Flag varieties and Gelfand-Zetlin polytopes

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I describe a relation between geometry of complete flag varieties and combinatorics of Gelfand–Zetlin polytopes. This is similar to the rich interplay between toric varieties and their Newton polytopes. For motivation, I will first recall some well-known results for toric varieties and then outline their partial extension to the setting where a toric variety is replaced by a *regular* compactification of an arbitrary reductive group.

Let X be a smooth complex toric variety of dimension n, and D a very ample divisor on X. Recall that with a pair (X, D) one can associate a convex lattice polytope $P_D \subset \mathbb{R}^n$ called the Newton polytope of X (e.g. P_D can be defined as the convex hull of all Laurent monomials occuring in the defining equation of D). Many geometric invariants of X can be computed explicitly in terms of the polytope P_D (see the list below). One of the key ingredients in such computations is a one-to-one correspondence between G–orbits in X and faces of P_D. This correspondence preserves dimensions and incidence relations.

- The self-intersection index D^n of the divisor D is equal to $n!$ times the volume of P_D [7].
- The Picard group of X is isomorphic to the group of virtual lattice polytopes analogous to P_D (i.e. having the same normal fan) modulo parallel translations.
- The Euler characteristic $\chi(D_1 \cap \ldots \cap D_m)$ of a complete intersection of hypersurfaces can be computed explicitly for any $m \leq n$ [7].
- There is an explicit description of the cohomology ring $H^*(X)$ by generators and relations [4]. In particular, there is the following formula for the intersection product of the divisor D with the G–orbit $\mathcal{O}_\Gamma$ corresponding to a face Γ.
$$D\overline{\mathcal{O}}_\Gamma = \sum_{\Delta \subset \Gamma} d(v, \Delta)\overline{\mathcal{O}}_\Delta,$$
where the sum is taken over the facets Δ of Γ. Here $v \in \Gamma \cap \mathbb{Z}^n \subset \mathbb{R}^n$ is any point on Γ with integer coordinates, and $d(v, \Delta)$ denotes the integral distance from v to the face Δ.

Consider now a more general case. Let G be an arbitrary connected complex reductive group of dimension n. Note that the left and right actions of G on itself are in general different so it makes sense to consider the action by the doubled group $G \times G$. Let X be a $G \times G$*–equivariant* compactification of G, that is, the group $G \times G$ acts on X with the open dense orbit isomorphic to G and on this orbit the action coincides with the action by left and right multiplications. As in the toric case, X will always consist of a finite number of $G \times G$–orbits. One way to construct such a compactification is to take a projectively faithful representation $\pi : G \to \mathrm{End}(V)$. Then the closure X_π of $\mathbb{P}(\pi(G))$ in the projective space $\mathbb{P}(\mathrm{End}(V))$ is a $G \times G$ equivariant compactification of G. In particular, when G is a complex torus all projective toric varieties can be obtained in this way.

An important class of $G \times G$–equivariant compactifications consists of *regular* compactifications introduced in [3]. These are the closest relatives of smooth toric varieties. In particular, the closures of all $G \times G$–orbits in a regular compactification are smooth and intersect each other transversally. Regular compactifications include all smooth toric varieties and wonderful compactifications of semisimple groups of adjoint type.

As in the toric case, with each very ample divisor D one can associate a convex lattice polytope $P_D \subset \mathbb{R}^k$. Here k is the rank of G, that is, the dimension of a maximal torus, and $\mathbb{Z}^k \subset \mathbb{R}^k$ is identified with the weight lattice of G. E.g. when $X = X_\pi$ and D is the divisor of hyperplane section then P_D is the weight polytope of π. There is a one-to-one correspondence between $G \times G$–orbits in X and orbits of the Weyl group of G acting on the faces of P_D. This correspondence preserves codimensions and incidence relations. In particular, vertices of P_D correspond to the closed orbits in X, which have dimension $n - k$ and are isomorphic to the product $G/B \times G/B$ of two flag varieties. Again there is a strong relation between geometry of X and combinatorics of P_D.

- Fix a fundamental Weyl chamber $\mathcal{D} \subset \mathbb{R}^k$. Then
$$D^n = n! \int_{P_D \cap \mathcal{D}} F(x)dx,$$
where F is a homogeneous polynomial function on $\mathbb{R}^k$ of degree $n - k$ that depends only on the group G and not on X and D [5, 2]. In particular, if G is a complex torus, then $F \equiv 1$.
- The Picard group of X is isomorphic to the group of virtual lattice polytopes analogous to P_D and invariant under the action of the Weyl group modulo parallel translations.
- The Euler characteristic $\chi(D_1 \cap \ldots \cap D_m)$ of a complete intersection of hypersurfaces can be computed explicitly for any $m \leq n$ [8, 9].

However, no description of $H^*(X)$ by generators and relations is known. In order to obtain such a description it might be useful to consider a bigger polytope $\widetilde{P}_D \subset \mathbb{R}^n$ that fibers over $P_D \cap \mathcal{D}$ with fibers equal to the product of two Gelfand-Zetlin polytopes. Such a polytope has been recently constructed in a much more general setting [6]. The bigger polytope $\widetilde{P}_D$ contains more information about the variety X. In particular, the self-intersection index D^n is equal to $n!$ times the volume of $\widetilde{P}_D$, exactly as in the toric case.

In a sense, a regular compactification X is made up of a toric variety (corresponding to the smaller polytope P_D) and the product of two flag varieties (corresponding to the product of two Gelfand-Zetlin polytopes). I hope that the relation between flag varieties and Gelfand-Zetlin polytopes will help to get new insights into geometry of regular compactifications of reductive groups.

I will now come to the main object of my talk. Let G be the group $GL_n(\mathbb{C})$, and $X = G/B$ the complete flag variety for G. Recall that with each strictly dominant weight λ of G one can associate the Gelfand-Zetlin polytope Q_λ so that the integral points inside and at the boundary of Q_λ parameterize a natural basis in the irreducible representation of G with the highest weight λ. The Gelfand-Zetlin polytope Q_λ is a convex polytope in $\mathbb{R}^d$ with vertices lying in the integral lattice $\mathbb{Z}^d \subset \mathbb{R}^d$. Here $d = n(n-1)/2$ denotes the dimension of X.

I have constructed a correspondence between the Schubert cycles in X and some special faces of the Gelfand-Zetlin polytope [10]. Namely, an l-dimensional face Γ of the Gelfand-Zetlin polytope is assigned to each l-dimensional Schubert cycle Z using *Demazure modules* for a Borel subgroup in G. There are some degrees of freedom in the construction, namely, the same Schubert cycle can be represented by different faces (different choices of a face correspond to different choices of a Borel subgroup containing a given maximal torus). For some Schubert cycles, it is possible to choose a face Γ so that combinatorics of Γ captures geometry of D very well (let us call such faces *admissible*). In particular, admissible faces behave well with respect to the incidence relation between Schubert cycles. Then the classical Chevalley formula [1] for the intersection product of Z with the divisor D_λ on X corresponding to the weight λ has the following interpretation in terms of an admissible face Γ.

$$D_\lambda Z_\Gamma = \sum_{\Delta\subset\Gamma} d(v,\Delta) Z_\Delta,$$

where the sum is taken over the facets Δ of Γ (these correspond to the Schubert cycles Z_Δ of codimension one at the boundary of Z_Γ). Here v is a fixed vertex of the face Γ. Note that in this form the formula is completely analogous to the formula for toric varieties mentioned above and to the analogous formula for regular compactifications of reductive groups [9].

Many Schubert cycles can be represented by an admissible face, but not all of them. In particular, all Schubert cycles that degenerate to a single toric variety under the Caldero's construction [11] of toric degenerations of flag varieties can be represented by admissible faces. However, there are many other examples of Schubert cycles represented by admissible faces. E.g. for the flag variety of GL_3 all Schubert cycles can be represented by admissible faces (although one of the 2-dimensional Schubert cycles degenerates into the union of two toric subvarieties under the Caldero's construction). For GL_4, exactly two Schubert cycles can not be represented by an admissible face. These two cycles are the homology classes of Schubert cells whose closures in the flag variety are not smooth. I conjecture that all Schubert cycles defined by Schubert cells with smooth closures can be represented by an admissible face. In our joint work with Evgeny Smirnov, we are currently proving this conjecture.

References

[1] I.N.Bernstein, I.M.Gelfand, S.I.Gelfand, *Schubert cells, and the cohomology of the spaces G/P,* Russian Math. Surveys **28** (1973), no. 3, 1–26

[2] Michel Brion, *Groupe de Picard et nombres caracteristiques des varietes spheriques*, Duke Math J. **58** (1989), no.2 , 397-/424

[3] C. De Concini and C. Procesi, *Complete symmetric varieties II Intersection theory,* Advanced Studies in Pure Mathematics **6** (1985), Algebraic groups and related topics, 481–513

[4] V. I. Danilov, *The geometry of toric varieties*, Russian Math. Surveys **33** no.2 (1978), 97–154

[5] B.Ya. Kazarnovskii, *Newton polyhedra and the Bezout formula for matrix-valued functions of finite-dimensional representations,* Functional Anal. Appl. **21** (1987), no. 4, 319–321

[6] Kiumars Kaveh, Askold Khovanskii, *Convex bodies and algebraic equations on affine varieties*, preprint arXiv:0804.4095v1[math.AG]

[7] A.G. Khovanskii, *Newton polyhedra, and the genus of complete intersections,* Functional Anal. Appl. **12** (1978), no. 1, 38–46

[8] Valentina Kiritchenko, *Chern classes for reductive groups and an adjunction formula*, Annales de l'Institut Fourier, **56** (2006), no. 3, 1225-1256

[9] Valentina Kiritchenko, *On intersection indices of subvarieties in reductive groups*, Moscow Mathematical Journal, **7** no.3 (2007), 489-505

[10] Valentina Kiritchenko, *Gelfand-Zetlin polytopes and geometry of flag varieties*, preprint http://guests.mpim-bonn.mpg.de/kirichen/prints.html

[11] Mikhail Kogan, Ezra Miller, *Toric degeneration of Schubert varieties and Gelfand-Tsetlin polytopes,* Adv. Math. **193** (2005), no. 1, 1–17