From moment polytopes to string bodies

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In toric geometry, a central role is played by moment (or Newton) polytopes of projective toric varieties. In the past decades, various analogs of Newton polytopes for other reductive group actions were constructed culminating in a recent construction of *string bodies* (special Newton-Okounkov convex bodies). My talk was mostly devoted to this construction [3]. A future objective is to use string bodies to study geometry of varieties with a reductive group action (as in the toric case). Below we discuss such an application in a non-toric example.

String bodies for the varieties of complete flags are just string polytopes (e.g. Gelfand–Zetlin polytopes in the case of $GL_n(\mathbb{C})$). Together with Evgeny Smirnov and Vladlen Timorin we develop a new approach to the Schubert calculus on the variety of complete flags in $\mathbb{C}^n$ using the volume polynomial on Gelfand–Zetlin polytopes. This approach allows us to compute the intersection product of Schubert cycles on the flag variety by intersecting faces of the Gelfand–Zetlin polytope. The Gelfand–Zetlin polytope thus gives a combinatorial model for the intersection theory on the flag variety.

First recall the definition of the volume polynomial. Consider the set of all convex polytopes in $\mathbb{R}^n$. This set can be endowed with the structure of an abelian semigroup using *Minkowski sum.* We can embed the semigroup of convex polytopes into its Grothendieck group V, which is a real (infinite-dimensional) vector space. The elements of V are called *virtual polytopes.* On the vector space V, there is a homogeneous polynomial *vol* of degree n, called the *volume polynomial.* It is uniquely characterized by the property that its value $vol(P)$ on any convex polytope P is equal to the volume of P. We will be interested in restrictions of $vol(P)$ to finite-dimensional subspaces V_P of V consisting of all virtual polytopes *analogous* to a given polytope P. Recall that two convex polytopes are called *analogous* if they have the same normal fan. It is easy to see that polytopes analogous to P also form a semigroup. Then V_P is defined as its Grothendieck group.

The volume polynomial on the spaces V_P was previously used by Pukhlikov and Khovanskii to describe the cohomology rings of smooth toric varieties. I briefly recall their result. As follows from the theory of toric varieties, each lattice polytope P defines a polarized toric variety X_P. If P is integrally simple (that is, only n edges meet at every vertex, and the primitive lattice vectors on these edges form a basis in $\mathbb{Z}^n \subset \mathbb{R}^n$), then X_P is smooth. In this case, the Chow ring of X (or equivalently, the cohomology ring $H^*(X_P, \mathbb{Z})$, which lives only in even degrees) is isomorphic to the quotient R_P of the ring of differential operators on V_P with constant integer coefficients. To get R_P we quotient by the operators that annihilate the volume polynomial. This description turned out to be very useful. First, it is functorial. Second, it is immediately clear from the definition that the ring R_P lives only in degrees up to n (since the volume polynomial has degree n) and that R_P has a non-degenerate pairing (Poincaré duality) defined by $(D_1, D_2) := D_1 D_2(vol) \in \mathbb{Z}$ for any two homogeneous operators D_1 and D_2 of complementary degrees. In fact, the Poincaré duality on the ring R_P is the key ingredient in the proof of the isomorphism between R_P and $H^{2*}(X_P, \mathbb{Z})$ (see [2] for more details).

Note that if P is not simple, we can still define the ring R_P, which will still live in degrees up to n and satisfy Poincaré duality. However, its relation to the Chow ring of (now singular) toric variety X_P is unclear. On the other hand, the ring R_P for non-simple polytopes is sometimes related to the Chow rings of smooth non-toric varieties.

We now consider the ring R_P for the *Gelfand–Zetlin polytope* $P = P_\lambda$ (which is not simple) associated with a strictly dominant weight $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ of the group $GL_n(\mathbb{C})$. Recall that the Gelfand-Zetlin polytope P_λ is a convex lattice polytope in $\mathbb{R}^d$, where $d = n(n-1)/2$, with the property that the integer points inside and at the boundary of P_λ parameterize a natural basis in the irreducible representation of $GL_n(\mathbb{C})$ with the highest weight λ. It can be simply defined by inequalities (see e.g. [4]). Note that Gelfand–Zetlin polytopes P_λ and P_μ are analogous for any two strictly dominant weights λ and μ, and hence define the same space V_P and the same ring R_P (so it does not matter which weight to choose). It turns out that the ring R_P is isomorphic to the Chow ring (or to the cohomology ring) of the complete flag variety X for $GL_n(\mathbb{C})$ (note that dim $X = d$) so that the differential operators $\frac{\partial}{\partial\lambda_1}, \ldots, \frac{\partial}{\partial\lambda_n}$ get mapped to the first Chern classes of the tautological line bundles on X. This follows immediately from the results of Kaveh [2] and can also be deduced directly from the Borel presentation for the cohomology ring $H^*(X, \mathbb{Z})$ using that the volume of P_λ (regarded as a function of λ) is equal to $\prod_{i<j}(\lambda_i - \lambda_j)$ times a constant.

We now discuss the most important for us feature of the isomorphism $R_P \simeq CH^*(X)$: the isomorphism allows us to identify the algebraic cycles on X with the linear combinations of the faces of P. We first recall the easier case of simple polytopes [7, §2]. If P is simple then the dimension of the space V_P is equal to the number l of facets of P (since we can move independently by parallel transport each of the hyperplanes containing the facets of P). Note that for non-simple P the dimension of V_P is strictly less than l (e.g. if P is an octahedron, then V_P is just one-dimensional). For simple P, the space V_P has natural coordinates $(H_1, \ldots, H_l)$ called the *support numbers*. They are defined by fixing l covectors $h_1,\ldots, h_l$ on $\mathbb{R}^n$ such that the facet Γ_i of P (for each $i = 1,\ldots, l$) is contained in the hyperplane $h_i(x) = H_i(P)$ for some constant $H_i(P)$ and the polytope P satisfies the inequalities $h_i(x) \le H_i(P)$. Then any collection of real numbers $(H_1, \ldots, H_l)$ uniquely defines a (possibly virtual) polytope in V_P by the inequalities $h_i(x) \le H_i$. The ring R_P then has multiplicative generators $\partial_1 := \frac{\partial}{\partial H_1}, \ldots, \partial_l := \frac{\partial}{\partial H_l}$. We now assign to each product $\partial_{i_1} \ldots \partial_{i_k}$ (for distinct $i_1,\ldots, i_k$) the face $\Gamma_{i_1} \cap \ldots \cap \Gamma_{i_k}$ of P (if we identify R_P with the Chow ring of the smooth toric variety X_P then this becomes the well-known correspondence between the (cycles of) torus orbits in X_P and the faces of P). Note that such a face will have codimension k since P is simple.

It easy to check that all linear relations between $\partial_1,\ldots,\partial_l$ have form $h_1(v)\partial_1 + \ldots + h_l(v)\partial_l = 0$, where $v \in \mathbb{Z}^n \subset \mathbb{R}^n$ (because the volume of a polytope does not change if the polytope is parallely transported by the vector v). Using these linear relations we can always reduce any monomial in $\partial_1,\ldots,\partial_l$ to the linear combination of monomials containing only pairwise distinct ∂_i. Geometrically, this corresponds to computing the intersection product of the closures of torus orbits by using linear equivalence relation on the closures of codimension one orbits (there is a well-known algorithm for this). Polytope P and ring R_P allows one to make these computations more explicit by using geometric invariants of P (such as volume of P, integer distances to the facets etc.).

If P is not simple, then things become more complicated. I now state our results in this case. It is still possible (though less straightforward) to identify each element of R_P with a linear combinations of faces of P, but not every face of P would correspond to an element of R_P. Namely, we embed the ring R_P into a certain R_P-module M_P whose elements can be regarded as linear combinations of arbitrary faces of P modulo some relations. The module M_P depends on the choice of a simple resolution $\widetilde{P}$ of P (that is, $\widetilde{P}$ is obtained from P by generic parallel transports of the hyperplanes containing the facets of P), and is also defined using the volume polynomial. The product of an element in M_P by an element of R_P can again be computed by intersecting faces (and applying linear relations if necessary to make the faces transverse). While all of these applies to any convex polytope P it is especially interesting to study the case where $P = P_\lambda$ is a Gelfand–Zetlin polytope due to the isomorphism $R_P \simeq CH^*(X)$ for the flag variety X. Recall that $CH^*(X)$ (as a group) is a free abelian group with the basis of Schubert cycles. We now give the answer to the following natural question: how to express Schubert cycles as linear combinations of faces of the Gelfand–Zetlin polytope?

The relation between Schubert cycles and faces of the Gelfand–Zetlin polytope was first investigated in [5], and then by different methods also in [6] and [4]. We noticed that the ring R_P and its realization by faces via the module M_P provide the uniform setting for all previously known results on the cycle-face correspondence as well as for some new results. In particular, we proved the following formula, which is formally similar to the Fomin–Kirillov theorem on Schubert polynomials and uses the correspondence between *rc-graphs* (or *reduced pipe-dreams*) and certain faces of the Gelfand–Zetlin polytope described in [5]. Denote by X_w the Schubert cycle corresponding to the permutation w as in [6, §4]. Then the following identity holds in M_P:

$$X_w = \sum_{w(\Gamma)=w} \Gamma, \qquad (1)$$

where the sum is taken over all *rc-faces* (see [6, §4] for the definition) of P_λ with permutation w. Note that (1) can not be deduced from the Fomin–Kirillov theorem because the faces Γ will not usually belong to R_P (only to M_P) and hence can not be identified with the monomials in the corresponding Schubert polynomial. Our proof of (1) uses simple convex geometry arguments.

Once we have identity (1) it is easy to get many other presentations of Schubert cycles via faces by applying to (1) the relations in R_P. We have described all linear relations between facets, which turned out to be quite simple and used them to represent each Schubert cycle as a sum of faces that are transverse to all rc-faces. Hence, the intersection of any two Schubert cycles can also be written as the sum of faces (that is, with nonnegative coefficients). We hope that further investigation will lead to a transparent Littlewood–Richardson rule (different from the one in [1]) for the varieties of complete flags. A simple example (for $n = 3$) illustrating our approach to Schubert calculus via Gelfand–Zetlin polytopes can be found in [4, §4].

References

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