\varpi(\Delta_n[-1])+\varpi(\tau\Delta_{n-1}[-1])+\ldots$$
$$+\varpi(\tau^{n-2}\Delta_2[-1])+\varpi(\tau^{n-1}\Delta_1[-1])+\tau^n,$$

where ϖ acts on the summands of $\tau^{n-k}\Delta_k[-1]$ as on products of n factors, i.e., it permutes τ with elements of $\mathfrak{gl}_n[-1]$.

Let $\theta\in\mathrm{Aut}(\mathfrak{g})$ be a Weyl involution. Then $\theta(\Delta_k)=(-1)^k\Delta_k$ for $\mathfrak{g}=\mathfrak{gl}_n$. Set $\theta(t^{-1})=t^{-1}$, then θ acts on $\widehat{\mathfrak{g}}^-$ and $\theta(\mathcal{H}[-1])=\mathcal{H}[-1]$. Thus θ acts on $\mathfrak{z}(\widehat{\mathfrak{g}})$. Hence there are $S_1,\ldots,S_n$ that are eigenvectors of θ, namely

$$S_n=\varpi(\Delta_n[-1])+\varpi(\tau^2\Delta_{n-2}[-1])\cdot 1+\ldots$$
$$+\varpi(\tau^{2r}\Delta_{n-2r}[-1])\cdot 1+\ldots\varpi(\tau^{2m-2}\Delta_2[-1])\cdot 1,$$

$$S_k=\varpi(\Delta_k[-1])+\sum_{1\leqslant r<k/2}\binom{n-k+2r}{2r}\varpi(\tau^{2r}\Delta_{k-2r}[-1])\cdot 1. \qquad (1)$$

$$S_k = \varpi(\Delta_k[-1]) + \sum_{1\leqslant r<k/2} \binom{n-k+2r}{2r} \varpi(\tau^{2r}\Delta_{k-2r}[-1])\cdot 1.$$

- This is not the original form,
  - no one was interested in the symmetrisation map,
  - the degrees of invariants that appear in the sum have one and the same parity.
- Is there any (reasonable) connection between Δ_k and Δ_{k-2}?
  - The G-invariant Laplacian brings one, $\nabla^2(\Delta_k) \in \mathbb{C}\Delta_{k-2}$, but this does not help to understand the formula.
  - There is a certain map m that does help.

For $\mathfrak{gl}_N = \mathfrak{gl}_N(\mathbb{C}) = \mathrm{End}(\mathbb{C}^N)$ and $1 \leqslant r \leqslant k$, consider the linear map $\mathsf{m}_r \colon \mathfrak{gl}_N^{\otimes k} \to \mathfrak{gl}_N^{\otimes(k-r+1)}$ s.t. $y_1 \otimes \ldots \otimes y_k \mapsto y_1 y_2 \ldots y_r \otimes y_{r+1} \otimes \ldots \otimes y_k$.
Clearly $\mathsf{m}_r \circ \mathsf{m}_s = \mathsf{m}_{r+s-1}$.
Via $\mathrm{ad} \colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, the construction leads to $\mathsf{m}_r \colon \mathfrak{g}^{\otimes k} \to \mathfrak{gl}(\mathfrak{g}) \otimes \mathfrak{g}^{\otimes(k-r)}$.
Observe that

$$\mathrm{ad}(y_1)\mathrm{ad}(y_2)\ldots\mathrm{ad}(y_{2r+1}) + \mathrm{ad}(y_{2r+1})\ldots\mathrm{ad}(y_2)\mathrm{ad}(y_1) \in \mathfrak{so}(\mathfrak{g}) \cong \Lambda^2\mathfrak{g}.$$

We embed $\mathcal{S}^k(\mathfrak{g})$ in $\mathfrak{g}^{\otimes k}$ via ϖ and for each odd $2r+1 \leqslant k$, obtain a G-equivariant map $\mathsf{m}_{2r+1} \colon \mathcal{S}^k(\mathfrak{g}) \to \Lambda^2\mathfrak{g} \otimes \mathcal{S}^{k-2r-1}(\mathfrak{g}) \subset \Lambda^2\mathfrak{g} \otimes \mathfrak{g}^{\otimes(k-2r-1)}$.

Set $\mathsf{m} = \mathsf{m}_3$. Then $\mathsf{m} \colon \mathcal{S}^k(\mathfrak{g}) \to \Lambda^2\mathfrak{g} \otimes \mathcal{S}^{k-3}(\mathfrak{g})$. For example, if $Y = y_1 y_2 y_3 \in \mathcal{S}^3(\mathfrak{g})$, then $\mathsf{m}(Y) \in \mathfrak{so}(\mathfrak{g})$ is equal to

$$\frac{1}{6}\big(\mathrm{ad}(y_1)\mathrm{ad}(y_2)\mathrm{ad}(y_3) + \mathrm{ad}(y_3)\mathrm{ad}(y_2)\mathrm{ad}(y_1) + \mathrm{ad}(y_1)\mathrm{ad}(y_3)\mathrm{ad}(y_2) +$$
$$+ \mathrm{ad}(y_2)\mathrm{ad}(y_3)\mathrm{ad}(y_1) + \mathrm{ad}(y_2)\mathrm{ad}(y_1)\mathrm{ad}(y_3) + \mathrm{ad}(y_3)\mathrm{ad}(y_1)\mathrm{ad}(y_2)\big).$$

For convenience, put $\mathsf{m}(\mathcal{S}^k(\mathfrak{g})) = 0$ for $k \leqslant 2$.

We have defined the maps $\mathrm{m}_{2r+1}\colon \mathcal{S}^k(\mathfrak{g}) \to \Lambda^2\mathfrak{g} \otimes \mathcal{S}^{k-2r-1}(\mathfrak{g})$ and set $\mathrm{m} = \mathrm{m}_3$.

Suppose that $\mathfrak{g}$ is simple (and non-Abelian). Then $\mathrm{ad}\colon \mathfrak{g} \hookrightarrow \mathfrak{so}(\mathfrak{g})$ and this is the unique copy of $\mathfrak{g}$ in $\mathfrak{so}(\mathfrak{g}) \cong \Lambda^2\mathfrak{g}$.

For certain elements $H \in \mathcal{S}^k(\mathfrak{g})$, we have $\mathrm{m}(H) \in \mathfrak{g} \otimes \mathcal{S}^{k-3}(\mathfrak{g})$. If $\mathrm{m}(H) \in \mathcal{S}^{k-2}(\mathfrak{g})$, then $\mathrm{m}_{2r+1}(H) = \mathrm{m}_{2\mathrm{r}-1} \circ \mathrm{m}(H)$.

If $H \in \mathcal{S}^k(\mathfrak{g})^G$, then $\mathrm{m}(H)$ is also a G-invariant. We will be looking for $H \in \mathcal{S}(\mathfrak{g})^G$ such that $\mathrm{m}(H) \in \mathcal{S}^{k-2}(\mathfrak{g})^G$.

Note that $\mathrm{m}(\mathcal{S}^3(\mathfrak{g})^{\mathfrak{g}}) = 0$, since $(\Lambda^2\mathfrak{g})^{\mathfrak{g}} = 0$.

Back to type A

Take $\mathfrak{g} = \mathfrak{sl}_n \subset \mathfrak{gl}_n$. Set $\tilde{\Delta}_k = \Delta_k|_{\mathfrak{sl}_n}$. Then $\tilde{\Delta}_k$ can be inserted into (1), i.e., $\tilde{S}_{k-1} = \varpi(\tilde{\Delta}_k[-1]) + \sum_{1\leqslant r<k/2} \binom{n-k+2r}{2r} \varpi(\tau^{2r}\tilde{\Delta}_{k-2r}[-1]){\cdot}1 \in \mathfrak{z}(\widehat{\mathfrak{g}})$.

Proposition (Y., 2019)

We have $\mathrm{m}_{2r+1}(\tilde{\Delta}_k) = \frac{(2r)!(k-2r)!}{k!}\binom{n-k+2r}{2r}\tilde{\Delta}_{k-2r}$ *if* $k-2r>1$ *and* $\mathrm{m}(\tilde{\Delta}_3) = \mathrm{m}(\Delta_3) = 0$.

Next put this into the formula for $\tilde{S}_{k-1}$. Then

$$\tilde{S}_{k-1} = \varpi(H[-1]) + \sum_{1\leqslant r<(k-1)/2} \binom{k}{2r} \varpi(\tau^{2r}\mathrm{m}_{2r+1}(H)[-1]){\cdot}1$$

for $H = \tilde{\Delta}_k$.

$\tilde{S}_{k-1} = \varpi(H[-1]) + \sum_{1\leqslant r<(k-1)/2} \binom{k}{2r}\varpi(\tau^{2r}\mathrm{m}_{2r+1}(H)[-1])\cdot 1$ for $H = \tilde{\Delta}_k$ in type A.

Theorem (Y., 2019)

Suppose that for some $H \in \mathcal{S}^k(\mathfrak{g})^G$, we have $\mathrm{m}_{2r+1}(H) \in \mathcal{S}^{k-2r}(\mathfrak{g})^G$ for each $r \geqslant 1$. Then

$$S = \varpi(H[-1]) + \sum_{1\leqslant r<(k-1)/2} \binom{k}{2r}\varpi(\tau^{2r}\mathrm{m}_{2r+1}(H)[-1])\cdot 1 \in \mathfrak{z}(\widehat{\mathfrak{g}}).$$

Theorem (Y., 2019)

If $F \in \mathcal{S}^k(\mathfrak{g})^G$, then $\varpi(F[-1]) \in \mathfrak{z}(\widehat{\mathfrak{g}}) \iff \mathrm{m}(F) = 0$.

The next question is: do such invariants H exist outside type A?

Yes! For example, if $\mathrm{Pf} \in \mathcal{S}^{\ell}(\mathfrak{so}_{2\ell})$ is the Pfaffian, then $\mathrm{m}(\mathrm{Pf}) = 0$.

If $\mathfrak{g} = \mathfrak{sp}_{2n}$ or $\mathfrak{g} = \mathfrak{so}_n$, then it has a standard basis $F_{ij} = E_{ij} - \epsilon_i\epsilon_j E_{j'i'}$. For $\mathfrak{sp}_{2n}$, we have $\mathcal{S}(\mathfrak{g})^G = \mathbb{C}[H_1, \ldots, H_n]$, where $H_k = \Delta_{2k}$ are coefficients of $\det(\lambda I_{2n} + (F_{ij}))$; for $\mathfrak{so}_n$, consider $\Phi_{2k} \in \mathcal{S}^{2k}(\mathfrak{g})^G$ arising from

$$\det(I_n - q(F_{ij}))^{-1} = 1 + \Phi_2 q^2 + \Phi_4 q^4 + \ldots + \Phi_{2k} q^{2k} + \ldots .$$

Then $\mathrm{m}_{2r+1}(\Delta_{2k}) = \frac{(2k-2r)!(2r)!}{(2k)!}\binom{2n-2k+2r+1}{2r}\Delta_{2k-2r}$ and $\mathrm{m}_{2r+1}(\Phi_{2k}) = \frac{(2k-2r)!(2r)!}{(2k)!}\binom{n+2k-2}{2r}\Phi_{2k-2r}$.

Theorem (Y., 2019)

There are the following complete sets of Segal–Sugawara vectors:

$$\{S_k = \varpi(\Delta_{2k}[-1]) + \sum_{1\leqslant r<k}\binom{2n-2k+2r+1}{2r}\varpi(\tau^{2r}\Delta_{2k-2r}[-1])\cdot 1 \mid 1 \leqslant k \leqslant n\}$$

in type C_n*;*

$$\{S_k = \varpi(\Phi_{2k}[-1]) + \sum_{1\leqslant r<k}\binom{n+2k-2}{2r}\varpi(\tau^{2r}\Phi_{2k-2r}[-1])\cdot 1 \mid 1 \leqslant k < \ell\}$$ *for* $\mathfrak{so}_n$ *with* $n = 2\ell - 1$ *with the addition of* $S_\ell = \varpi(\mathrm{Pf}[-1])$ *for* $\mathfrak{so}_n$ *with* $n = 2\ell$.

Theorem (Y., 2019)

There are the following complete sets of Segal–Sugawara vectors:

$$\{S_k = \varpi(\Delta_{2k}[-1]) + \sum_{1\leqslant r<k} \binom{2n-2k+2r+1}{2r} \varpi(\tau^{2r}\Delta_{2k-2r}[-1])\cdot 1 \mid 1 \leqslant k \leqslant n\}$$

in type C_n*;*

$$\{S_k = \varpi(\Phi_{2k}[-1]) + \sum_{1\leqslant r<k} \binom{n+2k-2}{2r} \varpi(\tau^{2r}\Phi_{2k-2r}[-1])\cdot 1 \mid 1 \leqslant k < \ell\}$$ *for*

$\mathfrak{so}_n$ *with* $n = 2\ell - 1$ *with the addition of* $S_\ell = \varpi(\mathrm{Pf}[-1])$ *for* $\mathfrak{so}_n$ *with* $n = 2\ell$*.*

Remark

First explicit formulas for complete sets of Segal–Sugawara vectors for $\mathfrak{sp}_{2n}$ and $\mathfrak{so}_n$ were produced by Molev in 2013. His construction involved the Brauer algebra. In August 2020, he showed that his elements coincide with the ones presented above.

Can we say something about the exceptional types?

Recall that $\{x_i\}$ is an orthonormal basis of $\mathfrak{g}$; set $\mathcal{H} = \sum_{i=1}^{\dim\mathfrak{g}} x_i^2$.
Assume now that $\mathfrak{g}$ is simple and exceptional.

Proposition (Y., 2019)

There are a nonzero $H \in \mathcal{S}^6(\mathfrak{g})^G$ *and* $R(1), R(2) \in \mathbb{C}$ *such that*
$S = \varpi(H[-1]) + R(1)\varpi(\tau^2\mathcal{H}^2[-1])\cdot 1 + R(2)\varpi(\tau^4\mathcal{H}[-1])\cdot 1 \in \mathfrak{z}(\widehat{\mathfrak{g}})$.

Example (obtained by hand in type G_2)

Suppose $\mathfrak{g}$ is of type G_2. Then $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}} = \{\Delta_2, \Delta_6\}$, where $\Delta_2 \in \mathbb{C}\mathcal{H}$.
Choose the normalisation such that $\Delta_2|_{\mathfrak{sl}_3} = -2\tilde{\Delta}_2$, $\Delta_6|_{\mathfrak{sl}_3} = -\tilde{\Delta}_3^2$. Then
$$S_2 = \varpi((\Delta_6 - \tfrac{25}{108}\Delta_2^3)[-1]) - \tfrac{65}{4}\varpi(\tau^2\Delta_2^2[-1])\cdot 1 - \tfrac{325}{3}\varpi(\tau^4\Delta_2[-1])\cdot 1$$
and $S_1 = \mathcal{H}[-1]$ form a complete set of Segal–Sugawara vectors for $\mathfrak{g}$.

If $\mathfrak{g}$ is of type E_6 and $F \in \mathcal{S}^5(\mathfrak{g})^G$, then $\mathrm{m}(F) = 0$ and $\varpi(F[-1]) \in \mathfrak{z}(\widehat{\mathfrak{g}})$.

Polasirations and symmetrisations

For $Y=\prod_{i=1}^k y_i \in \mathcal{S}^k(\mathfrak{g})$ and $\vec{a}=(a_1,\ldots,a_k)\in\mathbb{Z}^k_{<0}$, let

$$Y[\vec{a}] := \frac{1}{k!}\sum_{\sigma\in\mathrm{S}_k} y_1[\sigma(a_1)]\ldots y_k[\sigma(a_k)] \in \mathcal{S}^k(\widehat{\mathfrak{g}}^-)$$

be the *$\vec{a}$-polarisation* of Y. We extend this notion to all $F\in\mathcal{S}^k(\mathfrak{g})$ by linearity.

Our formulas for Segal–Sugawara vectors have terms $\varpi(\tau^{2r}H[-1]){\cdot}1$ with $H\in\mathcal{S}(\mathfrak{g})^G$.

An expression $\varpi(\tau^r F[-1]){\cdot}1$ encodes a sum of $\frac{1}{(k+r)!}c(r,\vec{a})\varpi(F[\vec{a}])$, where $c(r,\vec{a})\in\mathbb{N}$ are certain combinatorially defined coefficients.

The elements $S\in\mathfrak{z}(\widehat{\mathfrak{g}})$ that we have seen in this talk are of the form

$$\varpi(H[-1]) + \sum_{(k/2)>r\geqslant 1;\vec{a}} C_{r,\vec{a}}\,\varpi(H_r[\vec{a}]),\quad \text{where}\quad H_r=\mathrm{m}^r(H)\in\mathcal{S}^{k-2r}(\mathfrak{g})^G,$$

$H\in\mathcal{S}^k(\mathfrak{g})^G$, $\vec{a}\in\mathbb{Z}^{k-2r}_{<0}$, and $\sum_{j=1}^{k-2r} a_j = -k-2r$.

In conclusion,

- a better understanding of the map m would lead to a better understanding of $\mathfrak{z}(\widehat{\mathfrak{g}})$;
- conjecturally, each exceptional Lie algebra possesses a set $\{H_k\}$ of generating symmetric invariants such that $\mathrm{m}^d(H_k) \in \mathcal{S}(\mathfrak{g})$ for all k, d;
- it is quite probable, that one can handle types F_4 and E_6 on a computer.