A GAUSS-BONNET THEOREM, CHERN CLASSES AND AN ADJUNCTION FORMULA FOR REDUCTIVE GROUPS

by

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A Gauss-Bonnet theorem, Chern classes and an adjunction formula for reductive groups

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Abstract

My thesis is about geometry of reductive groups. The purpose is to extend some well-known geometric results that hold for the complex torus $(\mathbb{C}^*)^n$ to the case of an arbitrary complex reductive group. The thesis consists of two parts.

The first part contains an analog of the Gauss-Bonnet theorem for constructible sheaves equivariant under the adjoint action. This theorem relates the Euler characteristic of a sheaf to the Gaussian degrees of the components of its characteristic cycle. As a corollary I get that a perverse sheaf equivariant under the adjoint action has nonnegative Euler characteristic.

In the second part, I modify one of the classical definitions of the Chern classes to construct noncompact analogs of the Chern classes for equivariant vector bundles over reductive groups. These Chern classes have the same properties as the usual Chern classes. Using the Chern classes of the tangent bundle, I obtain an adjunction formula for the Euler characteristic of hypersurfaces in arbitrary reductive groups.

The common feature of the results and constructions of my thesis is that I use

a group action to extend to noncompact setting such classical results as the Gauss-Bonnet theorem and the adjunction formula.

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Chapter 1

Introduction and main results

A Gauss-Bonnet theorem for reductive groups. The classical Gauss-Bonnet theorem states that the Euler characteristic of a compact oriented hypersurface in $\mathbb{R}^n$ coincides up to a sign with the degree of its Gauss map. For arbitrary algebraic group and its possibly noncompact subvariety one can also define a Gauss map using the parallel transport given by the group multiplication. It is natural to ask if the degree of such Gauss map coincides with the Euler characteristic. J. Franecki and M. Kapranov proved that it is true for complex subvarieties of the complex torus $(\mathbb{C}^*)^n$ and, more generally, for constructible sheaves on the torus. However, their result does not hold for arbitrary constructible sheaves on a noncommutative algebraic group. Kapranov conjectured that it is still true for constructible sheaves on reductive groups if we consider only sheaves equivariant under the adjoint action. I proved this conjecture.

The precise statement is as follows. Let G be a complex reductive group, and let $\mathcal{F}$ be a constructible complex of sheaves on G. For a subvariety $X \subset G$, denote by $\mathrm{gdeg}(X)$ the *Gaussian degree* of X. It is is equal to the number of zeros of a generic left-invariant differential 1-form on G restricted to X. It is easy to show that the Gaussian degree is well-defined [7, 8]. E.g. the Gaussian degree of a hypersurface is equal to the degree of its Gauss map.

Theorem 1.1. [26] *If $\mathcal{F}$ is equivariant under the adjoint action of G, then its Euler characteristic can be computed in terms of its characteristic cycle $\sum c_\alpha \overline{\mathrm{T}^*_{X_\alpha} G}$ by the following formula*

$$\chi(G, \mathcal{F}) = \sum c_\alpha \mathrm{gdeg}(X_\alpha),$$

where coefficients c_α are the multiplicities of the characteristic cycle.

In other words, the Euler characteristic of a sheaf is equal to the intersection index of the characteristic cycle with the graph of a generic left-invariant 1-form on the group G. The Gauss map provides a definition of such intersection index in the noncompact situation. In this form, Theorem 1.1 is very similar to the well-known Kashiwara's index theorem, which holds for compact manifolds.

Theorem 1.1 relates two geometric invariants of a sheaf: the Euler characteristic and the characteristic cycle. In some cases this relation can be used to estimate one of these invariants if the other is known (see Corollary 1.3). This works especially well if the sheaf $\mathcal{F}$ is perverse (see Corollary 1.2). In this case the multiplicities c_α of the characteristic cycle are nonnegative.

Theorem 1.1 gives a tool to classify adjoint equivariant perverse sheaves with a given Euler characteristic. E.g. if the Euler characteristic is small then the characteristic cycle cannot have components with large Gaussian degrees. This gives some essential restrictions on the characteristic cycle. One of my future plans is to explore the geometry of subvarieties with a given Gaussian degree and to give a geometric description of perverse sheaves with a given Euler characteristic. I am especially interested in perverse sheaves with Euler characteristic 1. When G is a complex torus such sheaves (if irreducible) are identified with complexes of solutions of the generalized hypergeometric systems [11]. Irreducible hypersurfaces in $(\mathbb{C}^*)^n$ of the Gaussian degree 1 are identified with discriminantal hypersurfaces of the generalized hypergeometric functions [19].

Theorem 1.1 can also be used to obtain a lower estimate for the Euler characteristic of a perverse sheaf. In particular, since the Gaussian degree is always nonnegative by its definition, Theorem 1.1 immediately gives the following corollary.

Corollary 1.2. [26] *If $\mathcal{F}$ is a perverse sheaf equivariant under the adjoint action, then its Euler characteristic is nonnegative.*

In the case when G is a complex torus this statement was proved by F. Loeser and C. Sabbah using the theory of $\mathcal{D}$-modules [29]. Another proof was given by O. Gabber and F. Loeser for l-adic perverse sheaves [11]. In the case of an arbitrary reductive group, a result of A. Braverman implies this statement for a special class of perverse sheaves equivariant under the adjoint action [1].

The main example of a perverse sheaf is given by the shifted complex of intersection chains of a closed subvariety $X \subset G$. Suppose that X is invariant under the adjoint action. Then Corollary 1.2 implies that the Euler characteristic of X computed using the intersection cohomology has sign $(-1)^{\dim X}$ (because of the shift).

Adjoint equivariant sheaves arising naturally in representation theory sometimes have very special characteristic cycles. E.g. this is the case for character sheaves introduced by Lusztig [32]. For character sheaves, Theorem 1.1 immediately implies that their Euler characteristic vanishes. I also proved the following generalization of this classical result.

Corollary 1.3. *If the characteristic cycle of $\mathcal{F}$ is supported on the set of nonsemisimple elements of the group G, then the Euler characteristic of $\mathcal{F}$ vanishes.*

Theorem 1.1 stated for a stratified subvariety $X \subset G$ relates the Euler characteristic of X to the Euler characteristics of complex links of the strata as follows. Let $X = \bigsqcup X_\alpha$ be a finite Whitney stratification of X. Denote by e_α the Euler characteristic of the complex link of a stratum X_α. This number measures the singularity of a subvariety X along the stratum X_α. E.g. if X_α lies in the smooth locus of X, then e_α is equal to 1. If X_α is open and dense in X, put $\mathrm{e}_\alpha = 0$

Corollary 1.4. *If $X \subset G$ is a closed subvariety invariant under the adjoint action, then the topological Euler characteristic of X can be computed as follows*

$$\chi(X) = \sum (-1)^{\dim X_\alpha}(1 - \mathrm{e}_\alpha)\mathrm{gdeg}(X_\alpha).$$

In particular, if X is smooth, then $\chi(X) = (-1)^{\dim X}\mathrm{gdeg}(X)$, which is a noncompact analog of the Gauss-Bonnet theorem. It also reminds of the classical Hopf theorem which states that the Euler characteristic of a compact oriented C^∞-manifold M is equal to $(-1)^{\dim M}$ times the number of zeros of a generic 1-form on M, counted with signs.

Chern classes and the Euler characteristic of a hyperplane section. In the second part of my thesis I construct noncompact analogs of Chern classes of reductive groups and use them in the following problem. Consider a finite-dimensional representation $\pi : G \to \mathrm{End}(V)$ of a reductive group G in a vector space V. The problem is to find the Euler characteristic $\chi(\pi)$ of a generic hyperplane section of the image $\pi(G)$ in the space $\mathrm{End}(V)$. This problem is closely related with generalized hypergeometric functions. Let me discuss this relation.

The generalized hypergeometric functions can be described as solutions of certain holonomic systems of linear PDE's associated with representations of reductive groups. For a complex torus, these systems were studied by I.Gelfand, M.Kapranov, A.Zelevinsky. Later Kapranov defined them for arbitrary reductive groups [20]. A finite-dimensional representation $\pi : G \to \mathrm{End}(V)$ of a reductive group G in a vector space V gives rise to a family of such systems. Let $\mathcal{D}$ be a generic system from this family. In the torus case, the number of linearly independent solutions of $\mathcal{D}$ is equal to the degree $\deg(\pi)$ of the image of G in $\mathrm{End}(V)$ [13]. For other reductive groups, the question is open.

One of essential features of generalized hypergeometric functions is that they can be represented by Euler integrals. For example, for classical hypergeometric equations this enables us to construct global solutions and to find the monodromy group. In the torus case, the number of linearly independent solutions of the system $\mathcal{D}$ that are represented by Euler integrals is equal up to a sign to the Euler characteristic $\chi(\pi)$ of a generic hyperplane section of $\pi(G)$ in the space $\mathrm{End}(V)$. In the torus case, the equality $|\chi(\pi)| = \deg(\pi)$ implies that Euler integrals span the space of all solutions [14, 12]. It seems that for other reductive groups, one can also construct exactly $|\chi(\pi)|$ linearly independent Euler integrals satisfying the system $\mathcal{D}$. Thus it is very appealing to find this number and relate it to the number of all linearly independent solutions.

Another motivation comes from the beautiful explicit formula for $\chi(\pi)$, when G is a complex torus. In this case, the Euler characteristic $\chi(\pi)$ is equal to $(-1)^{\dim G-1} \cdot \deg(\pi)$ [24]. The degree $\deg(\pi)$ has a nice combinatorial description via the volume of the weight polytope of the representation. This result is one in a series of theorems due to D.Bernstein, A.Koushnirenko, A.Khovanskii. They expressed a number of invariants of hypersurfaces in $(\mathbb{C}^*)^n$ in terms of the corresponding Newton polytopes. For arbitrary reductive groups the only result in this direction is the similar combinatorial description for the intersection index of several hyperplane sections due to B.Kazarnovskii [23] and M.Brion [2]. In particular, this allows one to compute the degree $\deg(\pi)$ (which can be viewed as the self-intersection index of a hyperplane section). An interesting interpretation of this result in terms of the volume of the Gelfand-Zetlin polytope was given by A.Okounkov [34, 35].

For other reductive groups the Euler characteristic is no longer equal to the degree. Even for $SL_2(\mathbb{C})$ the answer is already more complicated [22]. I have proved a formula that expresses the Euler characteristic $\chi(\pi)$ via the degrees of certain interesting subvarieties of the group G (see Theorem 1.5).

While computing the Euler characteristic I discovered analogs of Chern classes of a reductive group. To construct them I considered the spherical action of $G \times G$ on a group G by left and right multiplication. This action provides a natural class of linear vector fields on G which I employed to define the subvarieties S_i dual to the "Chern classes". Denote by n and k the dimension and the rank of G, respectively. My construction repeats the usual construction of Chern classes via degeneracy loci of vector fields. E.g. the hypersurface S_1 consists of all points where n vector fields are linearly dependent, S_2 consists of all points where first $(n-1)$ vector fields are linearly dependent and so on. I proved that a subvariety S_i is nonempty only if $i \leq n - k$. E.g. if G is a torus then all subvarieties S_i are empty. In the reductive case, the subvarieties S_i are responsible for the discrepancy between the Euler characteristic and the degree. Namely, the following analog of an adjunction formula holds.

Theorem 1.5. *The Euler characteristic $\chi(\pi)$ of a generic hyperplane section is equal to the alternating sum of the degrees of $\pi(S_i)$:*

$$\chi(\pi) = \sum_{i=0}^{n-k} (-1)^{n-i-1} \mathrm{deg}\ \pi(S_i).$$

Clearly, when G is a torus, only the first term of the sum is left and the formula gives the right answer.

The formula from Theorem 1.5 is very similar to the adjunction formula that expresses the Euler class of a divisor on a compact manifold via the Chern classes of this manifold. In fact, there is a relation between the subvarieties S_i and the Chern classes of equivariant compactifications of G.

I also give a definition of Chern classes for all vector bundles over G that are equivariant under the action of $G \times G$ (the tangent bundle is an example of such bundles). Their construction is completely analogous to the one mentioned above. It provides the well-defined elements of the ring of conditions of G introduced by De

Concini and Procesi [5, 3]. This ring is an analog of cohomology ring for reductive groups, and was initially devised to solve enumerative problems.

An interesting problem is to find an explicit combinatorial answer for the degrees of Chern classes in the spirit of Brion-Kazarnovskii theorem. So far I have been able to find the degrees of the first and of the last Chern classes.

Chapter 2

A Gauss-Bonnet theorem for constructible sheaves on reductive groups

2.1 Introduction

In this chapter, we prove an analog of the Gauss-Bonnet formula for constructible sheaves on reductive groups. This formula holds for all constructible sheaves equivariant under the adjoint action and expresses the Euler characteristic of a sheaf in terms of its characteristic cycle. As a corollary from this formula we get that if a perverse sheaf on a reductive group is equivariant under the adjoint action, then its Euler characteristic is nonnegative.

We now give the basic definitions and then formulate the main results.

In the sequel, by a constructible complex we will always mean a bounded complex of sheaves of $\mathbb{C}$-vector spaces whose cohomology sheaves are constructible with respect to some finite algebraic stratification (see [21]). For any constructible complex $\mathcal{F}$ on a smooth complex variety X one can define the following geometric invariants: the global Euler characteristic $\chi(X, \mathcal{F})$ and the characteristic cycle $CC(\mathcal{F})$.

The global Euler characteristic $\chi(X, \mathcal{F})$ is defined as the the Euler characteristic of the complex of cohomology groups $H^i(X, \mathcal{F})$. The latter are obtained by taking the derived functor of global sections of $\mathcal{F}$ (see [21]).

There exists a morphism from the derived category of constructible complexes on a smooth complex variety X to the group of Lagrangian cycles in the cotangent bundle T^*X (see [21] Section 9.4). It has nice functorial properties. The characteristic cycle $CC(\mathcal{F})$ of a constructible complex $\mathcal{F}$ is the image of $\mathcal{F}$ under this morphism.

A constructible complex $\mathcal{F}$ on X is called a perverse sheaf if it satisfies the following two conditions. First, the local cohomology $H^i(\mathcal{F}_x)$ are supported on a subset of dimension at most $-i$. Second, the local cohomology $H^i_c(\mathcal{F}_x)$ with compact support are supported on a subset of dimension at most i ([21, 18, 31]).

We now formulate the main results. Let G be a complex reductive group, and let $\mathcal{F}$ be a constructible complex on G. The characteristic cycle of $\mathcal{F}$ is a linear combination of Lagrangian subvarieties $CC(\mathcal{F}) = \sum c_\alpha \mathrm{T}^*_{X_\alpha} G$. Here and in the sequel $\mathrm{T}^*_X G$ denotes the closure of the conormal bundle to the smooth locus of a subvariety $X \subset G$. With X one can associate a nonnegative number $\mathrm{gdeg}(X)$ called the *Gaussian degree* of X. It is equal to the number of zeros of a generic left-invariant differential 1-form on G restricted to X. The precise definitions of the Gaussian degree and of the Gauss map are given in section 2.2.

Theorem 2.1. *If $\mathcal{F}$ is equivariant under the adjoint action of G, then its Euler characteristic can be computed in terms of the characteristic cycle by the following formula*

$$\chi(G, \mathcal{F}) = \sum c_\alpha \mathrm{gdeg}(X_\alpha).$$

For a perverse sheaf the multiplicities c_α of its characteristic cycle are nonnegative

[15]. The Gaussian degrees of X_α are also nonnegative by their definition, see below. Thus Theorem 2.1 immediately implies the following important corollary.

Corollary 2.2. *If $\mathcal{F}$ is a perverse sheaf equivariant under the adjoint action of G, then its Euler characteristic is nonnegative.*

In particular, let $\underline{\mathbb{C}}_X$ be a constant sheaf on a subvariety $X \subset G$ extended by 0 to G. Applying the above statements to this sheaf we get the following corollary.

Corollary 2.3. *If $X \subset G$ is a closed smooth subvariety invariant under the adjoint action of G, then $\chi(X) = (-1)^{\dim X}\mathrm{gdeg}(X)$. Thus the number $(-1)^{\dim X}\chi(X)$ is nonnegative.*

Indeed, the characteristic cycle of $\underline{\mathbb{C}}_X$ coincides with $(-1)^{\dim X}\mathrm{T}^*_X G$. There is an analogous formula for the Euler characteristic of any closed (not necessarily smooth) subvariety invariant under the adjoint action (Corollary 2.18).

Another corollary proves the vanishing of the Euler characteristic of a sheaf $\mathcal{F}$ in the case when the characteristic cycle of $\mathcal{F}$ is supported on the set of nonsemisimple elements of the group G (Corollary 2.19). E.g. the characteristic cycles of Lusztig's character sheaves satisfy this condition.

For the case when $G = (\mathbb{C}^*)^n$ is a torus, Corollary 2.2 was first proved by F. Loeser and C. Sabbah [29] with another proof given by O. Gabber and F. Loeser [11]. In the case of arbitrary reductive groups, A.Braverman proved nonnegativity of the Euler characteristic for some class of Ad G-equivariant l-adic sheaves in finite characteristic [1]. For complex ground field his result implies, in particular, Corollary 2.2 in the case, when a perverse sheaf coincides with its Deligne-Goreski–MacPherson extension from the set of all regular semisimple elements of G.

Theorem 2.1 was proved in the torus case by J. Franecki and M. Kapranov [8]. Theorem 2.1 holds for all constructible sheaves on a torus. However, it does not hold for arbitrary constructible sheaves on a noncommutative algebraic group. There is a simple counterexample [8] (see also section 2.2). Kapranov conjectured that it may be still true for constructible sheaves on reductive groups if we consider only sheaves equivariant under the adjoint action. I proved this conjecture [26].

The main step in the proof of Theorem 2.1 is to reduce the problem to the case of a maximal torus $T \subset G$. Since $\mathcal{F}$ is Ad G-equivariant, it is constructible with respect to some Whitney stratification $\mathcal{S}$ with Ad G-invariant strata. In section 2.3 we prove that the Euler characteristic of a stratum $X \in \mathcal{S}$ coincides with that of the intersection $X \cap T$. This implies that the sheaf $\mathcal{F}$ restricted onto the maximal torus T has the same Euler characteristic as $\mathcal{F}$. In section 2.2 we recall some facts about Euler characteristic needed for the proof. In section 2.4 we prove that the Gaussian degrees of X and $X \cap T$ coincide.

To deal with the characteristic cycle we use the Dubson-Kashiwara index formula that expresses the multiplicities c_α in terms of the local Euler characteristic of $\mathcal{F}$ along each stratum and some topological data depending on the stratification only (section 2.2). This data is given by the Euler characteristics with compact support of complex links. In our case we can choose a complex link to be invariant under the action of some compact torus and thus simplify computation of its Euler characteristic. This approach is taken from [6]. In section 2.5 we prove that for any stratum $X_\beta \in \mathcal{S}$ and any semisimple stratum $X_\alpha \in \mathcal{S}$, such that $X_\alpha \subset \overline{X_\beta}$, the Euler characteristic with compact support of their complex link coincides with that of the complex link of the strata $X_\alpha \cap T$ and $X_\beta \cap T$. This allows us to view the formula from Theorem 2.1 as the same formula for the restriction of $\mathcal{F}$ onto T (section 2.6). Then we apply the result of [8].

2.2 Preliminaries

Gaussian degree. We now define the (left) Gauss map and the Gaussian degree. The material of this subsection is taken from [8]. For more details see [8, 7].

Let G be a complex algebraic group with Lie algebra $\mathfrak{g}$, and let X be its subvariety of the dimension k. Denote by $\mathrm{G}(k, \mathfrak{g})$ the Grassmannian of k-dimensional subspaces in $\mathfrak{g}$. For any point $x \in G$, there is a natural isomorphism between the tangent space $\mathrm{T}_x G$ and $\mathfrak{g}$ given by the left multiplication by x^{-1}:

$$L_x : y \mapsto x^{-1} y; \quad d_x L_x : \mathrm{T}_x G \to \mathfrak{g}.$$

The left Gauss map $\Gamma_X : X \to \mathrm{G}(k, \mathfrak{g})$ is defined as follows:

$$\Gamma_X(x) = d_x L_x(\mathrm{T}_x X).$$

The Gauss map is rational and regular on the smooth locus X^{sm} of X. If X is a hypersurface, Γ_X maps X to $\mathbb{P}(\mathfrak{g}^*)$, which has the same dimension as X. In this case we define the Gaussian degree of X to be the degree of its Gauss map. By the degree of a rational map $X \to Y$ we mean the number of preimages of a generic point in Y (see [33], Proposition 3.17). In general case the Gaussian degree is the degree of a rational map $\tilde{\Gamma}_X : \tilde{X} \to \mathbb{P}(\mathfrak{g}^*)$, where $\tilde{X}$ and $\tilde{\Gamma}_X$ are defined as follows. The variety $\tilde{X}$ is a fiber bundle over X^{sm}, whose fiber at a point x consists of all hyperplanes in $\mathfrak{g}$ that contain a subspace $\Gamma_X(x)$, i.e. $\tilde{X} = \{(x, y) \in X^{sm} \times \mathbb{P}(\mathfrak{g}^*) : \Gamma_X(x) \subset y\}$. Then $\tilde{\Gamma}_X(x, y) = y$. Note that $\tilde{X}$ and $\mathbb{P}(\mathfrak{g}^*)$ have the same dimension. It is clear from the definition that the Gaussian degree is a birational invariant of a subvariety.

In the sequel we will use another description of the Gaussian degree. Let ω be a generic left-invariant differential 1-form on G (for reductive groups we define a generic 1-form explicitly in section 2.4). We call a point $x \in X$ a *zero* of ω, if ω restricted to the tangent space $\mathrm{T}_x X$ is zero. Then it is easy to verify that the Gaussian degree of X is equal to the number of zeros of ω on X.

Counterexample We now show that the statement of Theorem 1 is no longer true if one drops either the assumption of the equivariance under the adjoint action or the assumption that the group G is reductive. The following counterexample is taken from [8]. Any noncommutative complex linear algebraic group G contains a subgroup X isomorphic to the affine space $\mathbb{C}^k$ for some $k > 0$. E.g. one can take the maximal unipotent subgroup. It follows from the definition of the Gaussian degree that the Gaussian degree of any subgroup with positive dimension is zero. Indeed, a generic left-invariant 1-form on G restricted to a subgroup coincides with a non-zero left-invariant 1-form on this subgroup. Hence, it vanishes nowhere on the subgroup. It follows that $\mathrm{gdeg}(X) = 0$, and the Euler characteristic of X is 1.

If G is not reductive, then a subgroup X with the above property can be chosen to be invariant under the adjoint action. Namely, the radical of G is not diagonalizable in this case, hence the maximal unipotent subgroup of the radical is non-trivial. It is invariant under the adjoint action.

Euler characteristic. Let T be a torus (it may be a complex torus $(\mathbb{C}^*)^n$ as well as a compact one $(S^1)^n$). Consider its linear algebraic action on $\mathbb{C}^N$, and a locally closed semialgebraic subset $X \subset \mathbb{C}^N$ invariant under this action. Let $X^T \subset X$ be the set of the fixed points. In what follows χ denotes the usual topological Euler characteristic and χ^c the Euler characteristic computed using cohomology with compact support. The following simple and well-known fact plays the crucial role in the sequel.

Proposition 2.4. *The spaces X and X^T have the same Euler characteristic with compact support:*

$$\chi^c(X) = \chi^c(X^T). \qquad \square$$

The following statement is also well-known, but the author could not find an appropriate reference.

Proposition 2.5. *If X is a complex algebraic variety, then $\chi^c(X) = \chi(X)$.*

Proof. Applying Proposition 2.6 to the constant sheaf $\underline{\mathbb{C}}_X$ and using additivity of the Euler characteristic with compact support, one can deduce this equality from the following fact. For any point $x \in X$ the Euler characteristic with compact support of a small open neighborhood of x is equal to 1. To prove this fact we use the induction by the dimension of X.

We may assume that a neighborhood of x is embedded in $\mathbb{C}^N$. Take a generic holomorphic function f on X such that $f(x) = 0$, and an open neighborhood $C = f^{-1}(D) \cap B$, where $D \subset \mathbb{C}$ is a small open disk with the center at 0 and $B \subset \mathbb{C}^N$ is a small ball with the center at x. Then $f : C \setminus f^{-1}(0) \to D \setminus \{0\}$ is a fiber bundle (see [16], Section 2.4). Thus $\chi^c(C \setminus f^{-1}(0)) = 0$, and $\chi^c(C) = \chi^c(f^{-1}(0))$. The dimension of $f^{-1}(0)$ is already less than that of X. □

The Euler characteristic of sheaves. We now recall a formula for the Euler characteristic of constructible sheaves on varieties. Let $X \subset \mathbb{C}^N$ be a smooth subvariety, and let $\mathcal{F}$ be a constructible complex on X. With any point $x \in X$ one can associate the local Euler characteristic $\chi(\mathcal{F}_x)$ of $\mathcal{F}$ at this point (see [21] Section 9.1). Thus $\mathcal{F}$ gives rise to the constructible function $\chi(\mathcal{F})$ on X by the formula $\chi(\mathcal{F})(x) = \chi(\mathcal{F}_x)$.

There is the concept of the direct image of a constructible function (see [9] and [21], Section 9.7). It is defined for any morphism of algebraic varieties $X \to Y$ and a constructible function on X. We use the more suggestive notation $\int_X f(x) d\chi$ for the direct image of f under the morphism $X \to pt$, since this direct image may be also defined as the integral of f over the Euler characteristic [37, 25].

Proposition 2.6. *The global Euler characteristic $\chi(X, \mathcal{F}) = \sum (-1)^i H^i(X, \mathcal{F})$ is equal to the following integral over the Euler characteristic*

$$\chi(X, \mathcal{F}) = \int_X \chi(\mathcal{F}) d\chi$$

In other words, if we fix a finite algebraic stratification $X = \bigsqcup X_\alpha, \alpha \in \mathcal{S}$, such that the function $\chi(\mathcal{F})$ is constant along each stratum, we get

$$\chi(X, \mathcal{F}) = \sum_{\alpha \in \mathcal{S}} \chi_\alpha(\mathcal{F}) \chi^c(X_\alpha),$$

where $\chi_\alpha(\mathcal{F})$ is the value of $\chi(\mathcal{F})$ at any point of a stratum X_α.

See [21], Section 9.7 for the proof.

Complex links and characteristic cycles. We use the notation of the previous subsection. Suppose that $\mathcal{S}$ is a Whitney stratification of X. Let $X_\alpha, X_\beta, \alpha, \beta \in \mathcal{S}$, be two strata such that $X_\alpha \subset \overline{X_\beta}$. Choose a point $a \in X_\alpha$ and any normal slice $N \subset X$ to X_α at the point a. Consider a holomorphic function l on N such that $l(a) = 0$ and its differential $d_a l$ is a generic covector in the cotangent space $\mathrm{T}^*_a N$ (i.e. $d_a l$ belongs to some open dense subset of this space that depends on the stratification $\mathcal{S}$). Let $h(\cdot, \cdot)$ be a Hermitian metric in $\mathbb{C}^N$ and $B = \{x \in \mathbb{C}^N : h(x - a, x - a) \leq \mathit{const}\}$ a small ball with the center at a.

We now define the complex link L of the strata X_α, X_β as $L = B \cap l^{-1}(\varepsilon) \cap X_\beta$. If the radius of the ball B is small enough and the absolute value of ε is small enough with respect to the radius, then the result up to a homeomorphism does not depend on any of the choices involved (see [16], Section 2.3 for the proof). We will use the notation $\mathrm{e}(\alpha, \beta)$ as well as $\mathrm{e}(X_\alpha, X_\beta)$ for the Euler characteristic of L with compact support. We also set $\mathrm{e}(\alpha, \alpha) = -1$.

In section 2.7, we will also use the notion of the complete complex link of a stratum X_α. It is defined as $B \cap l^{-1}(\varepsilon) \cap X$ in the previous notation. In other words, it is the union of complex links of strata X_α and X_β over all strata X_β whose closure contains X_α. Denote the Euler characteristic of the complete complex link of a stratum X_α by e_α.

The numbers $\mathrm{e}(\alpha, \beta)$ are useful when one need to find the multiplicities of the characteristic cycle $CC(\mathcal{F})$. Multiplicities are recovered from the constructible function $\chi(\mathcal{F})$ by the following theorem of Dubson and Kashiwara.

Theorem 2.7. *The characteristic cycle of $\mathcal{F}$ is the linear combination of Lagrangian subvarieties $\mathrm{T}^*_{X_\alpha}X, \alpha \in \mathcal{S}$, with coefficients*

$$c_\alpha = (-1)^{\dim X_\alpha + 1} \sum_{X_\alpha \subset \overline{X_\beta}} \mathrm{e}(\alpha, \beta)\chi_\beta(\mathcal{F}).$$

See [15], Theorem 8.2 for the proof.

2.3 Euler characteristic of invariant subvarieties

Let G be a connected reductive group over $\mathbb{C}$, and T a maximal complex torus in G. Consider a subvariety $X \subset G$ invariant under the adjoint action of G.

Proposition 2.8. *The varieties X and $X \cap T$ have the same Euler characteristic with compact support. Moreover, $\chi(X) = \chi(X \cap T)$.*

Proof. The subvariety X is invariant under the adjoint action of G. In particular, it is invariant under the adjoint action of the maximal torus T. The set $G^T \subset G$ of the fixed points under this action coincides with T, since the centralizer of the maximal torus coincides with the maximal torus itself. Thus by Proposition 2.4 the varieties X and $X \cap T$ have the same Euler characteristic with compact support. Combining this result with Proposition 2.5, we get that $\chi(X) = \chi(X \cap T)$. $\square$

Example 1. Let $X = \mathcal{O}_a$ be the orbit of an element $a \in G$ under the adjoint action of G. Then proposition 2.8 implies that if a is semisimple, then $\chi(\mathcal{O}_a)$ is equal to the number $|\mathcal{O}_a \cap T|$ of the intersection points. We may choose the maximal torus T such that $a \in T$. Since the orbit of a under the action of the Weyl group W on T

coincides with $\mathcal{O}_a \cap T$, we obtain that $\chi(\mathcal{O}_a) = |W|/|Stab\ a|$, where $Stab\ a \subset W$ is the stabilizer of a in W. If a is not semisimple, then $\chi(\mathcal{O}_a) = 0$.

Let $\mathcal{F}$ be a constructible complex on G.

Proposition 2.9. *Suppose that $\mathcal{F}$ is equivariant under the adjoint action of G. Let $\mathcal{F}_T$ be a restriction of $\mathcal{F}$ onto $T \subset G$. Then the sheaves $\mathcal{F}$ and $\mathcal{F}_T$ have the same Euler characteristic:*

$$\chi(G, \mathcal{F}) = \chi(T, \mathcal{F}_T).$$

Proof. The sheaves $\mathcal{F}$ and $\mathcal{F}_T$ have the same local Euler characteristic at a point $x \in T$, since $\mathcal{F}_T$ is the restriction of $\mathcal{F}$ onto T. Thus the Euler characteristic $\chi(T, \mathcal{F}_T)$ is equal to $\int_T \chi(\mathcal{F}) d\chi$ by Proposition 2.6. The function $\chi(\mathcal{F})$ is invariant under the adjoint action of G, and Proposition 2.8 implies

$$\int_T \chi(\mathcal{F}) d\chi = \int_G \chi(\mathcal{F}) d\chi.$$

The last integral is equal to the Euler characteristic $\chi(G, \mathcal{F})$ by Proposition 2.6. $\square$

2.4 Gaussian degree of invariant subvarieties

We now compare the Gaussian degrees of X in G and of $X \cap T$ in T. Clearly, the Gaussian degree of a k-dimensional subvariety is equal to the sum of the Gaussian degrees of its k-dimensional irreducible components. Thus we can assume that X is irreducible. There are two cases: the set of all nonsemisimple elements of X has codimension either 0 or at least 1. In what follows we prove that in the first case $\mathrm{gdeg}(X) = 0$, and in the second case $\mathrm{gdeg}(X) = \mathrm{gdeg}(X \cap T)$. In particular, the Gaussian degree of any orbit $\mathcal{O}_a \subset G$ coincides with the Gaussian degree of its intersection with the maximal torus T.

Any reductive group G admits an embedding in $GL_N(\mathbb{C})$ for some N, such that the inner product $\mathrm{tr}(Y_1Y_2)$ is nondegenerate on Lie algebra $\mathfrak{g}$. Let us fix such an embedding. Then $\mathfrak{g}$ may be identified with the space of all left-invariant differential 1-forms on G: an element $S \in \mathfrak{g}$ gives rise to a 1-form ω by the formula

$$\omega(Y) = \mathrm{tr}(x^{-1}YS), \tag{1}$$

where $x \in G$ and $Y \in \mathrm{T}_xG$. We will call such a form *generic*, if S is regular semisimple.

Lemma 2.10. *All generic left invariant* 1*-forms form an open dense subset in the space of all left-invariant* 1*-forms.*

Proof. All regular semisimple elements form an open dense subset in $\mathfrak{g}$. This implies the statement of the lemma. □

Proposition 2.11. *The Gaussian degree of an orbit $\mathcal{O}_a$ is equal to the number of the intersection points $\mathcal{O}_a \cap T$. In particular, if a is a nonsemisimple element, then* $\deg(\mathcal{O}_a) = 0$.

Proof. Consider the map

$$\varphi : G \to \mathcal{O}_a; \quad \varphi : g \mapsto gag^{-1}.$$

Since φ is smooth and surjective, the tangent space $\mathrm{T}_x\mathcal{O}_a$ is the image of the induced map $d\varphi$. A simple computation shows that $\mathrm{T}_x\mathcal{O}_a = [\mathfrak{g}, x]$. Let ω be a generic left-invariant differential 1-form on G given by the formula (1). Then $\omega = 0$ on $\mathrm{T}_x\mathcal{O}_a$ is equivalent to $\mathrm{tr}(x^{-1}YxS - YS) = 0$ for any $Y \in \mathfrak{g}$. Since the form $\mathrm{tr}(Y_1Y_2)$ is Ad G-invariant, we have $\mathrm{tr}(x^{-1}YxS - YS) = \mathrm{tr}(Y(xSx^{-1} - S))$. The form $\mathrm{tr}(Y_1Y_2)$ is nondegenerate on $\mathfrak{g}$. Thus x and S commute, and x belongs to some maximal torus T_S that depends on S. □

Remark 2.12. *Suppose that an element a lies in the maximal torus T. Then the space $\mathrm{T}_a\mathcal{O}_a = [\mathfrak{g}, a]$ is orthogonal to the tangent space T_aT with respect to the form $(Y_1, Y_2) \mapsto \mathrm{tr}(a^{-1}Y_1 \cdot a^{-1}Y_2)$. Since this form is nondegenerate on T_aT, we get that at any point $x \in \mathcal{O}_a \cap T$ the intersection of tangent spaces $\mathrm{T}_x\mathcal{O}_a$ and T_xT is zero.*

Corollary 2.13. *Let $Z \subset G$ be an irreducible subvariety invariant under the adjoint action of G, such that the set Z_n of all nonsemisimple elements of Z is a Zariski open nonempty subset in Z. Then* $\deg(Z) = 0$.

Proof. The Gaussian degree of a subvariety $Z \subset G$ is birationally invariant. Therefore, it suffices to compute it for Z_n. Let ω be a generic left-invariant differential 1-form on G given by the formula (1). For any smooth point $a \in Z_n$ the restriction of this form to the subspace $\mathrm{T}_a\mathcal{O}_a \subset \mathrm{T}_aZ_n$ of the tangent space T_aZ_n is already nonzero by Proposition 2.11. Thus the form ω does not vanish in any smooth point of Z_n, and $\mathrm{gdeg}(Z) = \mathrm{gdeg}(Z_n) = 0$. □

Proposition 2.14. *Let X be an irreducible subvariety of G invariant under the adjoint action of G such that the subset of all nonsemisimple elements of X has codimension at least* 1 *in X. Then the Gaussian degrees of X in G and of $X \cap T$ in T coincide.*

Proof. Let k be the maximal dimension of a semisimple orbit in X. Denote by X_s the set of all semisimple elements in X, whose orbits have dimension k. Then X_s is a Zariski open subset of X. Consider the map

$$\varphi : G \times (X \cap T) \to X, \quad (g, t) \mapsto gtg^{-1}.$$

The image of φ contains X_s. Since X_s is a Zariski open nonempty subset of X, $\deg(X) = \deg(X_s)$. For any smooth point $x \in X_s$ the tangent space T_xX is again the image of the induced map

$$d\varphi : \mathrm{T}_gG \times \mathrm{T}_t(X \cap T) \to \mathrm{T}_xX,$$

where $gtg^{-1} = x$. By calculating $d\varphi$ we obtain that $\mathrm{T}_x X = [\mathfrak{g}, x] \oplus g\mathrm{T}_t(X \cap T)g^{-1}$. Let ω be a generic left-invariant differential 1-form on G given by the formula (1). Then $\omega = 0$ on $\mathrm{T}_x X$ is equivalent to $\omega = 0$ on $[\mathfrak{g}, x]$ and $\omega = 0$ on $g\mathrm{T}_t(X \cap T)g^{-1}$. The first identity holds if and only if x belongs to the maximal torus T_S (see the proof of Proposition 2.11). Denote by ω_T the restriction of ω to T^*T_S. If $x \in T_S$, then $g\mathrm{T}_t(X \cap T)g^{-1} = \mathrm{T}_x(X \cap T_S)$. Thus the form ω vanishes on $\mathrm{T}_x X$ if and only if the form ω_T vanishes on $\mathrm{T}_x(X \cap T_S)$. It follows that $\deg(X) = \deg(X \cap T_S) = \deg(X \cap T)$, since all maximal tori are conjugate. $\square$

2.5 The Euler characteristic of the complex link

We now compute the Euler characteristic with compact support of a complex link for a certain class of stratifications of G. For any $a \in G$ we define the rank of a to be the dimension of its centralizer in G. A Whitney stratification $\mathcal{S}$ of G is called *admissible* if the following conditions hold. For every $\alpha \in \mathcal{S}$

- the stratum X_α is invariant under the adjoint action of G,
- elements of X_α are either all semisimple of the fixed rank or all nonsemisimple.

Denote by $\mathcal{S}_0 \subset \mathcal{S}$ the subset of all semisimple strata. Due to the second condition and the Remark 2.12, for any semisimple stratum $X_\alpha \in \mathcal{S}$ intersection $X_\alpha \cap T$ is smooth, and at any point $x \in X_\alpha \cap T$ the intersection of the tangent spaces $\mathrm{T}_x X_\alpha \cap \mathrm{T}_x T$ coincides with the tangent space $\mathrm{T}_x(X_\alpha \cap T)$. Thus we can consider an induced Whitney stratification $\mathcal{S}_T$ of the maximal torus T, namely, $T = \bigsqcup (X_\alpha \cap T), \alpha \in \mathcal{S}_0$.

Proposition 2.15. *Consider two strata X_α and X_β such that X_α belongs to the closure of X_β. If X_α is semisimple and X_β is not, then* $\mathrm{e}(\alpha, \beta) = 0$*. If both X_α, X_β are semisimple, then* $\mathrm{e}(\alpha, \beta) = \mathrm{e}(X_\alpha \cap T, X_\beta \cap T)$*, where the complex link of $X_\alpha \cap T$ and $X_\beta \cap T$ is taken in the torus T.*

Proof. Let $Z \subset G$ be the centralizer of an element $a \in X_\alpha$. Then Z is again a reductive group. Since the tangent spaces $\mathrm{T}_a Z$ and $\mathrm{T}_a \mathcal{O}_a$ are orthogonal with respect to the form $(Y_1, Y_2) \mapsto \mathrm{tr}(a^{-1}Y_1 \cdot a^{-1}Y_2)$, and this form is nondegenerate on $\mathrm{T}_a Z$, we get that Z is the normal slice to the orbit $\mathcal{O}_a \subset X_\alpha$. Thus any normal slice to $X_\alpha \cap Z$ in Z will also be the normal slice to X_α in G. Let us construct a normal slice $N \subset Z$ invariant under the adjoint action of Z.

Let k be the dimension of $X_\alpha \cap Z$. Some neighborhood of a in $X_\alpha \cap Z$ lies in the center of Z, because all elements of X_α have the same rank. Thus we can find k characters $\varphi_1, \ldots, \varphi_k$ of the group Z such that their differentials $d_a\varphi_1, \ldots, d_a\varphi_k$ restricted to the tangent space $\mathrm{T}_a(X_\alpha \cap Z)$ are linearly independent. Let $N \subset Z$ be the set of common zeros of the system $\varphi_1(za^{-1}) = \ldots = \varphi_k(za^{-1}) = 1$.

Example 2. a) Let G be $GL_N(\mathbb{C})$ and let $X_\alpha = Z(GL_N) = \mathbb{C}^*$ be the center of GL_N. Then $Z = G$, and the only characters of Z are the powers of determinant. We have $d_e\det = \mathrm{tr}$ for the identity element $e \in GL_N$, and tr is a nonzero linear function on $\mathbb{C}^* e$. Thus at the point $e \in X_\alpha$ we can take $N = SL_N(\mathbb{C})$.

b) Let G be any reductive group, and let X_α be a stratum consisting of regular semisimple elements. Then Z is a maximal torus. Thus any normal slice to $X_\alpha \cap Z$ in Z is invariant under the adjoint action of Z.

We now continue the proof of Proposition 2.15. Consider a generic linear function l on N given by the formula $l(x) = \mathrm{tr}((a^{-1}x - e)S)$, where $S \in \mathrm{Lie}\ Z$ is regular semisimple. There exists a maximal torus $T \subset Z$ centralizing S. Since a is semisimple, T is also a maximal torus in G. For any ε the set $l^{-1}(\varepsilon)$ is invariant under the adjoint action of T. Denote by T_c the compact form of T. Choose a Hermitian inner product $h(\cdot,\cdot)$ on $\mathfrak{gl}_N$ invariant under the adjoint action of T_c and a small ball $B = \{x \in \mathfrak{gl}_N : h(x-a, x-a) \le const\}$.

Thus with a generic vector $S \in \mathrm{Lie}\ Z$ we associate the complex link $L = B \cap$

$l^{-1}(\varepsilon) \cap X_\beta$ of the strata X_α and X_β. The complex link L is invariant under the adjoint action of the torus T_c by the construction. Thus by Proposition 2.8 we get $\chi^c(L) = \chi^c(L^{T_c}) = \chi^c(L \cap Z^{T_c})$. Note that $Z^{T_c} = Z^T = T$. If X_β is nonsemisimple, $X_\beta \cap T$ is empty, thus $L \cap T$ is empty. It follows that $\mathrm{e}(\alpha, \beta) = 0$ in this case. If X_β is semisimple, then $L \cap T$ is a complex link for $X_\alpha \cap T, X_\beta \cap T$ in the torus T. $\square$

Corollary 2.16. *If X is a smooth irreducible subvariety invariant under the adjoint action of G, then either X consists of nonsemisimple elements only or the set of all semisimple elements in X is dense. In particular, if in addition X is closed, then it contains a dense subset of semisimple elements. The Gaussian degrees of X and of $X \cap T$ coincide.*

Proof. Let us prove the first statement by contradiction. Let $\mathcal{S}$ be an admissible stratification of G subordinate to X, and let $X_n \subset X$ be a maximal open stratum in X, such that X_n is nonsemisimple. Then $\dim X_n = \dim X$, and $X - X_n$ contains at least one semisimple stratum X_s. The number $\mathrm{e}(X_s, X_n)$ is zero by Proposition 2.15. That contradicts to the smoothness of X. Combining the first statement with the results of Section 2.4, we get the last statement. $\square$

2.6 Proof of Theorem 2.1 (a Gauss-Bonnet theorem)

Since $\mathcal{F}$ is constructible and equivariant under the adjoint action, there exists some finite algebraic Whitney stratification $\mathcal{S}$ subordinate to $\mathcal{F}$ such that each stratum is invariant under the adjoint action. Stratifying each stratum if necessary we may assume that $\mathcal{S}$ is admissible. Let us apply Theorem 2.7 and Proposition 2.15 to the characteristic cycles of $\mathcal{F}$ and of $\mathcal{F}_T$. Notice that for a semisimple stratum $X_\alpha \in \mathcal{S}$ the difference dim $X_\alpha - \dim(X_\alpha \cap \mathcal{S})$ is equal to dim $\mathcal{O}_a, a \in X_\alpha$, and the latter is even. As a straightforward corollary we get

Corollary 2.17. *Let $X_\alpha \in \mathcal{S}$ be a semisimple stratum. The multiplicities of characteristic cycles of $\mathcal{F}$ and $\mathcal{F}_T$ along the strata X_α and $X_\alpha \cap T$, respectively, coincide.*

Example 3. Suppose that the support of the constructible function $\chi(\mathcal{F})$ lies in the closure of an orbit $\mathcal{O}_a, a \in G$. This kind of sheaves is studied in [6] for unipotent orbits. In this case the strata of an admissible stratification that contribute to the characteristic cycle are the orbits in $\overline{\mathcal{O}_a}$. Let $a_s, a_n \in G$ be the semisimple and unipotent elements respectively such that $a = a_s \cdot a_n$. Then $X_\alpha = \mathcal{O}_{a_s}$ is the only semisimple stratum in $\overline{\mathcal{O}_a}$. Thus for the multiplicity $c_\alpha(\mathcal{F})$ of $CC(\mathcal{F})$ along this stratum we get $c_\alpha(\mathcal{F}) = \chi_\alpha(\mathcal{F}) = c_\alpha(\mathcal{F}_T)$.

Now the formula of Theorem 2.1 reduces to the same formula for the sheaf $\mathcal{F}_T$ and the stratification $\mathcal{S}_T$. First, $\chi(\mathcal{F}, G) = \chi(\mathcal{F}_T, T)$ by Proposition 2.9. Second, for all nonsemisimple strata $X_\alpha, \alpha \in \mathcal{S}$, we have $\operatorname{gdeg}(X_\alpha) = 0$ by Corollary 2.13. Thus the right hand side of the formula may be considered as the sum over semisimple strata only, i.e.

$$\chi(X, \mathcal{F}) = \sum_{\alpha \in \mathcal{S}_0} c_\alpha(\mathcal{F}) \operatorname{gdeg}(X_\alpha).$$

By Corollary 2.17 this is equivalent to the formula

$$\chi(T, \mathcal{F}_T) = \sum_{\alpha \in \mathcal{S}_T} c_\alpha(\mathcal{F}_T) \operatorname{gdeg}(X_\alpha \cap T),$$

since $\operatorname{gdeg}(X_\alpha) = \operatorname{gdeg}(X_\alpha \cap T)$ by Proposition 2.14. To prove the latter formula we apply Theorem 1.3 from [8].

2.7 Applications

Let us deduce from Theorem 2.1 the formula for the Euler characteristic of a closed (possibly singular) subvariety $X \subset G$ invariant under the adjoint action. Denote by

$\underline{\mathbb{C}}_X$ the constant sheaf on X extended by 0 to G. Let us compute the coefficients of the characteristic cycle of $\underline{\mathbb{C}}_X$. Fix some Whitney stratification $\mathcal{S}$ of X. Then Theorem 2.7 gives the following formula for the multiplicity of the characteristic cycle $CC(\underline{\mathbb{C}}_X)$ along a stratum $X_\alpha \in \mathcal{S}$

$$c_\alpha(\underline{\mathbb{C}}_X) = (-1)^{\dim X_\alpha + 1} \sum_{X_\alpha \subset \overline{X_\beta}} \mathrm{e}(\alpha, \beta)\chi_\beta(\underline{\mathbb{C}}_X).$$

It is easy to see that the local Euler characteristic of $\underline{\mathbb{C}}_X$ at a point x is equal to the Euler characteristic with compact support of a small open neighborhood of x. As follows from the proof of Proposition 2.5, it equals to 1 for all $x \in X$. Thus we get that c_α coincides with $(-1)^{\dim X_\alpha + 1}(-1 + \mathrm{e}_\alpha)$, where e_α is the Euler characteristic of the total complex link of X_α.

Corollary 2.18. *If $X \subset G$ is a closed subvariety invariant under the adjoint action, then the topological Euler characteristic of X can be computed as follows*

$$\chi(X) = \sum (-1)^{\dim X_\alpha}(1 - \mathrm{e}_\alpha)\mathrm{gdeg}(X_\alpha).$$

We now compute the Euler characteristic of sheaves with special characteristic cycles. Namely, assume that the multiplicity c_α of the characteristic cycle $CC(\mathcal{F})$ along a stratum X_α is nonzero only if the stratum X_α is nonsemisimple. Then by Corollary 2.13 the Gaussian degree of X_α is zero. Hence Theorem 2.1 immediately implies the following corollary.

Corollary 2.19. *If the characteristic cycle of $\mathcal{F}$ is supported on the set of non-semisimple elements of the group G, then the Euler characteristic of $\mathcal{F}$ vanishes.*

Chapter 3

Chern classes of reductive groups and an adjunction formula

3.1 Introduction

Let G be a connected reductive group. Consider its finite-dimensional representation $\pi : G \to \mathrm{End}(V)$ in a vector space V. Let $H \subset \mathrm{End}(V)$ be a generic hyperplane. The main problem that I will discuss in this chapter is how to find the Euler characteristic of the hyperplane section $\pi(G) \cap H$. This problem also motivates the construction of Chern classes of equivariant bundles over reductive groups. The main result involving these Chern classes is an adjunction formula for the Euler characteristic of a hyperplane section.

Denote by $\chi(\pi)$ the Euler characteristic of a generic hyperplane section $\pi(G) \cap H$. When $G = (\mathbb{C}^*)^n$ is a complex torus, $\chi(\pi)$ was computed explicitly by D.Bernstein, A.Khovanskii and A.Koushnirenko [24]. This beautiful result relates $\chi(\pi)$ to combinatorial invariants of the representation π. The proof uses two facts:

- There is an explicit relation between the Euler characteristic $\chi(\pi)$ and the degree of the subvariety $\pi(G)$ in $\mathrm{End}(V)$

$$\chi(\pi) = (-1)^{n-1}\deg\, \pi(G). \tag{1}$$

- For the degree $\deg\, \pi(G)$ there is an explicit formula proved by Koushnirenko.

However, when G is arbitrary reductive group, only the second fact survives. B.Kazarnovskii found an explicit formula for the degree $\deg\, \pi(G)$ that generalizes Koushnirenko's formula [23]. Later M.Brion established an analogous result for all spherical homogeneous spaces [2].

As for the first fact, it is already wrong for $SL_2(\mathbb{C})$. K.Kaveh in his thesis computed explicitly $\chi(\pi)$ and $\deg\, \pi(G)$ for all representations π of $SL_2(\mathbb{C})$. His computation shows that, in general, there is a discrepancy between these two numbers. Kaveh also listed some special representations of reductive groups, for which these numbers still coincide [22].

In this chapter, I will present a formula that generalizes formula (1) to the case of arbitrary reductive groups. To do this I will construct subvarieties $S_i \subset G$, whose degrees fill the gap between the Euler characteristic and the degree. My construction reminds the construction of the Chern classes of a vector bundle. The subvarieties S_i can be thought of as the Chern classes of the tangent bundle of G. I will also construct the Chern classes of more general equivariant vector bundles over G (section 3.3). These Chern classes are in many aspects similar to the usual Chern classes of compact manifolds. There is an analog of cohomology ring of G, where the Chern classes of equivariant bundles live. This analog is the ring of conditions constructed by De Concini and Procesi [5, 3](see section 3.2 for a brief reminder). It is useful in solving enumerative problems. The intersection index in this ring is well-defined. In particular, it makes sense to speak of the degree of $\pi(S_i)$ in $\mathrm{End}(V)$.

Denote by n and k the dimension and the rank of G, respectively. In the case of the tangent bundle, it turns out (see Lemma 3.11) that the subvarieties S_i are nonempty only for $i \leq n-k$. E.g. if G is a torus then all subvarieties S_i are empty. For arbitrary reductive group G subvarieties S_i are nontrivial because of the noncommutative part of G.

The main result of this chapter is the following adjunction formula. Set $S_0 = G$.

Theorem 3.1. *Let π be a faithful representation of a reductive group G. The Euler characteristic $\chi(\pi)$ of a generic hyperplane section is equal to the alternating sum of the degrees of $\pi(S_i)$:*

$$\chi(\pi) = \sum_{i=0}^{n-k} (-1)^{n-i-1} \deg\ \pi(S_i). \tag{2}$$

My proof (section 3.4) of this formula is very similar to D.Bernstein's proof of formula (1) in the torus case. Chern classes S_i appear naturally, when one tries to generalize his proof to the case of arbitrary reductive groups.

To explain this motivation let me briefly recall Bernstein's proof of formula (1) in the torus case. The Euler characteristic $\chi(\pi)$ is equal up to a sign to the number of critical points of a generic linear functional $f \in \mathrm{End}^*(V)$ restricted to $\pi(G)$ (this follows from noncompact Morse theory and holds for arbitrary reductive groups as well). To find these critical points use the left action of $\pi(G)$ on $\mathrm{End}(V)$. This action gives rise to n left-invariant linear vector fields $v_1, \ldots, v_n$, which span the tangent space to G at each point. Then the critical points of f are exactly the points of intersection of $\pi(G)$ with a subspace of complimentary dimension given by equations $f(v_1(x)) = \ldots = f(v_n(x)) = 0$. The difference between torus and reductive cases is that in the first case, this subspace is generic, while in the second case it usually has huge intersection with $\pi(G)$ at infinity. The latter happens because the left action of G at infinity gives the infinite number of orbits, and at each such orbit f has in some sense critical points.

To avoid this difficulty it is natural to consider the action of $G \times G$ on $\mathrm{End}(V)$ by left and right multiplications (in the torus case this is the same as the left action because of commutativity). The advantage of this action is that it has the finite number of orbits at infinity. However, n generic vector fields coming from this action do not necessarily span the tangent space to G at each point. Thus we arrive to the classical notion of the Chern classes as the degeneracy loci of generic sections of the tangent bundle.

The remaining problem is to compute the degrees $\deg \pi(S_i)$ of Chern classes. In section 3.5, I will compute the degrees of the first and the of last Chern classes. Some examples will be listed in section 3.6.

The following remarks concern notations. In this chapter, the term equivariant (e.g. equivariant compactification, bundle, etc.) will always mean equivariant under the action of the doubled group $G \times G$, unless otherwise stated. By $\mathfrak{g}$ denote the Lie algebra of G. I also fix an embedding $G \subset GL(W)$ for some vector space W. Then for $g \in G$ and $A \in \mathfrak{g}$ notations Ag and gA mean the product of linear operators in $\mathrm{End}(W)$.

3.2 Equivariant compactifications and the ring of conditions

This section contains some classical notions and theorems, which will be used in the sequel. First, I will define the notion of spherical action and describe equivariant compactifications of reductive groups following [5], [20]. For more details see also [38]. Then I define and classify equivariant vector bundles over reductive groups. Finally, I state Kleiman's transversality theorem [28] and recall the definition of the ring of conditions [5, 3].

Spherical action. Reductive groups are partial cases of more general *spherical* homogeneous spaces. They are defined as follows. Let G be a connected reductive group, and let M be its homogeneous space. The action of G on M is called *spherical*, if a Borel subgroup of G has an open dense orbit in M. In this case, the homogeneous space M is also called spherical. An important and very useful property, which characterizes a spherical homogeneous space, is that any its compactification equivariant under the action of G contains only finite number of orbits [36, 30].

There is a natural action of the group $G\times G$ on G by left and right multiplications. Namely, an element $(g_1, g_2) \in G \times G$ maps an element $g \in G$ to $g_1 g g_2^{-1}$. This action is spherical as follows from the Bruhat decomposition of G with respect to some Borel subgroup. Thus the group G can be considered as a spherical homogeneous space of the doubled group $G\times G$ with respect to this action. For any representation $\pi : G \to \mathrm{End}(V)$ this action can be obviously extended to the action of $\pi(G)\times\pi(G)$ on the whole $\mathrm{End}(V)$ by left and right multiplications. I will call such actions standard.

Equivariant compactifications. With any representation π one can associate the following compactification of $\pi(G)$. Take a cone over $\pi(G)$ (consisting of all points $x \in \mathrm{End}(V)$ such that $\lambda\cdot x$ belongs to $\pi(G)$ for some $\lambda \in \mathbb{C}^*$), take its projectivization and then take its closure in $\mathbb{P}(\mathrm{End}(V))$. We obtain a compact projective variety $X_\pi \subset \mathbb{P}(\mathrm{End}(V))$ with a natural action of $G \times G$ coming from the standard action of $\pi(G) \times \pi(G)$ on $\mathrm{End}(V)$. Below I will list some important properties of this variety.

Without loss of generality one can assume that $\pi(G)$ is isomorphic to G. Fix a maximal torus $T \subset G$. Consider all weights of representation π, i.e. all characters of the maximal torus T occurring in π. Take their convex hull P_π in the lattice of all characters of T. Then it is easy to see that P_π is a polytope invariant under the action of the Weyl group of G. It is called the *weight polytope* of the representation π. The polytope P_π contains a lot of information about the compactification X_π.

Theorem 3.2. *1) [20] The subvariety X_π consists of the finite number of $G \times G$-orbits. These orbits are in one-to-one correspondence with the orbits of the Weyl* *group acting on the faces of the polytope P_π. The codimension of an orbit in X_π equals to the codimension of the corresponding face in P_π.*

2)[5] Let ρ be another representation of G. The subvarieties X_π and X_ρ are isomorphic if and only if the normal fans corresponding to the polytopes X_π and X_ρ coincide. If the first fan is a subdivision of the second, then there exists the equivariant map $X_\pi \to X_\rho$.

In particular, suppose that the group G is of adjoint type, i.e. it has a trivial center. Then G has one distinguished faithful representation, namely, the adjoint representation

$$\mathrm{Ad} : G \to \mathrm{End}(\mathfrak{g}); \quad \mathrm{Ad}(g)X = gXg^{-1}, X \in \mathfrak{g}.$$

The corresponding compactification X_{Ad} of the group G is called the wonderful compactification. It was introduced by De Concini and Procesi [5]. The wonderful compactification is smooth, and $X_{\mathrm{Ad}} \setminus G$ is a divisor with normal crossings. There are k orbits $\mathcal{O}_1, \ldots, \mathcal{O}_k$ of codimension 1 in X_{Ad}. The other orbits are obtained as the intersections of the closures $\overline{\mathcal{O}}_1, \ldots, \overline{\mathcal{O}}_k$. More precisely, to any subset $\{i_1, i_2, \ldots, i_m\} \subset \{1, \ldots, n\}$ corresponds an orbit $\overline{\mathcal{O}}_{i_1} \cap \overline{\mathcal{O}}_{i_2} \cap \ldots \cap \overline{\mathcal{O}}_{i_m}$ of codimension m. So the number of orbits equals to 2^k. There is a unique closed orbit $\mathcal{O}_1 \cap \ldots \cap \mathcal{O}_k$, which is isomorphic to the product of two flag varieties $G/B \times G/B$. Here B is a Borel subgroup of G.

In fact, if one takes any irreducible representation π with the strictly dominant highest weight, then the corresponding compactification X_π is isomorphic to the wonderful compactification. It is an immediate corollary from the second part of Theorem 3.2, since in this case, the normal fan of the weight polytope P_π is the fan of Weyl chambers. Hence, it is the same as for the weight polytope P_{Ad}.

Equivariant vector bundles. Let L be a vector bundle over G of rank d. Denote by $V_g \subset L$ a fiber of L lying over an element $g \in G$. Assume that the standard action of $G \times G$ on G can be linearly extended to L. More precisely, there exists a homomorphism $A : G \times G \to \mathrm{Aut}(L)$ such that $A(g_1, g_2)$ restricted to a fiber V_g is a linear operator from V_g to $V_{g_1 g g_2^{-1}}$. If these conditions are satisfied, then the vector bundle L is said to be *equivariant* under the action of $G \times G$.

Two equivariant vector bundles L_1, L_2 are equivalent if there exists an isomorphism between L_1 and L_2 that is compatible with the structure of a fiber bundle and with the action of $G \times G$. The following simple proposition describes equivariant vector bundles on G up to this equivalence relation.

Proposition 3.3. *The classes of equivalent equivariant vector bundles of rank d are in one-to-one correspondence with the linear representations of G of dimension d.*

Proof. Assign to a vector bundle L a representation $\pi : G \to \mathrm{End}(V_e)$ as follows: $\pi(g)$ is the restriction of $A(g, g)$ on V_e. Vice versa, with each representation $\pi : G \to V$ on can associate a bundle L isomorphic to $G \times V$ with the following action of $G \times G$:

$$A(g_1, g_2) : (g, v) \to (g_1 g g_2^{-1}, \pi(g_1)v).$$

E.g. a constant section $v(g) = v$ for $v \in V$ is right-invariant. It is easy to check that this correspondence is indeed one-to-one. □

E.g. the tangent bundle $\mathrm{T}G$ on G is clearly equivariant and corresponds to the adjoint representation of G on the Lie algebra $\mathfrak{g} = \mathrm{T}G_e$. This example will be important in section 3.4.

The ring of conditions. The following theorem gives a tool to define the intersection index on a noncompact group, or more generally, on a homogeneous space.

Theorem 3.4. *(Kleiman's transversality theorem)[28] Let H be a connected algebraic group, and let M be its homogeneous space. Take two algebraic subvarieties $X, Y \subset M$. Denote by gX a left translate of X by an element $g \in G$. There exists an open* *dense subset of H such that for all elements g from this subset the intersection $gX \cap Y$ either has dimension* $\dim X + \dim Y - \dim H$ *or is empty. If X and Y are smooth, then $gX \cap Y$ is transverse.*

In particular, if X and Y have complimentary dimensions (but are not necessarily smooth), then $gX \cap Y$ consists of the finite number of points, and this number is constant.

If X and Y have complimentary dimensions, define the *intersection index* (X, Y) as the number of intersection points $\#(gX \cap Y)$ for a generic $g \in H$. If one is interested in solving enumerative problems, then it is natural to consider algebraic subvarieties of M up to the following equivalence. Two subvarieties X_1, X_2 of the same dimension are equivalent if and only if for any subvariety Y of complimentary dimension the intersection indices (X_1, Y) and (X_2, Y) coincide. This relation is similar to the numerical equivalence in algebraic geometry (see [10], Chapter 19). Consider all formal linear combinations of algebraic subvarieties of M modulo this equivalence relation. Then the resulting group $C^*(M)$ is called the group of conditions of M.

One can define an *intersection product* of two subvarieties $X, Y \subset M$ by setting $X \cdot Y = gX \cap Y$, where $g \in G$ is generic. However, the intersection product sometimes is not well-defined on the group of conditions. There is the following simple counterexample [5]. Suppose that $G = \mathbb{C}^n$ is an affine space. Then two affine subspaces represent the same class in $\mathbb{C}^N$ if and only if they are parallel. Indeed, if they are not parallel, then there exists a line parallel to one subspace and intersecting the other. Then generic parallel shifts of this line do not intersect the first subspace but do intersect the other at one point. In an affine space $\mathbb{C}^3$ with coordinates (x, y, z), consider a quadric Y given by the equation $x = yz$. Take any two different planes parallel to a coordinate plane $X = \{y = 0\} \subset \mathbb{C}^3$. Their intersections with Y give two lines, which are not parallel. Hence, they do not belong to the same class in the group of conditions, and the class of $X \cap Y$ is not defined.

The remarkable fact is that for spherical homogeneous spaces the intersection product is well-defined, i.e. if one takes different representatives of the same classes, then the class of their product will be the same [5, 3]. The corresponding ring $C^*(M)$ is called the *ring of conditions*.

In particular, the group of conditions $C^*(G)$ of a reductive group is a ring. De Concini and Procesi related the ring of conditions to the cohomology rings of equivariant compactifications as follows. Consider the set $\mathcal{S}$ of all equivariant compactifications X_π of the group G corresponding to its representations π. This set has a natural partial order. Namely, a compactification X_ρ is greater than X_π if there exists an equivariant map $X_\rho \to X_\pi$ commuting with the action of $G \times G$. Clearly, such map is unique, and it induces a map of cohomology rings $H^*(X_\pi) \to H^*(X_\rho)$.

Theorem 3.5. *[5, 3] The ring of conditions $C^*(G)$ is isomorphic to the direct limit over the set $\mathcal{S}$ of the cohomology rings $H^*(X_\pi)$.*

De Concini and Procesi proved this theorem in [5] for symmetric spaces. In [3] De Concini noted that their arguments go verbatim for arbitrary spherical homogeneous spaces, in particular, for arbitrary reductive groups.

3.3 Chern classes with the values in the ring of conditions

In this section, I deal with vector bundles over G equivariant under the action of the doubled group $G \times G$. For such bundles I define their Chern classes with the values in the ring of conditions $C^*(G)$. Unlike the usual Chern classes in compact situation, these Chern classes measure the complexity of the action of $G \times G$ and not the topological complexity (topologically any $G \times G$-equivariant vector bundle over G

is trivial). However, they preserve many properties of the usual Chern classes. There is also a relation between these classes and the usual Chern classes of certain bundles over equivariant compactifications of the group G.

In the subsequent sections, I will mostly use the Chern classes of the tangent bundle. Their main application is the formula for the Euler characteristic of a hyperplane section. One of possible applications of the Chern classes of other equivariant bundles is to obtain an explicit description of the ring of conditions $C^*(G)$ in terms of these Chern classes.

Throughout this section, L denotes an equivariant vector bundle over G of rank d corresponding to a representation $\pi : G \to \mathrm{End}(V)$.

Definition of Chern classes. Among all global sections of an equivariant bundle L there are two distinguished subspaces, namely, the subspaces of left- and right-invariant sections. They consists of sections that are invariant under the action of the subgroups $G \times e \subset G \times G$ and $e \times G \subset G \times G$, respectively. Both of this spaces can be canonically identified with the vector space $V = V_e$. Denote by $\Gamma(L)$ the space of all global sections of L that are obtained as a sum of left- and right-invariant sections. If the representation π does not contain any trivial sub-representations, then $\Gamma(L)$ is canonically isomorphic to the direct sum of two copies of V. Otherwise, $\Gamma(L)$ is the quotient space $(V \oplus V)/E$, where $E \subset V \oplus V$ is the diagonal embedding of the maximal trivial sub-representation of π.

When $L = \mathrm{T}G$ is the tangent bundle, $\Gamma(L)$ is a very natural class of global sections. Namely, it consists of all vector fields coming from the standard action of $G \times G$ on G. By this I mean that with any element $(X, Y) \in \mathfrak{g} \oplus \mathfrak{g}$ one can associate a vector field $v \in \Gamma(L)$ as follows:

$$v(x) = \frac{d}{dt}\bigg|_{t=0} [e^{tX} x e^{-tY}] = Xx - xY.$$

This example suggests that one represent elements of $\Gamma(L)$ not as sums but as differences of left- and right-invariant sections.

The space $\Gamma(L)$ can be employed to define Chern classes of L as usual. Take d generic sections $v_1, \ldots, v_d \in \Gamma(L)$. Then the $i-$th Chern class is the $i-$th degeneracy locus of these sections. More precisely, the first Chern class $S_1(L) \subset G$ consists of all points $g \in G$ where all d sections $v_1(g), \ldots, v_d(g)$ are linearly dependent, $S_2(L)$ — of all points where first $d-1$ sections $v_1(g), \ldots, v_{d-1}(g)$ are dependent, and $S_i(L)$ — of all points where first $d-i+1$ sections $v_1(g), \ldots, v_{d-i+1}(g)$ are dependent. This definition almost repeats one of the classical definitions of the Chern classes in the compact setting (see [17]). The only difference is that global sections used in definition are not generic in the space of all sections. They are generic sections of the special subspace $\Gamma(L)$. If one drops this restriction and applies the same definition, then the result will be trivial, since the bundle L is topologically trivial. In some sense, the Chern classes will sit at infinity in this case (the precise meaning becomes clear from the second part of this section). The purpose of my definition is to pull them back to the finite part.

Thus for each $i = 1, \ldots, n$ we get a family $\mathcal{S}_i(L)$ of subvarieties $S_i(L)$ parameterized by collections of i elements from $\Gamma(L)$. In compact situation, all generic members of similar family would represent the same class in the cohomology ring. The same is true here, if one uses the ring of conditions as an analog of the cohomology ring in the noncompact setting. This is the content of the next lemma. Note that sections $v_1, \ldots, v_d \in \Gamma(L)$ are uniquely defined by d vectors $A_1, \ldots, A_d \in V \oplus V$.

Lemma 3.6. *For all collections $A_1, \ldots, A_d$ belonging to some open dense subset of $(V \oplus V)^d$ the class of the corresponding subvariety $S_i(L)$ in the ring of conditions $C^*(G)$ is the same. The class is the image of the class in $H^*(X_\pi)$ under the isomorphism between $C^*(G)$ and the direct limit over $\mathcal{S}$ of the cohomology rings $H^*(X_\rho)$ (see Theorem 3.5). Recall that π is the representation of G corresponding to the vector* *bundle L.*

Proof. The idea of the proof is to consider the closures of $\pi(S_i(L))$ in X_π. They represent the same class in cohomology ring of X_π by continuity. It is enough to prove that they have proper intersection with $G \times G$-orbits in X_π, i.e. the codimension of the intersection with each orbit in this orbit is less than or equal to the codimension of $S_i(L)$ in G. Then Kleiman's transversality theorem will imply that $S_i(L)$ represent the same class in $C^*(G)$ as well. To prove that the intersections are proper I reduce everything to the case of the equivariant bundle over $GL(V)$ corresponding to the tautological representation.

First, describe all left- and right-invariant global sections of L. Consider the representation $\pi : G \to \mathrm{End}(V)$ corresponding to a vector bundle L. Any vector $X \in V$ defines a right-invariant section $v_r(g) = X$ as in the proof of Proposition 3.3. Then it is easy to see that any left-invariant section v_l is given by the formula $v_l(g) = \pi(g)Y$ for $Y \in V$. Hence, any section $v = v_l - v_r$ that belongs to $\Gamma(L)$ can be written as $v(g) = \pi(g)X - Y$ for some $X, Y \in V$. Let us write the sections $v_1, \ldots, v_d$ in this form: $v_1(g) = \pi(g)X_1 - Y_1, \ldots, v_d(g) = \pi(g)X_d - Y_d$.

Now describe $S_i(L)$. First of all, it is clear that the subvarieties $S_i(L)$ depend only on the choice of a flag $F = \{\Lambda^1 \subset \ldots \subset \Lambda^d \subset V \oplus V\}$ where Λ^j is spanned by $A_1, \ldots, A_j$. E.g. S_1 depends only on the choice of a subspace $\Lambda^n \subset V \oplus V$. Consider the graph Λ_g of $\pi(g)$ in $V \oplus V$. This is a subspace of dimension d consisting of vectors $(X, \pi(g)X)$ for $X \in V$. Then g belongs to $S_i(L)$ (i.e. the vectors $v_1(g) = \pi(g)X_1 - Y_1, \ldots, v_d(g) = \pi(g)X_{n-i+1} - Y_{n-i+1}$ are linearly dependent) if and only if the subspaces Λ_g and Λ^{d-i+1} have nonzero intersection. This also implies the following simple relation. Consider a vector bundle U_d over $GL(V)$ corresponding to the tautological representation of $GL(V)$ on the space V. Then $S_i(L)$ is the preimage of $S_i(U_d)$ under the map $\pi : G \to GL(V)$:

$$S_i(L) = \pi^{-1}(S_i(U_d)).$$

Note that this is the same relation that holds for the usual Chern classes in compact situation since the vector bundle L is the pull-back of U_d.

Let us prove Lemma 3.6 for the group $GL(V)$ and the vector bundle $L = U_d$. In this case, $S_i(U_d)$ consists of all elements $g \in GL(V)$ such that the graph of g in $V \oplus V$ has nonzero intersection with Λ^{d-i+1}. Take another flag $F' = \{\Lambda'^1 \subset \ldots \subset \Lambda'^d\}$ and consider the corresponding subvarieties $S_i'(U_d)$. Clearly, if subspaces Λ^i and Λ'^i are generic, e.g. each of them intersects V_1 and V_2 only at the origin, then there exists an operator $h = (h_1, h_2) \in GL(V_1) \times GL(V_2)$ such that $h(\Lambda^i) = \Lambda'^i$ for all $i = 1, \ldots, d$. Hence, the subvariety S_i' constructed via the flag F coincides with the shift $h_1 S_i(U_d) h_2$ of the subvariety $S_i(U_d)$ constructed via the flag F'. In particular, it follows that they represent the same class in the ring of conditions of $GL(V)$.

Remark 3.7. There is another description of $S_i(U_d)$ and of $S_i(L)$. Define an operator $A \in GL(V)$ by setting $A(X_i) = Y_i$. Denote by V^i a subspace of V spanned by the elements $X_1, \ldots, X_i$ (i.e. V^i is the projection of Λ^i onto the first summand of $V \oplus V$). For each $g \in G$ consider a linear operator $(\pi(g) - A)$. It is clear that $S_i(L)$ consists of all elements $g \in G$ such that the operator $(\pi(g) - A)$ restricted to the subspace V^i has a nontrivial kernel. In particular, a subvariety $S_1(L)$ is given by the equation $det(\pi(g) - A) = 0$. Similarly, $S_i(U_d)$ consists of all elements $x \in GL(V)$ such that the operator $(x - A)$ restricted to the subspace V^i has a nontrivial kernel. This description easily implies that the closure of a generic $S_i(U_d)$ in $\mathbb{P}(\mathrm{End}(V))$ is smooth and intersects $GL(V) \times GL(V)$-orbits transversally. It is also clear that the codimension of $S_i(U_d)$ in $GL(V)$ is equal to i.

We now conclude the proof of Lemma 3.6. It follows from Kleiman's transversality theorem applied to $GL(V) \times GL(V)$-orbits in $\mathbb{P}(\mathrm{End}(V))$ that if h_1 and h_2 are generic, then the subvariety $h_1S_i(U_d)h_2$ intersects the $G\times G$-orbits of X_π transversally. This fact combined with the relation $S_i(L)=\pi^{-1}(S_i(U_d))$ implies the statement of Lemma 3.6 for G and L. $\square$

Hence, we proved that the family $\mathcal{S}_i(L)$ of subvarieties $S_i(L)\subset G$ parameterized by elements of $(V\oplus V)^d$ provides a well-defined class $[S_i(L)]$ in the ring of conditions $C(G)$.

Remark 3.8. It follows from the proof of Lemma 3.6 that the closure of a generic $S_i(L)\in\mathcal{S}_i(L)$ in the compactification X_π is smooth and intersects all $G\times G$-orbits transversally. There is also the following criterion for choosing a generic $S_i(L)$. The class of a given subvariety $S_i(L)$ in the ring of conditions coincides with $[S_i(L)]$ if and only if the closure of $S_i(L)$ in X_π intersects all $G\times G$-orbits by subvarieties of codimension i.

In particular, if L is the tangent bundle, then we get that for a generic $S_i(TG)$ the closure of $\mathrm{Ad}(S_i(\mathrm{T}G))$ in the wonderful compactification intersects all orbits transversally.

Definition 1. *The class $[S_i(L)]\in C^*(G)$ defined by the family $\mathcal{S}_i(L)$ is called the i-th Chern class of a fiber bundle L with the value in the ring of conditions.*

From now by $S_i(L)$ I will always mean any subvariety of the family $\mathcal{S}_i(L)$, whose class in the ring of conditions coincides with the Chern class $[S_i(L)]$.

Embeddings to Grassmannians. For an equivariant vector bundle I will now construct an equivariant map of the group G to a Grassmannian. This construction shows that the Chern classes of L are very related to the usual Chern classes of a certain vector bundle over an equivariant compactification of G.

Assume for simplicity that $\pi: G\to \mathrm{End}(V)$ is an embedding. With each $g\in G$ one can associate the graph $\Lambda_g\subset V\oplus V$ of $\pi(g)$. This is a subspace of dimension

d consisting of vectors $(X, \pi(g)X)$ for $X \in V$. Hence, there is a map φ_L from the group G to the Grassmannian $\mathrm{G}(d, 2d)$ of d-dimensional subspaces in a $2d$-dimensional vector space

$$\varphi_L : G \to \mathrm{G}(d, 2d); \quad \varphi_L : g \mapsto \Lambda_g.$$

Note that the restriction to $\varphi_L(G) \simeq G$ of the tautological quotient vector bundle over $\mathrm{G}(d, 2d)$ is isomorphic to L.

Remark 3.9. This construction repeats the following well-known construction (see [17]). Let M be a smooth variety, S be a vector bundle of rank d over M, and $\Gamma(S)$ be a subspace of dimension N in the space of all global sections of S. Suppose that at each point $x \in M$ the sections of $\Gamma(S)$ span the fiber of S at the point x. Then one can map M to the Grassmannian $\mathrm{G}(N-d, N)$ by assigning to each point $x \in M$ the subspace of all sections from $\Gamma(L)$ that vanish at x. Clearly, the vector bundle S coincides with the pull-back of the tautological quotient vector bundle over the Grassmannian $\mathrm{G}(N-d, N)$.

Denote by X_L the closure of $\varphi_L(G)$ in $\mathrm{G}(d, 2d)$. This is a compact projective variety. It is equivariant under an action of $G \times G$ coming from its representation $\pi \oplus \pi$ in the space $V \oplus V$. Namely, an element $(g_1, g_2) \in G \times G$ takes a subspace $\Lambda \subset V \oplus V$ to a subspace $(\pi(g_1), \pi(g_2))(\Lambda)$. The open dense $G \times G$-orbit in X_L is clearly isomorphic to $\pi(G) \simeq G$.

Example 1. Demazure embedding. Let G be a group of adjoint type, and let π be its adjoint representation on the Lie algebra $\mathfrak{g}$. Then as I already noted the corresponding vector bundle L coincides with the tangent bundle of G. The corresponding embedding $\varphi_L : G \to \mathrm{G}(n, \mathfrak{g} \oplus \mathfrak{g})$ coincides with the one constructed by Demazure [4]. Demazure's construction is as follows. An element $x \in G$ goes to the Lie algebra of the stabilizer of x under the standard action of $G \times G$. E.g. the identity element gets mapped to the Lie algebra $\mathfrak{g} \subset \mathfrak{g} \oplus \mathfrak{g}$ embedded diagonally. Then the conjugation by an element $(g_1, g_2) \in G \times G$ maps the Lie algebra $\mathfrak{g} \subset \mathfrak{g} \oplus \mathfrak{g}$ to the Lie algebra of the stabilizer of an element $g_1 g_2^{-1}$. It is now easy to see that the embedding φ_L and Demazure embedding are the same. The compactification X_L in this case is isomorphic to the wonderful compactification of G [4].

Example 2. a) Let G be $GL(V)$ and let π be its tautological representation on a space V of dimension d. Then φ_L is an embedding of $GL(V)$ into the Grassmannian $\mathrm{G}(d, 2d)$. Notice that the dimensions of both varieties are the same. Hence, the compactification X_L coincides with $\mathrm{G}(d, 2d)$.

b) Take $SL(V)$ instead of $GL(V)$ in the previous example. Its compactification X_L is a hypersurface in the Grassmannian $\mathrm{G}(d, 2d)$ which can be described as follows. Consider the Plücker embedding $p : \mathrm{G}(d, 2d) \to \mathbb{P}(\Lambda^d(V_1 \oplus V_2))$ (V_1 and V_2 are two copies of V). Then $p(X_L)$ is a special hyperplane section of $p(\mathrm{G}(d, 2d))$. Namely, the decomposition $V_1 \oplus V_2$ yields a decomposition of $\Lambda^d(V_1 \oplus V_2)$ into the direct sum. This sum contains two one-dimensional components $p(V_1)$ and $p(V_2)$ (which are considered as lines in $\Lambda^d(V_1 \oplus V_2)$). In particular, for any vector in $\Lambda^d(V_1 \oplus V_2)$ it makes sense to speak of its projections to $p(V_1)$ and $p(V_2)$. On V_1 and V_2 there are two special $n-$forms, preserved by $SL(V)$. These forms give rise to two 1-forms l_1 and l_2 on $p(V_1)$ and $p(V_2)$, respectively. Consider a hyperplane H in $\Lambda^d(V_1 \oplus V_2)$ consisting of all vectors v such that the functionals l_1, l_2 take the same values on the projections of v to $p(V_1)$ and $p(V_2)$, respectively. Then it is easy to check that $p(X_L) = p(\mathrm{G}(d, 2d)) \cap \mathbb{P}(H)$.

Denote by L_X a restriction to X_L of the tautological quotient vector bundle U_d over the Grassmannian $\mathrm{G}(d, 2d)$. Assume that X_L is smooth. Let $c_1, \ldots, c_d \in H_*(X_L)$ be the subvarieties dual to the Chern classes of L_X. I will also call them Chern classes.

Proposition 3.10. *The homology class of the closure of $\varphi_L(S_i(L))$ in X_L coincides* *with the i-th Chern class of* L_X

$$[\overline{\varphi_L(S_i(L))}] = c_i(X_L).$$

Proof. The Chern classes of L_X can be obtained as the intersections of X_L with the Chern classes of U_d. The latter have nice representatives $C_1,\ldots,C_d$ which are the closures of certain Schubert cycles corresponding to any partial flag $F = \{\Lambda^1 \subset \ldots \Lambda^d \subset V \oplus V\}$ (see [17]). Namely, C_i consists of all subspaces $\Lambda \in \mathrm{G}(d,2d)$ such that the intersection $\Lambda \cap \Lambda^{n-i+1}$ is non-trivial, i.e. has the dimension at least 1. For a generic flag F the intersection $C_i \cap X_L$ is transverse, and hence, it represents the $i-$th Chern class of X_L. On the other hand, $C_i \cap \varphi_L(G)$ coincides with $\varphi_L(S_i(L))$. $\square$

Properties of the Chern classes of reductive groups The next lemma computes the dimensions of the Chern classes. It also shows that if the action of $\pi(G)$ on V is not transitive, then the higher Chern classes automatically vanish. Denote by $d(\pi)$ the dimension of a generic orbit of $\pi(G)$ in V. In particular, if $\pi(G)$ acts transitively on V, then $d(\pi) = d$.

Lemma 3.11. *If $i > d(\pi)$, then $S_i(L)$ is empty, and if $i \leq d(\pi)$ then the dimension of $S_i(L)$ is equal to $n - i$.*

Proof. Consider the union $D \subset V \oplus V$ of all graphs Λ_g for $g \in G$. Clearly, D consists of all pairs $(X,Y) \in V \oplus V$ such that X and Y belong to the same orbit under the action of $\pi(G)$. I.e. if G acts transitively on V, then $D = V \oplus V$. Thus the codimension of D is equal to the codimension $d - d(\pi)$ of a generic orbit in V. In my main example, when π is the adjoint representation, the codimension of D is equal to the rank of G. Note also that D is conic (i.e. together with each point it contains a line passing through this point and the origin). Since Λ^{d-i+1} is a generic vector subspace, the dimension of $D \cap \Lambda^{d-i+1}$ equals to $d(\pi) - i + 1$, if $i \leq d(\pi)$. In particular, if $i = d(\pi)$, then $D \cap \Lambda^{d-i+1}$ consists of several lines whose number is equal to the degree of D. If $i > d(\pi)$, then $D \cap \Lambda^{d-i+1}$ contains only the origin. It follows that if $i > d(\pi)$, then $S_{n-i+1}(L)$ is empty. Consider the case when $i \leq d(\pi)$. If g belongs to $S_i(L)$ but does not belong to $S_{i+1}(L)$ then there exists exactly one (up to proportionality) element $A = (X, Y) \in D \cap \Lambda^{d-i+1}$ such that $A \in \Lambda_g$. Denote by l_A the line spanned by A. Thus there is a map

$$p : S_i(L) \setminus S_{i+1}(L) \to \mathbb{P}(D \cap \Lambda^{d-i+1}); \quad p : g \mapsto l_A.$$

The preimage $p^{-1}(l_A)$ consists of all $g \in G$ such that $\pi(g)X = Y$. Hence, it is isomorphic to the stabilizer of X under the action of $\pi(G)$. We get that the dimension of $S_i(L)$ is equal to dim $\mathbb{P}(D \cap \Lambda^{d-i+1}) +$ dim $p^{-1}(l_A) = n - i$. □

In fact, the second part of Lemma 3.11 immediately follows from Remark 3.7 combined with Kleiman's transversality theorem as soon as on shows that S_i is nonempty. However, I gave the above proof because it also implies the following corollary. Denote by $H \subset G$ the stabilizer of a generic element in V.

Corollary 3.12. *There exists an open dense subset $U_i \subset S_i(L)$ such that U_i admits a fibration with fibers that coincide with the shifts of H. In particular, the last Chern class $S_{d(\pi)}(L)$ coincides with the disjoint union of several shifts of H. Their number equals to the degree of a generic orbit of G in V.*

The last statement follows from the fact that the degree of D in $V \oplus V$ (see the proof of Lemma 3.11) is equal to the degree of a generic orbit of G in V.

In particular, let L be the tangent bundle. Then stabilizer of a generic element in $\mathfrak{g}$ is a maximal torus in G. Hence, the last Chern class $S_{n-k}(\mathrm{T}G)$ is the union of several shifted maximal tori.

Chern classes of equivariant bundles over G retain many properties of the usual Chern classes. These properties are listed below.

- **Vanishing.** If $i > d$ then the Chern class $[S_i(L)]$ vanishes. As follows from Lemma 3.11 even more precise statement is true. If $i > d(\pi)$, then the Chern class $[S_i(L)]$ vanishes.

- **Dimensions.** If $i \leq d(\pi)$, then the i-th Chern class $[S_i(L)]$ has the codimension i (see Lemma 3.11). In compact situation Definition 1 would rather be a definition of homology cycles dual to the usual Chern classes. The class $[S_i(L)]$ can also be viewed as a linear functional on $C^i(G)$: on each cycle $Y \in C^*(G)$ of dimension i the Chern class $S_i(L)$ takes the value $(S_i(L), Y)$.

  The remaining properties follow directly from the description of the Chern classes given in the proof of Lemma 3.6.

- **Pull-back.** Let $\varphi : G_1 \to G_2$ be a homomorphism of two reductive groups G_1 and G_2, and let L be an equivariant vector bundle on G_2 corresponding to a representation π. The pull-back φ^*L is an equivariant fiber bundle on G_1 corresponding to the representation $\pi \circ \varphi$. The Chern classes of L and of φ^*L are related as follows:
$$[S_i(\varphi^*L)] = \varphi^{-1}[S_i(L)].$$
  In particular, let us apply this formula to the homomorphism $\pi : G \to \pi(G)$. We immediately get that if the representation $\pi : G \to \mathrm{End}(V)$ corresponding to a vector bundle L has a nontrivial kernel, then $S_i(L)$ are invariant under left and right multiplications by elements of the kernel.

- **Line bundles.** Suppose that L has rank 1. Then L corresponds to a character $\pi : G \to \mathbb{C}^*$. The first Chern class of L coincides with the class of a hypersurface $\{\pi(g) = 1\}$.

3.4 Proof of Theorem 3.1 (an adjunction formula)

In this section, I will use only the Chern classes $S_i = S_i(\mathrm{T}G)$ of the tangent bundle.

The proof is a modification of Bernstein's proof of formula (1) in the torus case. I will use the same ideas.

Denote by f a linear functional which defines the hyperplane H, i.e. $H = \{x \in \mathrm{End}(V) : f(x) = C\}$ for some constant C. It follows from noncompact Morse theory [18] that $(-1)^{n-1}\chi(\pi)$ is equal to the number of critical points of f restricted to $\pi(G)$ (see [22] for the proof of exactly this statement by various methods). Let us express the number of the critical points in terms of the degrees of $\pi(S_i)$.

Recall that vector fields $v_1, \ldots, v_n$ on G were given by the formula $v_i(g) = X_i g - gY_i$. Note that for any representation π of the group G the direct images $\pi_* v_i$ are well-defined (they are given by the formula $d\pi(X_i)\pi(g) - \pi(g)d\pi(Y_i)$, where $d\pi : \mathfrak{g} \to \mathrm{End}(V)$ is the derived map) and can be straightforwardly extended to the linear vector fields on the whole $\mathrm{End}(V)$. Namely, for $x \in \mathrm{End}(V)$ set $\pi_* v_i(x) = d\pi(X_i)x - xd\pi(Y_i)$.

Abusing notation I will denote vector fields $\pi_* v_i$ again by v_i and will write S_i instead of $\pi(S_i)$ during the proof.

Since S_1 has a codimension 1 in G one can always choose a hyperplane H in such a way that all critical points of f restricted to G lie outside S_1. Clearly, each critical point x satisfy n linear equations of the form $f(v_i(x)) = 0$. The converse is also true for points outside S_1, since at these points vector fields $v_1, \ldots, v_n$ span the tangent space to G. Denote by $V_n(f)$ a vector subspace given by the equations $f(v_i(x)) = 0$ for $i = 1, \ldots, n$. It has codimension n. Denote by $S(f)$ the set of the critical points of the function f restricted to G. We get that $S(f) = G \cap V_n(f) \setminus S_1 \cap V_n(f)$. It turns out that if f is generic, then all the intersection points of $G \cap V_n(f)$ are transverse, and their number equals to the degree of G. Moreover, the following statement is true.

Lemma 3.13. *For $i = 1, \dots, n$ consider a subspace $V_i(f) \subset \mathrm{End}(V)$ of codimension i given by i equations $f(v_1(x)) = \dots = f(v_i(x)) = 0$. If f is generic, then*

1)all the intersection points of $S_{n-i} \cap V_i(f)$ are transverse, and their number equals to the degree of S_{n-i};

2)the intersection $S_{n-i+2} \cap V_i(f)$ is empty (or in other words, all points of the intersection $S_{n-i+1} \cap V_i(f)$ lie outside S_{n-i+2}).

Postpone the proof of this lemma until the next paragraph. It follows that $\#S(f) = \deg G - \#(S_1 \cap V_n(f))$. It remains to compute $\#(S_1 \cap V_n(f))$. Here we can use the induction argument. Indeed, the second part of Lemma 3.13 immediately implies that the intersection $S_{n-i+1} \cap V_i(f)$ coincides with the difference $S_{n-i+1} \cap V_{i-1}(f) \setminus S_{n-i+2} \cap V_{i-1}$. Thus $\#(S_1 \cap V_n(f)) = \deg(S_1) - \#(S_2 \cap V_{n-1}(f))$ and we can proceed by induction.

Hence, we proved the following formula

$$\#S(f) = \sum_{i=0}^{n-k} (-1)^{n-i+1} \#(S_i \cap V_{n-i}(f)).$$

By Lemma 3.13 the number $\#(S_i \cap V_{n-i}(f))$ is equal to $\deg(S_i)$. This completes the proof of Theorem 3.1.

Proof of Lemma 3.13. First, I consider more general situation. Let $X \subset W$ be an irreducible closed affine variety in an affine space $W = \mathbb{C}^N$, and let $l_1(x), \dots, l_m(x)$ be m linear vector fields on W. The next proposition is analogous to Lemma 3.13 and holds under condition that vector fields $l_1, \dots, l_m$ restricted to X are not too degenerate at infinity. This can be formalized as follows. Denote by $C_X \subset W$ an asymptotic cone of X, i.e. a conic variety whose projectivization $\mathbb{P}(C_X) \subset \mathbb{CP}^{N-1}$ coincides with the Zarisky boundary of X in $\mathbb{CP}^N$. Denote by $X_i \subset W$ the i-th degeneracy locus of the vector fields $l_1, \dots, l_m$, i.e. $X_i = \{x \in W : l_1(x), \dots, l_{m-i+1}(x)$ are linearly dependent$\}$.

Proposition 3.14. *Suppose that the following conditions are satisfied:*

1)at any point $x \in X_i \cap X$ the intersection of the tangent spaces $\mathrm{T}_x X_i$ and $\mathrm{T}_x X$ has codimension at least i,

2)the intersection $X_i \cap C_X$ has codimension at least i in C_X.

For a generic linear function f on W consider a vector subspace $V_f \subset \mathbb{CP}^N$ of codimension m given by the equations $f(l_1(x)) = \ldots = f(l_m(x)) = 0$. Then

1. *If* $\dim\ X = m$*, then all intersection points $X \cap V_f$ are transverse and their number is equal to the degree of X: $\#(X \cap V_f) = \deg X$.*

2. *If* $\dim\ X < m$*, then $X \cap V_f$ is empty.*

Proof. First, prove that for a generic f the subspace V_f does not intersect X at infinity, i.e. V_f intersects the asymptotic cone of X only at the origin. To do this consider the subvariety $D \subset W^*$ of all linear functions f on W such that $\mathbb{P}(V_f)$ does intersect $\mathbb{P}(C_X)$. It is enough to show that the dimension of D is less than N. Indeed, in this case the complement to D in W^* is an open dense subset and for any f from the complement, the subspace V_f intersects C_X only at the origin. Let us estimate the dimension of D. Consider a subvariety $\tilde{D} \subset \mathbb{P}(C_X) \times W^*$ consisting of all pairs (z, f) such that $\mathbb{P}(V_f)$ contains z. Then D is the image of $\tilde{D}$ under its projection onto W^*. Hence the dimension of D is less than or equal to the dimension of $\tilde{D}$. A filtration $X_m \cap C_X \subset X_{m-1} \cap C_X \subset \ldots \subset X_1 \cap C_X \subset C_X$ induces a filtration on $\mathbb{P}(C_X)$. Consider the projection of $\tilde{D}$ onto $\mathbb{P}(C_X)$. The preimage of $z \in \mathbb{P}((X_i \setminus X_{i+1}) \cap C_X)$ has dimension $N - m + i$. Indeed, it coincides with the subspace of W^* given by the linear system of rank $m - i$ (which consists of m linear equations $f(l_1(x)) = \ldots = f(l_m(x)) = 0$). Thus the dimension of the preimage of $\mathbb{P}((X_i \setminus X_{i+1}) \cap C_X)$ has dimension at most $(N - m + i) + (m - i - 1) = N - 1$. It follows that the dimension of $\tilde{D}$ is at most $N - 1$.

Note that exactly the same argument proves the second part of Proposition 3.14.

It remains to prove that all points of the intersection $X \cap V_f$ are transverse for a generic f. The first condition of Proposition 3.14 implies that for any point $x \in X$ the tangent space T_xX and X_i intersect each other transversally. Hence, we can apply to T_xX the first part of Proposition 3.14 and get that for a generic f the subspaces T_xX and V_f intersect each other at one point. □

To complete the proof of Lemma 3.13 apply Proposition 3.14 to $W = \mathrm{End}(V), X = S_i$ and vector fields $l_1 = v_1, \ldots, l_{n-i+1} = v_{n-i+1}$. All we need to show is that the vector fields $v_1, \ldots, v_n$ and the subvariety S_i corresponding to them satisfy the non-degeneracy conditions of Proposition 3.14. The first condition follows from the proof of Lemma 3.6 (see the Remark in the proof). The second condition can also be deduced from what we already proved. However, I will give a more direct self-contained proof. It explains why Bernstein's idea works for sums of left- and right-invariant vector fields although it does not work if one takes only left-(or right-)invariant vector fields.

Recall that there is a natural family of deformations of vector fields $v_1, \ldots, v_n$ parameterized by elements of $\mathfrak{g}\oplus\mathfrak{g}$. I will show that a generic perturbation of $v_1, \ldots, v_n$ inside this family satisfies the second non-degeneracy condition.

Let $C_\pi(S_i)$ be an asymptotic cone of $S_i \subset \mathrm{End}(V)$. The asymptotic cone $C_\pi(G)$ of the group itself consists of the finite union of orbits under the standard action of $G \times G$ on $\mathrm{End}(V)$. Note that vector fields $v_1, \ldots, v_n$ can be any fields coming from this action. Take any point $x \in C_\pi(S_i)$. Consider its orbit under the standard action. Clearly, any collection of n vectors in the tangent space to the orbit at the point x can be realized as the values of vector fields $v_1, \ldots, v_n$ at x. In particular, if x belongs to the intersection of $C_\pi(S_i)$ with an orbit of codimension less than or equal to i, then one can perturb $v_1, \ldots, v_{n-i+1}$ in such a way that they still be linearly dependent but $v_1, \ldots, v_{n-i}$ will not. Hence, the point x belongs to $C_\pi(S_i)$, but does not belong to $C_\pi(S_{i+1})$. Since the number of orbits is finite, it follows that the codimension of

$C_\pi(S_{i+1})$ in $C_\pi(S_i)$ is at least 1.

To get the second part of Lemma 3.13 apply Proposition 3.14 to S_i and $n-i+1$ vector fields $v_2, \ldots, v_{n-i+2}$. Perturbing the last vector field v_{n-i+2} we can make these vector fields linearly independent at a generic point of S_i without changing S_i.

3.5 Degree of the first and of the last Chern classes

Computing the Euler characteristic $\chi(\pi)$ of a generic hyperplane section we have expressed it via the degrees of the Chern classes S_i. The next question is how to compute these degrees. That means to compute the intersection index of $\pi(S_i)$ with $n-i$ generic hyperplane sections corresponding to the representation π. If S_i is a complete intersection of generic hyperplane sections corresponding to some representations of G, then the answer to this question is given by the Brion-Kazarnovskii formula. In this section, I will prove that this is the case for S_1. I can also compute the degree of the last Chern class S_{n-k} , because S_{n-k} is the union of several maximal tori (see Corollary 3.12). There is a hope that an explicit answer can be obtained for the other S_i as well.

It follows from the proof of Lemma 3.6 (see Remark 3.7) that $S_1 \subset G$ is given by the equation $det(\mathrm{Ad}(g)-A)=0$ for a generic $A \in \mathrm{End}(\mathfrak{g})$. The function $det(\mathrm{Ad}(g)-A)$ is a linear combination of matrix elements corresponding to all exterior powers of the adjoint representation. Hence, the equation of S_1 is the equation of a hyperplane section corresponding to the sum of all exterior powers of the adjoint representation. Denote this representation by σ. It is easy to check that the weight polytope P_σ coincides with the weight polytope of the irreducible representation θ with the highest weight 2ρ (here ρ is the sum of all fundamental weights). Hence, the degree of S_1 can be computed by Brion-Kazarnovskii formula [2, 23] for the representations θ and π. It remains to prove that S_1 is generic, which means that the closure of S_1 in X_σ

intersects all $G \times G$-orbits by subvarieties of codimension at least 1. The Remark after Lemma 3.6 implies that this is true for the wonderful compactification, and X_σ is isomorphic to the wonderful compactification by Theorem 3.2 (since $P_\sigma = P_\theta$).

The last Chern class S_{n-k} is the disjoint union of maximal tori. Their number is equal to the degree of a generic adjoint orbit in $\mathfrak{g}$. The latter is equal to the order of the Weyl group W. Denote by $[T]$ the class of a maximal torus in the ring of conditions $C^*(G)$. Then the following identity holds in $C^*(G)$:

$$[S_{n-k}] = |W|[T].$$

The degree of $\pi(T)$ can be computed using the formula of D.Bernstein, Khovanskii and Koushnirenko [24].

3.6 Examples

$\mathbf{G} = \mathbf{SL_2}(\mathbb{C})$**.** Consider a tautological embedding of G, namely, $G = \{(a,b,c,d) \in \mathbb{C}^4 : ad - bc = 1\}$. Since the dimension of G is 3 and the rank is 1, then by Lemma 3.11 we get that there are only two nontrivial Chern classes: S_1 and S_2. Let us apply the results of the preceding section to find them. The first S_1 is a generic hyperplane section corresponding to the second symmetric power of the tautological representation, i.e. to the representation $SL_2(\mathbb{C}) \to SO_3(\mathbb{C})$. In other words, it is the intersection of $SL_2(\mathbb{C})$ with a quadric in $\mathbb{C}^4$. The second Chern class S_2 (which is also the last one in this case) is the union of two shifted maximal tori.

Let π be a faithful representation of $SL_2(\mathbb{C})$. It is the sum of irreducible representations of SL_2. Any irreducible representation of SL_2 is isomorphic to the $i-$th symmetric power of the tautological representation for some i. Its weight polytope is a line segment $[-i, i]$. Hence the weight polytope of π is the line segment $[-n, n]$ where n is the greatest exponent of symmetric powers occurring in π. Then matrix elements of π are polynomials in a, b, c, d of degree n. In this case, it is easy to compute the degrees of the subvarieties G, S_1 and S_2 by Bezout theorem. One gets deg $\pi(G) = 2n^3$, deg $\pi(S_1) = 4n^2$, deg $\pi(S_2) = 4n$. Thus the Euler characteristic $\chi(\pi)$ is equal to $2n^3 - 4n^2 + 4n$. This answer was first obtained by K.Kaveh who used different methods [22].

If π is not faithful, i.e. $\pi(SL_2(\mathbb{C})) = SO_3(\mathbb{C})$, then clearly, $\chi(\pi)$ is two times smaller and equal to $n^3 - 2n^2 + 2n$.

$\mathbf{G} = (\mathbb{C}^*)^{\mathbf{n}}$ is a complex torus. In this case, all left-invariant vector fields are also right-invariant since the group is commutative. Hence, they are linearly independent at any point of $G = (\mathbb{C}^*)^n$ as long as their values at the identity are linearly independent. It follows that all subvarieties S_i are empty, and the only one term in the right hand side of formula (2) is left

$$\chi(\pi) = (-1)^{n-1}\text{deg } \pi(G).$$

This is exactly the formula proved by D.Bernstein, A.Khovanskii and A.Koushnirenko [24].

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