ON INTERSECTION INDICES OF SUBVARIETIES IN REDUCTIVE GROUPS

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To my Teacher, Askold Khovanskii

In this paper, I give an explicit formula for the intersection indices of the Chern classes (defined in [11]) of an arbitrary reductive group with hypersurfaces. This formula has the following applications. First, it allows to compute explicitly the Euler characteristic of complete intersections in reductive groups thus extending the beautiful result by D.Bernstein and Khovanskii, which holds for a complex torus. Second, for any regular compactification of a reductive group, it computes the intersection indices of the Chern classes of the compactification with hypersurfaces. The formula is similar to the Brion–Kazarnovskii formula for the intersection indices of hypersurfaces in reductive groups. The proof uses an algorithm of De Concini and Procesi for computing such intersection indices. In particular, it is shown that this algorithm produces the Brion–Kazarnovskii formula.

1. Introduction

Let G be a connected complex reductive group of dimension n, and let $\pi : G \to GL(V)$ be a faithful representation of G. A generic *hyperplane section* H_π corresponding to π is the preimage $\pi^{-1}(H)$ of the intersection of $\pi(G)$ with a generic affine hyperplane $H \subset \mathrm{End}(V)$. There is a nice explicit formula for the self-intersection index H_π^n of H_π in G, and more generally, for the intersection index of n generic hyperplane sections corresponding to different representations (see Theorem 1.1 below) in terms of the weight polytopes of the representations [3, 9]. In this paper, I give a similar formula for the intersection indices of the *Chern classes* of G (defined in [11]) with generic hyperplane sections (see Theorem 1.2).

The *Chern classes* of G can be defined using the Chern classes of the *logarithmic* tangent bundle over a *regular* compactification of G (see Section 3 for a precise definition). They were introduced in [11] as main ingredients in a formula for the Euler characteristic of a generic hyperplane section and of complete intersections of several hyperplane sections. In the case where a reductive group is a complex torus $(\mathbb{C}^*)^n$, there are beautiful explicit formulas for the Euler characteristic due to D.Bernstein and A.Khovanskii [10]. The result of the present paper combined with [11] provides analogous formulas in the case of an arbitrary reductive group.

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Denote by k the rank of G, i.e. the dimension of a maximal torus in G. Only the first $(n-k)$ Chern classes are not trivial [11]. These Chern classes are elements of the *ring of conditions* of G, which was introduced by C.De Concini and C.Procesi [7] (see also Subsection 2.4 for a brief reminder). They can be represented by subvarieties $S_1, \ldots, S_{n-k} \subset G$, where S_i has codimension i. All enumerative problems for G, such as the computation of the intersection index $S_i H_\pi^{n-i}$, make sense in the ring of conditions.

First, I recall the usual Brion–Kazarnovskii formula for the intersection indices of hyperplane sections. Choose a maximal torus $T \subset G$, and denote by L_T its character lattice. Choose also a Weyl chamber $\mathcal{D} \subset L_T \otimes \mathbb{R}$. Denote by R^+ the set of all positive roots of G and denote by ρ the half of the sum of all positive roots of G. The inner product $(\cdot,\cdot)$ on $L_T \otimes \mathbb{R}$ is given by a nondegenerate symmetric bilinear form on the Lie algebra of G that is invariant under the adjoint action of G (such a form exists since G is reductive).

Theorem 1.1. [3, 9] *If H_π is a hyperplane section corresponding to a representation π with the weight polytope $P_\pi \subset L_T \otimes \mathbb{R}$, then the self-intersection index H_π^n of H_π is equal to*

$$n! \int\limits_{P_\pi \cap \mathcal{D}} \prod_{\alpha \in R^+} \frac{(x,\alpha)^2}{(\rho,\alpha)^2} dx.$$

The measure dx on $L_T \otimes \mathbb{R}$ is normalized so that the covolume of L_T is 1.

This theorem was first proved by B.Kazarnovskii [9]. Later, M.Brion proved an analogous formula for arbitrary spherical varieties using a different method [3].

The integrand in this formula has the following interpretation. The direct sum $L_T \oplus L_T$ can be identified with the Picard group of the product $G/B \times G/B$ of two flag varieties. Here B is a Borel subgroup of G. Hence, to each lattice point $(\lambda_1, \lambda_2) \in L_T \oplus L_T$ one can assign the self-intersection index of the corresponding divisor in $G/B \times G/B$. The resulting function extends to the polynomial function $(n-k)!F$ on $(L_T \oplus L_T) \otimes \mathbb{R}$, where

$$F(x,y) = \prod_{\alpha \in R^+} \frac{(x,\alpha)(y,\alpha)}{(\rho,\alpha)^2}.$$

Note that the integrand is the restriction of F onto the diagonal $\{(x,x) : x \in L_T \otimes \mathbb{R}\}$.

This interpretation leads to another proof of the Brion–Kazarnovskii formula (different from those of Kazarnovskii and Brion). Namely, take any regular compactification X of G that lies over the compactification X_π corresponding to the representation π (see Subsection 2.2). Then reduce the computation of H_π^n to the computation of the intersection indices of divisors in the closed orbits of X (see Section 4). All closed orbits are isomorphic to the product of two flag varieties. The precise algorithm for doing this was given by De Concini and Procesi [6] in the case, where X is a wonderful compactification of a symmetric space. Then E.Bifet extended this algorithm to all regular compactifications of symmetric spaces [2]. I will show that in the case, where a symmetric space is a reductive group, this algorithm actually produces the Brion–Kazarnovskii formula if one uses the weight polytope of π to keep track of all transformations.

Moreover, the De Concini–Procesi algorithm works not only for divisors. It can also be carried over to the Chern classes of G (which are, in general, not linear combinations of complete intersections). In particular, there is the following explicit formula for the intersection indices of the Chern classes of G with hyperplane sections. Assign to each lattice point $(\lambda_1, \lambda_2) \in L_T \oplus L_T$ the intersection index of the i-th Chern class of the tangent bundle over $G/B \times G/B$ with the divisor $D(\lambda_1, \lambda_2)$ corresponding to (λ_1, λ_2), that is, the number $c_i(G/B \times G/B)D^{n-k-i}(\lambda_1, \lambda_2)$. Extend this function to the polynomial function on $(L_T \oplus L_T) \otimes \mathbb{R}$. Since the Chern classes of G/B are known the resulting function can be easily computed (see Section 4). The final formula is as follows.

Let $\mathbb{D}$ be the differential operator (on functions on $(L_T \oplus L_T) \otimes \mathbb{R}$) given by the formula

$$\mathbb{D} = \prod_{\alpha \in R^+} (1 + \partial_\alpha)(1 + \widetilde{\partial}_\alpha),$$

where ∂_α and $\widetilde{\partial}_\alpha$ are directional derivatives along the vectors $(\alpha, 0)$ and $(0, \alpha)$, respectively. Denote by $[\mathbb{D}]_i$ the i-th degree term in $\mathbb{D}$.

Theorem 1.2. *If H_π is a generic hyperplane section corresponding to a representation π with the weight polytope $P_\pi \subset L_T \otimes \mathbb{R}$, then the intersection index $S_i H_\pi^{n-i}$ of the i-th Chern class of G with H_π^{n-i} is equal to*

$$(n-i)! \int_{P_\pi \cap \mathcal{D}} [\mathbb{D}]_i F(x, x) dx.$$

The measure dx on $L_T \otimes \mathbb{R}$ is normalized so that the covolume of L_T is 1.

Of course, this formula also allows to compute the intersection index $S_i H_{\pi_1} \dots H_{\pi_{n-i}}$ for any $n - i$ generic hyperplane sections corresponding to different representations $\pi_1, \dots, \pi_{n-i}$.

Since, in general, the Chern classes of G are not complete intersections, this formula extends computation of the intersection indices to a bigger part of the ring of conditions of G. Theorem 1.2 also completes some results of [11]. Namely, the Chern classes $S_1, \dots, S_{n-k}$ were used there in the following adjunction formula for the topological Euler characteristic of complete intersections of hyperplane sections in G.

Theorem 1.3. [11] *Let $H_1, \dots, H_m$ be generic hyperplane sections corresponding to m (possibly different) representations of G. The Euler characteristic of the complete intersection $H_1 \cap \dots \cap H_m$ is equal to the term of degree n in the expansion of the* *following product:*

$$(1 + S_1 + \ldots + S_{n-k}) \cdot \prod_{i=1}^{m} H_i(1 + H_i)^{-1}.$$

The product in this formula is the intersection product in the ring of conditions.

Theorem 1.2 in the present paper allows to make this formula explicit, since it allows to compute all terms of the form $S_i H_1^{k_1} \ldots H_m^{k_m}$, where $k_1 + \ldots + k_m = n - i$. E.g. if a complete intersection is just one hyperplane section H_π, then

$$\chi(H_\pi) = (-1)^{n-1} \int\limits_{P_\pi \cap \mathcal{D}} (n! - (n-1)![\mathbb{D}]_1 + (n-2)![\mathbb{D}]_2 - \ldots + k![\mathbb{D}]_{n-k})\, F(x,x)dx.$$

There is also a formula for the Chern classes $c_i(X)$ of the tangent bundle over any regular compactification X of G in terms of $S_1, \ldots, S_{n-k}$ (see Corollary 4.4 in [11]). Theorem 1.2 allows to compute explicitly the intersection index of $c_i(X)$ with a complete intersection of complementary dimension in X.

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2. Preliminaries

In this section, I recall some well-known facts which are used in the proof of Theorem 1.2. In Subsection 2.2, I define the regular compactification X of G associated with a representation π and describe the orbit structure of X in terms of the weight polytope of the representation. In Subsection 2.3, the Picard group of X is related to the space of virtual polytopes analogous to the weight polytope of π. The notion of analogous polytopes is discussed in Subsection 2.1. In Subsection 2.4, I recall the definition of the ring of conditions of G. Subsection 2.5 contains a formula for the integral of a polynomial function over a simplex, which is used to interpret the computation of intersection indices in terms of integrals over the weight polytope.

2.1. **Polytopes.** Let $P \subset \mathbb{R}^k$ be a convex polytope. Define the *normal fan* P^* of P. This is a fan in the dual space $(\mathbb{R}^k)^*$. To each face $F^i \subset P$ of dimension i there corresponds a cone F_i^* of dimension $(n-i)$ in P^* defined as follows. The cone F_i^* consists of all linear functionals in $(\mathbb{R}^k)^*$ whose maximum value on P is attained on the interior of the face F^i. In particular, to each facet of P there corresponds a one-dimensional cone, i.e. a ray, in P^*. If the dual space $(\mathbb{R}^k)^*$ is identified with $\mathbb{R}^k$ by means of the Euclidean inner product, the ray corresponding to a facet is spanned by a normal vector to the facet.

Two convex polytopes are called *analogous* if they have the same normal fan. All polytopes analogous to a given polytope P form a semigroup S_P with respect to Minkowski sum. This semigroup is also endowed with the action of the multiplicative group $\mathbb{R}^{>0}$ (polytopes can be dilated). Hence, S_P can be regarded as a cone in the vector space V_P, where V_P is the minimal group containing S_P (i.e. the Grothendieck group of S_P). The elements of V_P are called *virtual* polytopes analogous to P.

We now introduce special coordinates in the vector space V_P in the case where P is *simple*. A polytope in $\mathbb{R}^k$ is called *simple* if it is generic with respect to parallel translations of its facets. Namely, exactly k facets must meet at each vertex. This implies that any other face is also the transverse intersection of those facets that contain it. Let $\Gamma_1, \ldots, \Gamma_l$ be the facets of P, and let $\Gamma_1^*, \ldots, \Gamma_l^*$ be the corresponding rays in P^*. Choose a non-zero functional $h_i \in \Gamma_i^*$ in each ray. Call h_i a *support function* corresponding to the facet Γ_i. For any polytope Q analogous to P, denote by $h_i(Q)$ the maximal value of h_i on the polytope Q. For instance, if h_i is normalized so that its value on the external unit normal to the facet Γ_i is 1, then $h_i(P)$ is up to a sign the distance from the origin to the hyperplane that contains the facet Γ_i (the sign is positive if the origin and the polytope P are to the same side of this hyperplane, and negative otherwise). The numbers $h_1(Q), \ldots, h_l(Q)$ are called the *support numbers* of Q. Clearly, the polytope Q is uniquely defined by its support numbers. The coordinates $h_1(Q), \ldots, h_l(Q)$ can be extended to the space V_P, providing the isomorphism between V_P and the coordinate space $\mathbb{R}^l$.

In what follows, we will deal with *integer* polytopes, i.e. polytopes whose vertices belong to a given lattice $\mathbb{Z}^k \subset \mathbb{R}^k$. For such polytopes, the natural way to normalize the support functions is to require that $h_i(P)$ be equal to the *integral* distance from the origin to the hyperplane that contains the facet Γ_i. Suppose that a hyperplane H not passing through the origin is spanned by lattice vectors. Then the *integral* distance from the origin to the hyperplane H is the index in $\mathbb{Z}^k$ of the subgroup spanned by $H \cap \mathbb{Z}^k$. To compute the integral distance one can apply a unimodular (with respect to the lattice $\mathbb{Z}^k$) linear transformation of $\mathbb{R}^k$ so that H becomes parallel to a coordinate hyperplane. Then the integral distance is the usual Euclidean distance from the origin to this coordinate hyperplane.

2.2. **Regular compactifications of reductive groups.** With any representation $\pi : G \to GL(V)$ one can associate the following compactification of $\pi(G)$. Take the projectivization $\mathbb{P}(\pi(G))$ of $\pi(G)$ (i.e. the set of all lines in $\mathrm{End}(V)$ passing through a point of $\pi(G)$ and the origin), and then take its closure in $\mathbb{P}(\mathrm{End}(V))$. We obtain a projective variety $X_\pi \subset \mathbb{P}(\mathrm{End}(V))$ with a natural action of $G \times G$ coming from the left and right action of $\pi(G) \times \pi(G)$ on $\mathrm{End}(V)$. E.g. when $G = (\mathbb{C}^*)^n$ is a complex torus, all projective toric varieties can be constructed in this way.

Assume that $\mathbb{P}(\pi(G))$ is isomorphic to G. Consider all weights of the representation π, i.e. all characters of the maximal torus T occurring in π. Take their convex hull P_π in $L_T \otimes \mathbb{R}$. Then it is easy to see that P_π is a polytope invariant under the action of the Weyl group of G. It is called the *weight polytope* of the representation π. The polytope P_π contains information about the compactification X_π.

Theorem 2.1. *1)* ([14], Proposition 8) *The subvariety X_π consists of a finite number of $G \times G$-orbits. These orbits are in one-to-one correspondence with the orbits of the Weyl group acting on the faces of the polytope P_π. This correspondence preserves* *incidence relations. I.e. if F_1, F_2 are faces such that $F_1 \subset F_2$, then the orbit corresponding to F_1 is contained in the closure of the orbit corresponding to F_2.*

2) Let σ be another representation of G. The normalizations of subvarieties X_π and X_σ are isomorphic if and only if the normal fans corresponding to the polytopes X_π and X_σ coincide. If the first fan is a subdivision of the second, then there exists a $G \times G$–equivariant map from the normalization of X_π to X_σ, and vice versa.

The second part of Theorem 2.1 follows from the general theory of spherical varieties (see [13], Theorem 5.1) combined with the description of compactifications X_π via colored fans (see [14], Sections 7, 8).

In what follows, we will only consider *regular* compactifications of G. The simplest example of a regular compactification is the *wonderful compactification* constructed by De Concini and Procesi. Suppose that the group G is of adjoint type, i.e. the center of G is trivial. Take any irreducible representation π with a strictly dominant highest weight. It is proved in [6] that the corresponding compactification X_π of the group G is always smooth and, hence, does not depend on the choice of a highest weight. Indeed, the normal fan of the weight polytope P_π coincides with the fan of the Weyl chambers and their faces, so the second part of Theorem 2.1 applies. This compactification is called the *wonderful compactification* and is denoted by X_{can}.

Other *regular* compactifications of G can be characterized as follows. The normalization X of X_π is *regular* if first, it is smooth, and second, there is a $(G \times G)$–equivariant map from X to X_{can}. These two conditions can be reformulated in terms of the weight polytope P_π. Namely, the first condition implies that P_π is *integrally* simple (see [14] Theorem 9), i.e. it is simple and the primitive vectors on the edges meeting at each vertex form a basis of L_T. The second condition implies that none of the vertices of P_π lies on the walls of the Weyl chambers, i.e. the normal fan of P_π subdivides the fan of the Weyl chambers and their faces.

A regular compactification X has the following nice properties (see [4] for details), which we will use in the sequel. The boundary divisor $X \setminus G$ is a divisor with normal crossings. The $G \times G$–orbits of codimension s correspond to the faces of P_π of codimension s and have rank $(k - s)$. Recall that each face $F \subset P_\pi$ is the transverse intersection of several facets of P_π (since P_π is simple). Then the closure of the orbit corresponding to F is the transverse intersection of the closures of the codimension one orbits that correspond to these facets. Each closed orbit of X (such orbits correspond to the vertices of P_π) is isomorphic to the product of two flag varieties $G/B \times G/B$.

2.3. **Picard group of compactifications.** Let X be the normalization of the compactification X_π of G. We assume that X is regular, and hence smooth. Then the second cohomology group $H^2(X)$ is isomorphic to the Picard group of X (see [2]). There is a description of the Picard group of a regular complete symmetric space due to Bifet (see [2], Theorem 2.4, see also [3], Proposition 3.2). In our case, this description can be reformulated as follows (such a reformulation is well-known in the toric case, and in the reductive case it was suggested by K.Kiumars). Denote by $V(\pi)$ the group of all integer virtual polytopes analogous to the weight polytope P_π and invariant under the action of the Weyl group.

Proposition 2.2. *The Picard group* $\mathrm{Pic}(X)$ *of* X *is canonically isomorphic to the quotient group of* $V(\pi)$ *modulo parallel translations. The isomorphism takes the hyperplane section corresponding to a representation* σ *to the weight polytope of* σ *and extends to the other divisors by linearity.*

In particular, if G is semisimple, then $\mathrm{Pic}(X) = V(\pi)$ (the only parallel translation taking a W–invariant polytope to a W–invariant polytope is the trivial one). Let us identify divisors in X with the corresponding polytopes using this isomorphism.

The variety X has l distinguished *boundary divisors* $\overline{\mathcal{O}}_1, \ldots, \overline{\mathcal{O}}_l$, which are the closures of codimension one orbits. Let us describe the corresponding virtual polytopes. Choose l facets $\Gamma_1, \ldots, \Gamma_l$ of P_π so that each orbit of the Weyl group acting on the facets of P_π contains exactly one Γ_i. E.g. take all facets that intersect the fundamental Weyl chamber. Choose the support functions $h_1, \ldots, h_l$ corresponding to these facets so that $h_i(P_\pi)$ is equal to the integral distance (with respect to the weight lattice L_T) from the origin to the facet Γ_i.

Lemma 2.3. *The closure* $\overline{\mathcal{O}}_i$ *of codimension one orbit corresponds to the virtual polytope whose i-th support number is* 1 *and the other support numbers are* 0.

Proof. Let σ be any representation of G whose weight polytope P is analogous to P_π. Then X is isomorphic to the normalization of the compactification X_σ. Thus a generic linear functional f on X_σ can also be regarded as a rational function on X. Let us find the zero and the pole divisors of f. The zero divisor D is the divisor corresponding to the weight polytope of σ. The pole divisor is a linear combination of the divisors $\overline{\mathcal{O}}_1, \ldots, \overline{\mathcal{O}}_l$. It is not hard to show that the coefficients are the support numbers $h_1(P_\sigma), \ldots, h_l(P_\sigma)$, i.e. the integral distances from the origin to the facets of P_σ corresponding to $\Gamma_1, \ldots, \Gamma_l$. Indeed, for toric varieties, this statement is well-known (see [8], Section 3.4). In particular, this holds for the closure $\overline{T}$ in X of the maximal torus $T \subset G$. Note that the toric variety $\overline{T}$ corresponds to the same polytope P_σ and the codimension one orbits of $\overline{T}$ are the irreducible components of $\overline{\mathcal{O}_i \cap T}$. Hence, in order to have the right coefficients in the decomposition of $\overline{D \cap T}$ along the hypersurfaces $\overline{\mathcal{O}}_1 \cap \overline{T}, \ldots, \overline{\mathcal{O}}_l \cap \overline{T}$ in $\overline{T}$, we must have

$$D = h_1(P_\sigma)\overline{\mathcal{O}}_1 + \ldots + h_l(P_\sigma)\overline{\mathcal{O}}_l.$$

It follows that $h_i(\overline{\mathcal{O}}_j) = 0$, unless $i = j$. □

Another useful collection of divisors consists of the closures in X of codimension one Bruhat cells in G. Denote these divisors by $D_1, \ldots, D_k$. They can also be described as follows. Denote by $\omega_1, \ldots, \omega_k$ the fundamental highest weights of G. Let X_i be the compactification corresponding to the irreducible representation π_i of G with the highest weights ω_i. Then by Theorem 2.1 there is an equivariant map from the wonderful compactification X_{can} to X_i, and hence, there is also an equivariant map $p : X \to X_i$. The divisor D_i is the preimage under the map p of the hyperplane section (regarded as a divisor in X_i) corresponding to the representation π_i.

To each dominant weight $\lambda = m_1\omega_1 + \ldots + m_k\omega_k$ there corresponds the *weight divisor* $D(\lambda) = m_1D_1 + \ldots + m_kD_k$. The polytope of this divisor is the weight polytope P_λ of the irreducible representation with the highest weight λ. Note that λ is the only vertex of P_λ inside the fundamental Weyl chamber. Hence, it belongs to all facets of P_λ corresponding to $\Gamma_1,\ldots,\Gamma_l$ (e.g. some of the facets might degenerate to the vertex λ). This implies the following lemma.

Lemma 2.4. *Let $D(\lambda)$ be the weight divisor corresponding to a weight $\lambda \in L_T$ and let P_λ be its polytope. Then $h_i(P_\lambda) = h_i(\lambda)$ for any $i = 1,\ldots,l$.*

Combination of these two lemmas leads to the following result.

Corollary 2.5. *Let D be the divisor on X corresponding to a polytope P. We assume that P is analogous to P_π and identify the respective facets. Then for any face $F \subset P$ of codimension s that intersects the fundamental Weyl chamber $\mathcal{D}$ and for any point $\lambda \in F \cap \mathcal{D}$, the divisor D can be written uniquely as a linear combination of $D(\lambda)$ and of boundary divisors $\overline{\mathcal{O}}_i$ such that the corresponding facets Γ_i do not contain F. Namely, if $F = \Gamma_{i_1} \cap \ldots \cap \Gamma_{i_s}$, then*

$$D = D(\lambda) + \sum_{j\in\{1,\ldots,l\}\setminus\{i_1,\ldots,i_s\}} [h_j(P) - h_j(\lambda)]\overline{\mathcal{O}}_j.$$

2.4. **Ring of conditions.** Let Z_1 and Z_2 be two algebraic subvarieties in G. One can define their intersection index Z_1Z_2 as the number of points in the intersection $gZ_1 \cap Z_2$ for a generic $g \in G$. Kleiman's transversality theorem ensures that such an intersection index is well-defined, i.e. for a generic $g \in G$ the intersection $gZ_1 \cap Z_2$ is transverse and its cardinality does not depend on the choice of g [12]. We now consider the group $C^*(G)$ of all formal linear combinations of algebraic subvarieties in G up to the following equivalence relation. Two subvarieties Z_1, Z_2 of the same dimension are equivalent if and only if for any subvariety Y of complementary dimension the intersection indices Z_1Y and Z_2Y coincide.

For any two equivalence classes in $C^*(G)$ represented by subvarieties Z_1 and Z_2 define their product as the class of the subvariety $gZ_1 \cap Z_2$ for a generic $g \in G$. De Concini and Procesi showed that this product is well defined in $C^*(G)$, i.e. it does not depend on the choice of representatives [7]. The ring $C^*(G)$ is called the *ring of conditions* of G.

In what follows, we will also use the relation between the ring of conditions and the (co)homology ring of a regular compactification. Namely, the subvarieties Z_1 and Z_2 represent the same class in the ring of conditions $C^*(G)$ if there exists a regular compactification X of the group G such that the closures of Z_1 and Z_2 in X represent the same homology class in X and have proper intersections with all $G \times G$–orbits [6]. In particular, we will use regular compactifications to compute the intersection indices of subvarieties in G.

2.5. **Integration of polynomials.** Let $f(x_1,\ldots,x_k)$ be a homogeneous polynomial function of degree d defined on a real affine space $\mathbb{R}^k$ with coordinates $(x_1,\ldots,x_k)$. There is a useful formula expressing the integral of f over a simplex in $\mathbb{R}^k$ in terms of the *polarization* of f. Recall that the *polarization* of f is the unique symmetric d-linear form f_{pol} on $\mathbb{R}^k$ such that the restriction of f_{pol} to the diagonal coincides with f. One can define f_{pol} explicitly as follows:

$$f_{pol}(v_1,\ldots,v_d)=\frac{1}{d!}\frac{\partial^d}{\partial_{v_1}\ldots\partial_{v_d}}f,$$

where ∂_{v_i} is the directional derivative along the vector v_i.

Let $\Delta\subset\mathbb{R}^k$ be a k-dimensional simplex with vertices $a_0,\ldots,a_k$ and let $dx=dx_1\wedge dx_2\wedge\ldots\wedge dx_k$ be the standard measure on $\mathbb{R}^k$.

Proposition 2.6. [3] *Let f_{pol} be the polarization of f. It can be regarded as a linear function on the d-th symmetric power of V. Then the average value of f on the simplex Δ coincides with the average value of f_{pol} on all symmetric products of d vectors from the set $\{a_0,\ \ldots,\ a_k\}$:*

$$\frac{1}{\mathrm{Vol}(\Delta)}\int\limits_{\Delta}f(x)dx=\frac{1}{\binom{d+k}{k}}\sum_{i_0+\ldots+i_k=d}f_{pol}(\underbrace{a_0,\ldots,a_0}_{i_0},\ldots,\underbrace{a_k,\ldots,a_k}_{i_k}).$$

3. Chern classes

In this section, I recall the definition of the Chern classes of spherical homogeneous spaces (see [11] for more details). In the sequel, only Chern classes of $G\times G$–orbits in regular compactifications of G will be used. For these Chern classes, I prove a vanishing result for their intersection indices with certain weight divisors in regular compactifications. This result will be important in Section 4 when applying the De Concini–Procesi algorithm to the Chern classes of G.

Let G/H be a spherical homogeneous space under G of dimension d. The i-th Chern class $S_i(G/H)$ of G/H is the i-th degeneracy locus of n generic vector fields $v_1,\ldots,\ v_n$ coming from the action of G, that is, $S_i(G/H)=\{x\in G/H:v_1(x),\ldots,v_{d-i+1}(x)$ are linearly dependent$\}$. In what follows we will use the following reformulation of this definition. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H, respectively, and denote by m the dimension of $\mathfrak{h}$. Define the *Demazure map* φ from G/H to the Grassmannian $\mathrm{G}(m,\mathfrak{g})$ of m-dimensional subspaces in $\mathfrak{g}$ as follows:

$$\varphi:G/H\to\mathrm{G}(m,\mathfrak{g});\quad \varphi:gH\to g\mathfrak{h}g^{-1}.$$

Let $C_i\subset\mathrm{G}(m,\mathfrak{g})$ be the Schubert cycle corresponding to a generic subspace $\Lambda_i\subset\mathfrak{g}$ of codimension $m+i-1$, i.e. $C_i=\{\Lambda\in\mathrm{G}(m,\mathfrak{g}):\dim(\Lambda\cap\Lambda_i)\geq 1\}$. Then it is easy to see that the i-th *Chern class* $S_i(G/H)$ of G/H is the preimage of C_i under the map φ:

$$S_i(G/H)=\varphi^{-1}(C_i).$$

The class of $S_i(G/H)$ in the ring of conditions of G/H is the same for all generic C_i [11]. It is related to the Chern classes of the tangent bundles over regular compactifications of G [5, 11]. Namely, if X is a regular compactification of G/H, then the closure of $S_i(G/H)$ in X is the i-th Chern class of the *logarithmic* tangent bundle over X that corresponds to the divisor $X \setminus (G/H)$. This vector bundle is generated by all vector fields on X that are tangent to G–orbits in X. In what follows, this bundle will be called the *Demazure bundle* of X.

Let $X = G/H$ and $Y = G/P$ be two spherical homogeneous spaces under G. Suppose that H is a subgroup of P. Consider the G–equivariant map

$$f : X \to Y; \quad f : gH \mapsto gP.$$

In general, it is not true that $S_i(X)$ is the inverse image under the map f of a subset in Y. However, under the assumption that H contains a regular element of G, the intersection of $S_i(X)$ (when it is nonempty) with a fiber of f has dimension at least $\mathrm{rk}(P) - \mathrm{rk}(H)$.

Example. In what follows, we will mostly deal with the case, where X and Y are spherical homogeneous spaces under the doubled group $G \times G$. Namely, X is a $G \times G$–orbit $\mathcal{O}$ of a regular compactification of the group G and Y is a partial flag variety constructed as follows. Let $H \subset G\times G$ be the stabilizer of a point in $\mathcal{O}$. Take the minimal parabolic subgroup $P \subset G\times G$ that contains H and set $Y = (G\times G)/P$. It easily follows from an explicit description of the stabilizer H (see [14], Theorem 8) that H does contain a regular element of $G \times G$.

Lemma 3.1. *For a generic $S_i(X)$, there exists an open dense subset of $S_i(X)$ such that for any element x of this subset the intersection of the fiber xP with $S_i(X)$ has dimension greater than or equal to the* $\mathrm{rk}(P) - \mathrm{rk}(H)$. *In particular, the dimension of $f(S_i(X))$ satisfies the inequality*

$$\dim\, f(S_i(X)) \leq \dim\, S_i(X) - (\mathrm{rk}(P) - \mathrm{rk}(H)).$$

Proof. Choose a generic vector space $\Lambda \subset \mathfrak{g}$ of codimension $\dim\, H + i - 1$. Denote by $\mathfrak{h}$ and $\mathfrak{p}$ the Lie algebras of H and P respectively. Then by definition $S_i(X)$ consists of all cosets gH such that $g\mathfrak{h}g^{-1}$ has a nontrivial intersection with Λ, or equivalently $\mathfrak{h} \cap g^{-1}\Lambda g$ is nontrivial.

Let gH be any element of $S_i(X)$. Estimate the dimension of the intersection of $S_i(X)$ with the fiber gP of the map f. Because of the assumption on H stated above, for all g from a dense open subset of $S_i(X)$, the intersection $\mathfrak{h} \cap g^{-1}\Lambda g$ contains an element v that is regular in $\mathfrak{g}$. Denote by C the centralizer in P of $v \in \mathfrak{h} \subset \mathfrak{p}$. Then $\dim(C \cap H) = \mathrm{rk}(H)$ while C has dimension at least $\mathrm{rk}(P)$. Note that for any $c \in C$ the coset gcH still belongs to $S_i(X)$ since $c^{-1}g^{-1}\Lambda gc$ contains $c^{-1}vc = v$. Hence, $S_i(X) \cap gP$ contains a set gCH of dimension at least $\mathrm{rk}(P) - \mathrm{rk}(H)$. □

Lemma 3.1 is crucial for proving the following two vanishing results, which extend Proposition 9.1 from [6] and rely on the same ideas. Let X be a regular compactification of G, and let $p : X \to X_{can}$ be its equivariant projection to the wonderful compactification. Denote by $c_1, \ldots, c_{n-k}$ the Chern classes of the Demazure vector bundle over X.

Lemma 3.2. *Let $\mathcal{O}$ be a $G \times G$–orbit in X of codimension $s < k$, and $\overline{\mathcal{O}} \subset X$ its closure. Suppose that the image $p(\mathcal{O})$ under the map $p : X \to X_{can}$ coincides with the closed orbit of X_{can}. In terms of polytopes, this means that the face corresponding to $\mathcal{O}$ does not intersect the walls of the Weyl chambers.*

Let λ be any weight of G, and $D(\lambda)$ the corresponding weight divisor. Then the homology class $c_iD^{n-i-s}(\lambda)$ vanishes on $\overline{\mathcal{O}}$, i.e. the following intersection index is zero:

$$c_iD(\lambda)^{n-i-s}\overline{\mathcal{O}} = 0.$$

Proof. First of all, the intersection product $c_i \cdot \overline{\mathcal{O}}$ is the i-th Chern class of the Demazure bundle over $\overline{\mathcal{O}}$ (see [1], Proposition 2.4.2). Hence, it can be realized as the closure in $\overline{\mathcal{O}}$ of the i-th Chern class $S_i(\mathcal{O})$ of the spherical homogeneous space $\mathcal{O}$. The computation of the intersection index $c_i\overline{\mathcal{O}}D(\lambda)^{n-i-s}$ in X thus reduces to the computation of the intersection index $\overline{S_i(\mathcal{O})}D(\lambda)^{n-i-s}$ in $\overline{\mathcal{O}}$. The latter is equal to the intersection index $S_i(\mathcal{O})D(\lambda)^{n-i-s}$ in the ring of conditions of $\mathcal{O}$ since $D(\lambda)$ and $\overline{S_i(\mathcal{O})}$ have proper intersections with the boundary $\overline{\mathcal{O}} \setminus \mathcal{O}$.

To compute $S_i(\mathcal{O})D(\lambda)^{n-i-s}$ we use the restriction of the map $p : X \to X_{can}$ to $\overline{\mathcal{O}}$. By the hypothesis the image $p(\overline{\mathcal{O}})$ is the closed orbit F in X_{can}, so it is isomorphic to the product $G/B \times G/B$ of two flag varieties. Then the divisor $D(\lambda)$ restricted to $\overline{\mathcal{O}}$ is the inverse image under the map p of the divisor $D(\lambda, \lambda)$ in F. Indeed, $D(\lambda) = p^{-1}(\widetilde{D}(\lambda))$, where $\widetilde{D}(\lambda)$ is the weight divisor in X_{can} corresponding to λ. It is easy to check that $\widetilde{D}(\lambda) \cap F = D(\lambda, \lambda)$ (see Proposition 8.1 in [6]).

Hence, all the intersection points in $S_i(\mathcal{O})D(\lambda)^{n-i-s}$ are contained in the preimage of $p(S_i(\mathcal{O}))D(\lambda, \lambda)^{n-i-s}$. But the latter is empty. Indeed, since $\mathcal{O}$ has positive rank and F has zero rank, Lemma 3.1 implies that

$$\dim\, p(S_i(\mathcal{O})) < \dim\, S_i(\mathcal{O}) = n - i - s.$$

□

It remains to deal with the orbits in X whose image under the map p is not the closed orbit in X_{can}. In this case, the face corresponding to such an orbit intersects the walls of the Weyl chambers, and hence, it is orthogonal to some of the fundamental weights $\omega_1, \ldots, \omega_k$. Note that the codimension one orbits $\mathcal{O}_1, \ldots, \mathcal{O}_k$ in X_{can} are in one-to-one correspondence with the fundamental weights $\omega_1, \ldots, \omega_k$. Namely, the facet corresponding to $\mathcal{O}_i$ is orthogonal to ω_i. Let $\overline{\mathcal{O}}_1, \ldots, \overline{\mathcal{O}}_k$ be the closures in X_{can} of $\mathcal{O}_1, \ldots, \mathcal{O}_k$, respectively.

Lemma 3.3. *Let $\mathcal{O}$ be a $G \times G$–orbit in X of codimension $s < k$. Suppose that the image $p(\mathcal{O})$ under the map $p : X \to X_{can}$ is not closed and lies in the intersection $\overline{\mathcal{O}}_{i_1} \cap \ldots \cap \overline{\mathcal{O}}_{i_s}$. In terms of polytopes, this means that the face corresponding to $\mathcal{O}$ is orthogonal to the weights $\omega_{i_1}, \ldots, \omega_{i_s}$.*

Let λ be any linear combination of the weights $\omega_{i_1},\ldots,\omega_{i_s}$. Then

$$c_iD(\lambda)^{n-i-s}\overline{\mathcal{O}}=0.$$

Proof. We use the $G\times G$–equivariant map r from $\overline{\mathcal{O}}_{i_1}\cap\ldots\cap\overline{\mathcal{O}}_{i_s}$ to a partial flag variety $G/P\times G/P$ constructed in [6] (see [6] Lemma 5.1 for details). Consider the compactification $X_{i_1,\ldots,i_s}$ of G corresponding to the irreducible representation $\pi_{i_1,\ldots,i_s}$ whose highest weight lies strictly inside the cone spanned by $\omega_{i_1},\ldots,\omega_{i_s}$. This compactification has a unique closed orbit $G/P\times G/P$, where $P\subset G$ is the stabilizer of the highest weight vector in the representation $\pi_{i_1,\ldots,i_s}$. Clearly, the fan of the Weyl chambers and their faces subdivides the normal fan of the weight polytope of $\pi_{i_1,\ldots,i_s}$. Hence, by Theorem 2.1 there is an equivariant map $r: X_{can}\to X_{i_1,\ldots,i_s}$. This map takes $\overline{\mathcal{O}}_{i_1}\cap\ldots\cap\overline{\mathcal{O}}_{i_s}$ to the closed orbit $G/P\times G/P$.

The composition rp maps the orbit $\mathcal{O}$ to the closed orbit $G/P\times G/P$ of $X_{i_1,\ldots,i_s}$. It is easy to show that the divisor $D(\lambda)$ restricted to $\mathcal{O}$ is the preimage of the divisor $D(\lambda,\lambda)\subset G/P\times G/P$ under this map (see [6] Section 8.1). Now repeat the arguments of the proof of Lemma 3.2.

□

These two lemmas imply the following vanishing result.

Corollary 3.4. *Let $\mathcal{O}$ be any $G\times G$–orbit in X of codimension $s<k$, and let F be the face of the polytope of X that corresponds to $\mathcal{O}$. The intersection index*

$$c_iD(\lambda_1)\ldots D(\lambda_{n-i-s})\overline{\mathcal{O}}$$

vanishes in the cohomology ring of X in the following two cases:

1) The face F does not intersect the walls of the Weyl chambers. Then weights $\lambda_1,\ldots,\lambda_{n-i-s}$ are any weights of G.

2) The face F intersects a wall of the Weyl chambers and weights $\lambda_1,\ldots,\lambda_{n-i-s}$ are orthogonal to F (with respect to the inner product $(\cdot,\cdot)$ on $L_T\otimes\mathbb{R}$ defined in the Introduction).

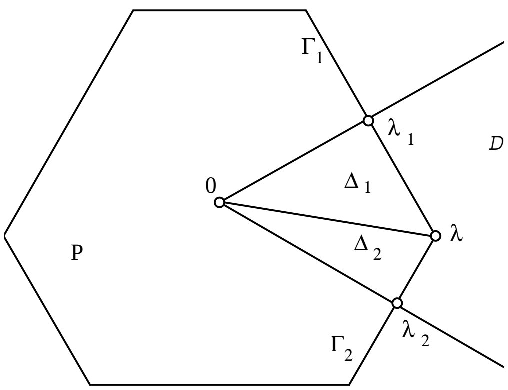
4. Proof of Theorem 1.2

We use notation of Subsections 2.2 and 2.3. Let X be any regular compactification lying over the compactification X_π. Then the closure $\overline{H}_\pi$ of H_π in X has proper intersections with all $G\times G$–orbits in X, and thus $S_iH_\pi^{n-i}$ coincides with the intersection index $\overline{S}_i\overline{H}_\pi^{n-i}$ in the cohomology ring of X.

Assume that X corresponds to a representation of G with the weight polytope P_0. Let us compute $\overline{S}_iD^{n-i}$ for a divisor D under the assumption that the polytope P corresponding to D is analogous to P_0. After we establish the formula of Theorem 1.2 for such divisors, it will automatically extend to the other divisors (in particular, for $\overline{H}_\pi$) since any virtual polytope analogous to P_0 is a linear combination of polytopes analogous to P_0. Since X is regular, P_0 and hence P are simple.

All computations are carried in the cohomology ring of X. First, break D^{n-i} into monomials of the form $\overline{\mathcal{O}}_{i_1}\ldots\overline{\mathcal{O}}_{i_k}D(\lambda_1)\ldots D(\lambda_{n-i-k})$, where $i_1,\ldots,i_k$ are distinct integers from 1 to l and $\lambda_1,\ldots,\ \lambda_{n-i-k}$ are weights. Then every such monomial can be computed explicitly, since the intersection $\overline{\mathcal{O}}_{i_1}\cap\ldots\cap\overline{\mathcal{O}}_{i_k}$ is either empty or isomorphic to the product of two flag varieties.

Since we are going to intersect D^{n-i} with $\overline{S}_i$ we can ignore all monomials that are annihilated by $\overline{S}_i$. Recall that $\overline{S}_i$ is the i-th Chern class of the Demazure bundle over X. In particular, Corollary 3.4 implies that $\overline{S}_i$ annihilates the ideal $I\subset H^*(X)$ generated by the monomials of the form $D(\lambda_1)\ldots D(\lambda_{n-i-s})\overline{\mathcal{O}}$ such that either the face of P corresponding to the codimension $s<k$ orbit $\mathcal{O}$ does not intersect the walls of the Weyl chambers or, if it does, the weights $\lambda_1,\ldots,\lambda_{n-i-s}$ are orthogonal to this face.



To keep track of our calculations we use a subdivision of the polytope $P\cap\mathcal{D}$ into simplices coming from the barycentric subdivision of P described below. For each face $F\subset P$ choose a point $\lambda_F\in F$ as follows. If F does not intersect the walls of the Weyl chamber $\mathcal{D}$, then λ_F is any point in the interior of the face. Otherwise, choose λ_F so that the corresponding vector is orthogonal to the face F (in particular, λ_F will belong to the intersection of the face with a wall of $\mathcal{D}$.) If $F=P$ take $\lambda_F=0$.

An s*–flag* $\mathcal{F}$ is the collection $\{F_1\supset\ldots\supset F_s\}$ of $s\le k$ nested faces of P such that each of them intersects $\mathcal{D}$, and F_i has codimension i in P. Denote by $\overline{\mathcal{O}}_{\mathcal{F}}$ the closure in X of the orbit corresponding to the last face F_s, and by $\Delta_{\mathcal{F}}$ the s–dimensional simplex with the vertices $0,\ \lambda_{F_1},\ \ldots,\ \lambda_{F_s}$. In particular, when $s=k$, the simplex $\Delta_{\mathcal{F}}$ has full dimension and the orbit $\mathcal{O}_{\mathcal{F}}$ is closed. The polytope $\mathcal{D}\cap P$ is the union of simplices $\Delta_{\mathcal{F}}$ over all possible k–flags $\mathcal{F}$.

Example. Take $G=PSL_3(\mathbb{C})$, and let $X=X_{can}$ be its wonderful compactification. Let divisor D be a hyperplane section corresponding to the irreducible representation with a strictly dominant highest weight λ. In this case, P is a hexagon symmetric under the action of the Weyl group, with two edges Γ_1 and Γ_2 intersecting $\mathcal{D}$. Then $\lambda_i=\lambda_{\Gamma_i}\in\Gamma_i$ is orthogonal to Γ_i for $i=1,2$ and $\Gamma_1\cap\Gamma_2=\lambda$. The subdivision of $P\cap\mathcal{D}$ into simplices consists of two triangles Δ_1 and Δ_2 with the vertices $0,\ \lambda_1,\ \lambda$ and $0,\ \lambda_2,\ \lambda$, respectively (see the figure).

Lemma 4.1. *Denote by $f_d(x_1,\ldots,x_k)$ the sum of all monomials of degree d in k variables $x_1,\ldots,\ x_k$. The following identity holds in the cohomology ring of X modulo the ideal I:*

$$D^{n-i}\equiv k!\sum_{\mathcal{F}}\mathrm{Vol}(\Delta_{\mathcal{F}})f_{n-k-i}(D,D(\lambda_{F_1}),\ldots,D(\lambda_{F_{k-1}}))\mathcal{O}_{\mathcal{F}}\pmod{I},$$

where the sum is taken over all possible k–flags $\mathcal{F} = \{F_1 \supset \ldots \supset F_k\}$. *The volume form* Vol *is normalized so that the covolume of* L_T *is equal to* 1.

Proof. We will prove the following more general statement for s-flags. Denote by $f_{d,s}(x_1, \ldots, x_s)$ the sum of all monomials of degree d in s variables.

Recall that $\Gamma_1, \ldots, \Gamma_l$ denote the facets of P that intersect the Weyl chamber $\mathcal{D}$. An s-flag can be alternatively described by an ordered collection of facets $\Gamma_{i_1}, \ldots, \Gamma_{i_s}$ such that their intersection $\Gamma_{i_1} \cap \ldots \cap \Gamma_{i_s}$ has codimension s. Then $F_j = \Gamma_{i_1} \cap \ldots \cap \Gamma_{i_j}$. This is a one-to-one correspondence, since the polytope P is simple. Assign to each s-flag $\mathcal{F}$ the following number

$$c_{\mathcal{F}} = h_{i_1}(P)[h_{i_2}(P) - h_{i_2}(\lambda_{F_1})] \ldots [h_{i_s}(P) - h_{i_s}(\lambda_{F_{s-1}})].$$

In particular, when $s = k$, i.e. F_s is just a vertex, the number $c_{\mathcal{F}}$ coincides with the volume of $\Delta_{\mathcal{F}}$ times $k!$. Indeed, by a unimodular linear transformation of $L_T \otimes \mathbb{R}$ we can map the hyperplanes containing the facets $\Gamma_{i_1}, \ldots, \Gamma_{i_s}$ to the coordinate hyperplanes. Then $[h_{i_j}(P) - h_{i_j}(\lambda_{F_{j-1}})]$ is just the Euclidean distance from the vertex $\lambda_{F_{j-1}}$ of $\Delta_{\mathcal{F}}$ to the hyperplane containing Γ_{i_j}. Note that to define volumes we do not use the inner product $(\cdot, \cdot)$ on the lattice L_T. We only use the lattice itself.

Then for any integer s such that $1 \leq s \leq k$ the following is true:

$$D^{n-i} \equiv \sum_{\mathcal{F}} c_{\mathcal{F}} f_{n-s-i,s}(D, D(\lambda_{F_1}), \ldots, D(\lambda_{F_{s-1}}))\overline{\mathcal{O}}_{\mathcal{F}} \pmod{I}, \tag{1}$$

where the sum is taken over all s–flags.

Example: If G is a complex torus, formula (1) is still meaningful but looks much simpler and reduces to

$$D^s = \sum_{\mathcal{F}} c_{\mathcal{F}} \overline{\mathcal{O}}_{\mathcal{F}}.$$

Prove formula (1) by induction on s. We use the notations of Subsection 2.3. For $s = 1$, the statement coincides with the decomposition $D = h_1(P)\overline{\mathcal{O}}_1 + \cdots + h_l(P)\overline{\mathcal{O}}_l$ from Lemma 2.3.

Assume that the formula is proved for some $s < k$. Prove it for $s + 1$. We now deal separately with each term on the right hand side of formula (1). First subtract from every term $f_{n-s-i,s}(D, D(\lambda_{F_1}), \ldots, D(\lambda_{F_{s-1}}))\overline{\mathcal{O}}_{\mathcal{F}}$ the element $f_{n-s-i,s}(D(\lambda_{F_s}), D(\lambda_{F_1}), \ldots, D(\lambda_{F_{s-1}}))\overline{\mathcal{O}}_{\mathcal{F}}$ of the ideal I. This operation does not change the identity (1). A simple calculation shows that

$$f_{n-s-i,s}(x, x_1, \ldots, x_{s-1}) - f_{n-s-i,s}(x_s, x_1, \ldots, x_{s-1}) = (x - x_s) f_{n-s-i-1,s+1}(x, x_1, \ldots, x_{s-1}, x_s).$$

Hence, after subtraction we can rewrite the difference as

$$(D - D(\lambda_{F_s})) f_{n-s-i-1,s+1}(D, D(\lambda_{F_1}), \ldots, D(\lambda_{F_s}))\overline{\mathcal{O}}_{\mathcal{F}}.$$

Since λ_s lies in the intersection of s facets $\Gamma_{i_1}, \ldots, \Gamma_{i_s}$, Corollary 2.5 implies that

$$(D - D(\lambda_{F_s}))\overline{\mathcal{O}}_{\mathcal{F}} = \sum_{j \neq i_1, \ldots, i_s} [h_j(P) - h_j(\lambda_{F_s})]\overline{\mathcal{O}}_j \overline{\mathcal{O}}_{\mathcal{F}}.$$

Note that $\overline{\mathcal{O}}_j\overline{\mathcal{O}}_{\mathcal{F}}$ is empty if and only if the intersection of Γ_j with $\Gamma_{i_1}\cap\ldots\cap\Gamma_{i_s}$ is empty. Hence,

$$(D-D(\lambda_{F_s}))\overline{\mathcal{O}}_{\mathcal{F}}=\sum_{\mathcal{F}'}[h_j(P)-h_j(\lambda_{F_s})]\overline{\mathcal{O}}_{\mathcal{F}'},$$

where the sum is taken over all $(s+1)$-flags $\mathcal{F}'$ that extend $\mathcal{F}$, i.e. $\mathcal{F}'=\{F_1\supset\ldots\supset F_s\supset F_s\cap\Gamma_j\}$. □

It remains to compute the term

$$\overline{S}_i\cdot f_{n-k-i}(D,D(\lambda_{F_1}),\ldots,D(\lambda_{F_{k-1}}))\mathcal{O}_{\mathcal{F}} \qquad (2)$$

for each k-flag $\mathcal{F}$. Suppose that the closed orbit $\mathcal{O}_{\mathcal{F}}$ is the intersection of k hypersurfaces $\overline{\mathcal{O}}_{i_1},\ldots,\ \overline{\mathcal{O}}_{i_k}$. Then for any other codimension 1 orbit $\mathcal{O}_j$ (such that $j\neq i_1,\ldots,i_k$), the intersection $\mathcal{O}_{\mathcal{F}}\cap\overline{\mathcal{O}}_j$ is empty. Hence, D in (2) can be replaced by $D(\lambda_{F_k})$ since

$$D=D(\lambda_{F_k})+\sum_{j\neq i_1,\ldots,i_k}(h_j(P)-h_j(\lambda_{F_k}))\overline{\mathcal{O}}_j.$$

Note also that the evaluation of (2) reduces to the computation of intersection indices in $\mathcal{O}_{\mathcal{F}}$, which is the product of two flag varieties. We have that $\overline{S}_i\cdot\mathcal{O}_{\mathcal{F}}=c_i(\mathcal{O}_{\mathcal{F}})$ and $D(\lambda)\cdot\mathcal{O}_{\mathcal{F}}=D(\lambda,\lambda)$. Here $c_i(\mathcal{O}_{\mathcal{F}})$ is the i-th Chern class of the tangent bundle of $\mathcal{O}_{\mathcal{F}}$, which coincides with the Demazure bundle over $\mathcal{O}_{\mathcal{F}}$ since $\mathcal{O}_{\mathcal{F}}$ is closed. Hence,

$$\overline{S}_if_{n-k-i}(D(\lambda_{F_k}),D(\lambda_{F_1}),\ldots,D(\lambda_{F_{k-1}}))\mathcal{O}_{\mathcal{F}}=$$
$$=c_i(G/B\times G/B)f_{n-k-i}(D(\lambda_{F_1},\lambda_{F_1}),\ldots,D(\lambda_{F_k},\lambda_{F_k})). \qquad (3)$$

The intersection product in the right hand side of this formula is taken in $G/B\times G/B$.

The function $F_i(\lambda)=c_i(G/B\times G/B)D(\lambda,\lambda)^{n-k-i}$ can be expressed explicitly in terms of the function F defined in the Introduction, since the i-th Chern class of $G/B\times G/B$ is the term of degree i in the intersection product

$$\prod_{\alpha\in R^+}(1+D(\alpha,0))(1+D(0,\alpha)).$$

One way to compute F_i is as follows. Let $\mathbb{D}$ and $[\mathbb{D}]_i$ be the differential operators defined in the Introduction. Then

$$F_i(x)=(n-k-i)![\mathbb{D}]_iF(x,x).$$

This easily follows from the formula for the polarization mentioned in Subsection 2.5 and the fact that $D^{n-k}(\lambda,\lambda)=(n-k)!F(\lambda,\lambda)$.

We can now apply Proposition 2.6 to convert the sum (3) into the integral over the simplex $\Delta_{\mathcal{F}}$. Indeed, by definition of the function f_{n-k-i} we have that (3) can be rewritten as

$$\sum_{i_1+\ldots+i_k=n-k-i}(F_i)_{pol}(\underbrace{\lambda_{F_1},\ldots,\lambda_{F_1}}_{i_1},\ldots,\underbrace{\lambda_{F_k},\ldots,\lambda_{F_k}}_{i_k}).$$

This is equal to the integral

$$\binom{n-i}{k}\int\limits_{\Delta_{\mathcal{F}}} F_i(x)dx/\mathrm{Vol}(\Delta_{\mathcal{F}})$$

by Proposition 2.6 applied to the simplex $\Delta_{\mathcal{F}}$ (with the vertices 0, λ_{F_1},..., λ_{F_k}) and to the function $F_i(x)$. Combining this with Lemma 4.1 we get

$$\overline{S}_i D^{n-i} = \frac{(n-i)!}{(n-k-i)!}\sum_{\mathcal{F}}\int\limits_{\Delta_{\mathcal{F}}} F_i(x)dx = (n-i)!\int\limits_{P\cap\mathcal{D}} [\mathbb{D}]_i F(x,x)dx.$$

Note that when $i=0$, we get the Brion–Kazarnovskii formula.

5. Example

In this section, I give an example of computation of the Euler characteristic using Theorems 1.2 and 1.3. Namely, for the group $G = SL_3(\mathbb{C})$, I compute the Euler characteristic of a hyperplane section corresponding to an irreducible representation.

Let π be the irreducible representation of $SL_3(\mathbb{C})$ with the highest weight $\lambda = m\omega_1 + n\omega_2$, where ω_1, ω_2 are the fundamental weights of $SL_3(\mathbb{C})$, and m and n are nonnegative integers. Take a generic hyperplane section H_π corresponding to the representation π. We first find $S_i H_\pi^{n-i}$. The dimension of $SL_3(\mathbb{C})$ is 8, and the rank is 2, so there are 6 nontrivial Chern classes S_1,..., S_6.

Let us compute all ingredients of the formula of Theorem 1.2. The weight polygon P_π of π is depicted on the figure above. The domain $P_\pi \cap \mathcal{D}$ is the union of two triangles Δ_1 and Δ_2. The positive roots of $SL_3(\mathbb{C})$ are $\alpha_1 = 2\omega_1 - \omega_2$, $\alpha_2 = 2\omega_2 - \omega_1$ and $\alpha_1 + \alpha_2$. Write $\mathbb{D}$ and F in the coordinates $(s,t;\widetilde{s},\widetilde{t})$ in $(L_T \oplus L_T)\otimes\mathbb{R}$ associated with the basis $\{\alpha_1,\alpha_2\}$ in $L_T \otimes \mathbb{R}$:

$$\mathbb{D} = (1+\partial_s)(1+\partial_t)(1+\partial_s+\partial_t)(1+\partial_{\widetilde{s}})(1+\partial_{\widetilde{t}})(1+\partial_{\widetilde{s}}+\partial_{\widetilde{t}}).$$

Since $\rho = \alpha_1 + \alpha_2$ and $(\alpha_1,\omega_1) = (\alpha_2,\omega_2) = 1$, $(\alpha_1,\omega_2) = (\alpha_2,\omega_1) = 0$ we have that

$$F = \frac{1}{4}(2s-t)(2t-s)(t+s)(2\widetilde{s}-\widetilde{t})(2\widetilde{t}-\widetilde{s})(\widetilde{t}+\widetilde{s}).$$

If we plug $\mathbb{D}$ and F in the formula of Theorem 1.2 and integrate, we get that

$H_\pi^8 = 3(m^8 + 16m^7n + 112m^6n^2 + 448m^5n^3 + 700m^4n^4 + 448m^3n^5 +$
$112m^2n^6 + 16mn^7 + n^8)$;

$S_1H_\pi^7 = 18(m+n)(m^6 + 13m^5n + 71m^4n^2 + 139m^3n^3 + 71m^2n^4 + 13mn^5 + n^6)$;

$S_2H_\pi^6 = 54(m^6 + 12m^5n + 50m^4n^2 + 80m^3n^3 + 50m^2n^4 + 12mn^5 + n^6)$;

$S_3H_\pi^5 = 90(m+n)(m^4 + 9m^3n + 19m^2n^2 + 9mn^3 + n^4)$;

$S_4H_\pi^4 = 18(5m^4 + 40m^3n + 72m^2n^2 + 40mn^3 + 5n^4)$;

$S_5H_\pi^3 = 54(m+n)(m^2 + 5mn + n^2)$;

$S_6 H_\pi^2 = 18(m^2 + 4mn + n^2)$.

We now apply Theorem 1.3 to obtain

$\chi(H_\pi) = -3(m^8+16m^7n+112m^6n^2+448m^5n^3+700m^4n^4+448m^3n^5+112m^2n^6+16mn^7+n^8+ 18(m^6+12m^5n+50m^4n^2+80m^3n^3+50m^2n^4+12mn^5+n^6)+ 6(5m^4+40m^3n+72m^2n^2+40mn^3+5n^4)+ 6(m^2+4mn+n^2) -6(m+n)(m^6+13m^5n+71m^4n^2+139m^3n^3+71m^2n^4+13mn^5+n^6+ 5(m^4+9m^3n+19m^2n^2+9mn^3+n^4)+3(m^2+5mn+n^2)))$.

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