[13] S. V. Fomin, On measures invariant under a certain group of transformations, Izv. Akad. Nauk SSSR Ser. Mat. 14 (1950), 261–274. MR 12 # 33.
[14] G. Goldin, Non-relativistic current algebras as unitary representations of groups, J. Mathematical Phys. 12 (1971), 462–488. MR 44 # 1330.
[15] G. Goldin, K. J. Grodnik, R. Powers and D. Sharp, Non-relativistic current algebra in the N/V limit, J. Mathematical Phys. 15 (1974), 88–100.
[16] A. Guichardet, Symmetric Hilbert spaces and related topics, Lecture Notes in Math. 261, Springer-Verlag, Berlin-Heidelberg-New York 1972.
[17] J. Kerstan, K. Mattes and J. Mecke, Unbegrenzt teilbare Punktprozesse, Berlin 1974.
[18] D. Knutson, λ-rings and the representation theory of the symmetric group, Lecture Notes in Math. 308, Springer-Verlag, Berlin-Heidelberg-New York, 1973.
[19] R. Menikoff, The hamiltonian and generating functional for a non-relativistic local current algebra, J. Mathematical Phys. 15 (1974), 1138–1152. MR 49 # 10285.
[20] R. Menikoff, Generating functionals determining representations of a non-relativistic local current algebra in the N/V limit, J. Mathematical Phys. 15 (1974), 1394–1408.
[21] J. Moser, On the volume elements on a manifold, Trans. Amer. Math. Soc. 120 (1965), 286–294. MR 32 # 409
[22] R. S. Ismagilov, Unitary representations of the group of diffeomorphisms of the space $\mathbf{R}^n$, $n \geq 2$, Funktsional. Anal. i Prilozhen. 9:2 (1975), 71–72. = Functional Anal. Appl. 9 (1975), 144–145.

Received by the Editors, 15 May 1975

Translated by A. West

I. N. Bernstein, I. M. Gelfand, S. I. Gelfand

AN INTRODUCTION TO THE PAPER 'SCHUBERT CELLS AND COHOMOLOGY OF THE SPACES G/P'

Graeme Segal

It is well known that a generic invertible matrix can be factorized as the product of an upper triangular and a lower triangular matrix. A more precise statement is that any invertible $n \times n$ matrix g can be written in the form $b_1 w b_2$, where b_1 and b_2 are upper triangular and w is a permutation matrix. Here w is uniquely determined by g, though b_1 and b_2 are not. The matrices g for which w is the order-reversing permutation $i \mapsto n - i + 1$ form a dense open set in $\mathrm{GL}_n(\mathbf{C})$.

This double-coset decomposition $\mathrm{GL}_n(\mathbf{C}) = \bigcup_w BwB$, where B denotes the upper triangular matrices and w runs through the permutation matrices, has an analogue for any connected affine algebraic group G. The role of B is played by a Borel subgroup (i.e. a maximal soluble subgroup), and the role of the permutation matrices by the Weyl group $W = N(H)/H$, where $H \cong (\mathbf{C}^\times)^l$ is a maximal algebraic torus in G and $N(H)$ is its normalizer. The decomposition is nowadays called the *Bruhat decomposition*; but Gelfand had earlier recognized its importance in his work on the representations of the classical groups.

It is best to think of the decomposition as the decomposition of the homogeneous space $X = G/B$ into the orbits of the left action of B. The space X plays a central role in representation theory. It turns out that it is a complex projective algebraic variety, and that the orbits of B are algebraic affine spaces $\mathbf{C}^m$ of various dimensions, the "Bruhat cells". The closures of the cells are algebraic subvarieties which in general have singularities. It is important that the maximal compact subgroup K of G acts transitively on X, so that X has an alternative description as K/T, where $T = K \cap B$ is a maximal torus of K.

If $G = \mathrm{GL}_n(\mathbf{C})$ then X is the *flag manifold*: a flag in $\mathbf{C}^n$ is an increasing sequence of subspaces

$$F = (F_1 \subset F_2 \subset \ldots \subset F_n = \mathbf{C}^n)$$

with $\dim(F_k) = k$. For $\mathrm{GL}_n(\mathbf{C})$ acts transitively on the set of all flags, and B is the isotropy group of the standard flag $\mathbf{C} \subset \mathbf{C}^2 \subset \ldots \subset \mathbf{C}^n$. In this case the cells are indexed by permutations w of $\{1, 2, \ldots, n\}$, and we can take as a representative point in the cell X_w the flag F^w such that F^w_k is spanned by $\{e_{w(1)}, e_{w(2)}, \ldots, e_{w(k)}\}$, where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbf{C}^n$. X_w can be defined by "Schubert conditions": it consists precisely of the flags F such that $\dim(F_k \cap \mathbf{C}^m) = \nu_{km}$, where

$$\nu_{km} = \mathrm{card}\,\{\, i : i \leq k,\ w(i) \leq m \,\}.$$

The dimension of X_w is the *length* $l(w)$ of w, defined by

$$l(w) = \sum_{i=1}^{n} |\, w(i) - i \,|.$$

Alternatively $l(w)$ is the number of pairs (i, j) such that $i < j$ but $w(i) > w(j)$. In fact if N_w is the subgroup of B consisting of matrices (a_{ij}) with diagonal elements $a_{ii} = 1$ and such that $a_{ij} = 0$ unless $i < j$ and $w(i) > w(j)$ then the map $N_w \to X_w$ given by $g \mapsto g \cdot F^w$ is an isomorphism of algebraic spaces.

An analogous discussion can be carried out for the other classical groups. If $\mathbf{C}^n$ has a non-degenerate bilinear form $\langle\ ,\ \rangle$, and G is the group of automorphisms of $\mathbf{C}^n$ which preserve the form, then G/B can be identified with the set of flags F such that $F_k^\perp = F_{n-k}$ for $k = 1, \ldots, n$.

Returning to the general case, we have a topological cell decomposition of X into cells $\{X_w\}_{w \in W}$ which are all of even dimension. The homology groups of X are therefore free abelian with the classes $[X_w]$ of the cells as a natural basis.

On the other hand there is a completely different way of describing the cohomology ring of X. For every algebraic homomorphism $\lambda: B \to \mathbf{C}^\times$, or equivalently for every character λ of the compact torus T, there is a holomorphic line bundle E_λ on X. Associating to λ the first Chern class $c_\lambda = c_1(E_\lambda)$ of E_λ gives an isomorphism $\hat{T} \to H^2(X; \mathbf{Z})$. ($\hat{T}$ is the lattice of *weights* of G: in the paper it is called $\mathfrak{h}^*_{\mathbf{Z}}$.) The classes c_λ generate the cohomology ring of X multiplicatively over the rationals, and $H^*(X; \mathbf{Q}) \cong R/J$, where R is the polynomial algebra over $\mathbf{Q}$ generated by the c_λ, and J is the ideal generated by the homogeneous W-invariant polynomials of positive degree. (When $G = \mathrm{GL}_n(\mathbf{C})$ there are n obvious line bundles $E_1, \ldots, E_n$ on X: the fibre of E_i at a flag F is F_i/F_{i-1}. The classes $x_i = c(E_i)$ span $H^2(X; \mathbf{Z})$. The elementary symmetric functions in the x_i vanish because they are the Chern classes of $E_1 \oplus \ldots \oplus E_n$, which is a trivial bundle.)

It is natural to ask for the relation between these descriptions of the homology and the cohomology. In other words if p is a homogeneous polynomial of degree k in the c_λ, and w is an element of W length k, what is the value $\langle p, [X_w]\rangle$ of p on the cell X_w? One can also ask how to express the cohomology class Poincare dual to a cell X_w as a polynomial in the Chern classes.

To answer these questions it is enough in principle to determine the cap-product $c_\lambda \cap [X_w] \in H_{2k-2}(X; \mathbf{Z})$ for each $\lambda \in \hat{T}$ and each $w \in W$ of length k. The paper uses a simple and very attractive geometrical argument to do this. One begins by observing that by linearity it is enough to consider weights λ which are in the interior of the positive Weyl chamber. In that case X can be embedded as a projective algebraic variety in $P(V_\lambda)$, the projective space of the irreducible representation V_λ of G with highest weight λ, as the orbit under G of the highest weight vector f_λ. (V_λ is the dual of the space of holomorphic sections of E_λ.) The cohomology class $c_\lambda = c_1(E_\lambda)$ is then the class dual to the intersection $X \cap \Pi$, where Π is a hyperplane in $P(V_\lambda)$; and the cap-product with c_λ can be interpreted as the geometric operation of intersection with Π. This is amenable to calculation because of the following properties of the embedding $X \to P(V_\lambda)$:

(i) the centre of the cell X_w maps to the point of $P(V_\lambda)$ represented by the weight vector $f_w \in V_\lambda$ of weight $w\lambda$,

(ii) $\bar{X}_w$ is precisely the intersection of X with a linear subspace of $P(V_\lambda)$, and

(iii) the boundary $\bar{X}_w - X_w$ of X_w is $\bar{X}_w \cap \Pi_w$, where Π_w is the hyperplane perpendicular of f_w.

Now let us recall that the Weyl group W - which we are regarding as a group of automorphisms of the lattice $\hat{T}$ – is generated by the reflections σ_γ in the hyperplanes of $\hat{T}$ perpendicular to the roots γ of G. If $w \in W$ has length k it turns out that the $(k-1)$-dimensional cells in the boundary of X_w are precisely the $X_{w\sigma_\gamma}$ such that $l(w\sigma_\gamma) = k - 1$. Thus the cap-product $c_\lambda \in [X_w]$ is necessarily of the form $\sum_\gamma n_\gamma [X_{w\sigma_\gamma}]$, where n_γ is a positive integer. To determine n_γ one must calculate the order to which the linear form $\langle f_w, \rangle$, when regarded as a function on $\bar{X}_w$, vanishes on the cell $X_{w\sigma_\gamma}$. That is easy to do because the formula

$$t \mapsto w\sigma_\gamma \exp(t E_{-\gamma}) f_e,$$

where $E_{-\gamma}$ is the standard element of $\mathfrak{g}$ in the $(-\gamma)$ root-space, defines a holomorphic curve in $\bar{X}_w$ which passes through the centre $f_{w\sigma_\gamma} = w\sigma_\gamma f_e$ of $X_{w\sigma_\gamma}$ when $t = 0$, and is transversal to $X_{w\sigma_\gamma}$. We calculate

$$\langle f_w, w\sigma_\gamma \exp(t E_{-\gamma}) f_e \rangle = \langle \sigma_\gamma f_e, \exp(t E_{-\gamma}) f_e \rangle = 0(t^{n_\gamma}),$$

where $n_\gamma = \langle \lambda, H_\gamma \rangle$, H_γ being the co-root associated to γ, i.e. the element of the dual lattice to $\hat{T}$ characterized by the property

$$\sigma_\gamma(\chi) = \chi - \langle \chi, H_\gamma \rangle \gamma$$

for all $\chi \in \hat{T}$.

The formula

$$c_\lambda \cap [X_w] = \sum_\lambda \langle \lambda, H_\gamma \rangle [X_{w\sigma_\gamma}]$$

gives us the pairing between homology and cohomology in the form

$$\langle c_{\lambda_1} c_{\lambda_2} \ldots c_{\lambda_n}, [X_w] \rangle = \Sigma \langle \lambda_1, H_{\gamma_1} \rangle \ldots \langle \lambda_k, H_{\gamma_1} \rangle,$$

where the sum is over all strings $\gamma_1, \ldots, \gamma_k$ of positive roots such that

$$w = \sigma_{\gamma_1} \sigma_{\gamma_2} \ldots \sigma_{\gamma_k}.$$

I shall not describe here the elegant algebraic formulations the authors derive from this.

It ought, however, to be mentioned that the methods apply equally well not only to the space G/B, but to G/P for every parabolic subgroup P of G. The most obvious case of this is the Grassmannian $\mathrm{Gr}_{k,n}$ of k-dimensional subspaces of $\mathbf{C}^n$, which is $\mathrm{GL}_n(\mathbf{C})/P$, where P is the appropriate group of echelon matrices.

(In terms of compact groups $Gr_{k,n} = U_n/U_k \times U_{n-k}$.) The analogue of $Gr_{k,n}$ for the orthogonal groups is the Grassmannian of *isotropic* k-dimensional subspaces of $\mathbf{C}^n$ for some non-degenerate quadratic form on $\mathbf{C}^n$: this space can be identified with $O_n/U_k \times O_{n-2k}$. When $k = 1$ it is a complex projective quadric hypersurface.

SCHUBERT CELLS AND COHOMOLOGY OF THE SPACES G/P

I. N. Bernstein, I. M. Gel'fand, S. I. Gel'fand

We study the homological properties of the factor space G/P, where G is a complex semisimple Lie group and P a parabolic subgroup of G. To this end we compare two descriptions of the cohomology of such spaces. One of these makes use of the partition of G/P into cells (Schubert cells), while the other consists in identifying the cohomology of G/P with certain polynomials on the Lie algebra of the Cartan subgroup H of G. The results obtained are used to describe the algebraic action of the Weyl group W of G on the cohomology of G/P.

Contents

Introduction

Let G be a linear semisimple algebraic group over the field C of complex numbers and assume that G is connected and simply-connected. Let B be a Borel subgroup of G and $X = G/B$ the fundamental projective space of G.

The study of the topology of X occurs, explicitly or otherwise, in a large number of different situations. Among these are the representation theory of semisimple complex and real groups, integral geometry and a number of problems in algebraic topology and algebraic geometry, in which analogous spaces figure as important and useful examples. The study of the homological properties of G/P can be carried out by two well-known methods. The first of these methods is due to A. Borel [1] and involves the identification of the cohomology ring of X with the quotient ring of the ring of polynomials on the Lie algebra $\mathfrak{h}$ of the Cartan subgroup