Chern classes of reductive groups and an adjunction formula

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In this paper, I construct noncompact analogs of the Chern classes for equivariant vector bundles over complex reductive groups. For the tangent bundle, these Chern classes yield an adjunction formula for the Euler characteristic of complete intersections in reductive groups. In the case where a complete intersection is a curve, this formula gives an explicit answer for the Euler characteristic and the genus of the curve.

1 Introduction and main results

Let G be a connected complex reductive group. Consider a faithful finite-dimensional representation $\pi : G \to GL(V)$ on a complex vector space V. Let $H \subset \mathrm{End}(V)$ be a generic affine hyperplane. The hypersurface $\pi^{-1}(\pi(G) \cap H) \subset G$ is called a *hyperplane section* corresponding to the representation π. The problem underlying this paper is how to find the Euler characteristic of a hyperplane section or, more generally, of the complete intersection of several hyperplane sections corresponding to different representations.

The motivation to study such question comes from the case when the group $G = (\mathbb{C}^*)^n$ is a complex torus. In this case, D.Bernstein, A.Khovanskii and A.Kouchnirenko found an explicit and very beautiful answer in terms of the weight polytopes of representations (see [18]). E.g. the Euler characteristic $\chi(\pi)$ of a hyperplane section corresponding to the representation π is equal to $(-1)^n$ times the normalized volume of the weight polytope of π. The proof uses an explicit relation between the Euler characteristic $\chi(\pi)$ and the degree of the affine subvariety $\pi(G)$ in $\mathrm{End}(V)$:

$$\chi(\pi) = (-1)^{n-1}\mathrm{deg}\ \pi(G). \tag{1}$$

The degree is defined as usual. Namely, the degree of an affine subvariety $X \subset \mathbb{C}^N$ equals to the number of the intersection points of X with a generic affine subspace in $\mathbb{C}^N$ of complementary dimension. For the degree $\deg \pi(G)$ (that can also be interpreted as the self-intersection index of a hyperplane section corresponding to the representation π) there is an explicit formula proved by Kouchnirenko. Later D.Bernstein, and Khovanskii found an analogous formula for the intersection index of hyperplane sections corresponding to different representations.

How to extend these results to the case of arbitrary reductive groups? It turned out that the formulas for the intersection indices of several hyperplane sections can be generalized to reductive groups and, more generally, to spherical homogeneous spaces. For reductive groups, this was done by B.Kazarnovskii [17]. Later, M.Brion established an analogous result for all spherical homogeneous spaces [4]. For reductive groups, the Brion–Kazarnovskii theorem allows to compute explicitly the intersection index of n generic hyperplane sections corresponding to different representations. The precise definition of the intersection index is given in Section 2.

However, when G is an arbitrary reductive group, it is no longer true that $\chi(\pi) = (-1)^{n-1}\deg \pi(G)$. K.Kaveh computed explicitly $\chi(\pi)$ and $\deg \pi(G)$ for all representations π of $SL_2(\mathbb{C})$. His computation shows that, in general, there is a discrepancy between these two numbers. Kaveh also listed some special representations of reductive groups, for which these numbers still coincide [16].

In this paper, I will present a formula that, in particular, generalizes formula (1) to the case of arbitrary reductive groups. To do this I will construct algebraic subvarieties $S_i \subset G$, whose degrees fill the gap between the Euler characteristic and the degree. My construction is similar to one of the classical constructions of the Chern classes of a vector bundle in the compact setting (Subsection 3.1). The subvarieties S_i can be thought of as Chern classes of the tangent bundle of G. I will also construct Chern classes of more general equivariant vector bundles over G (Subsection 3.2). These Chern classes are in many aspects similar to the usual Chern classes of compact manifolds. There is an analog of the cohomology ring for G, where the Chern classes of equivariant bundles live. This analog is the ring of conditions constructed by C.De Concini and C.Procesi [10, 8](see Section 2 for a reminder). It is useful in solving enumerative problems. In particular, the intersection product in this ring is well-defined.

I now formulate the main results. Denote by n and k the dimension and the rank of G, respectively. Recall that the rank is the dimension of a maximal torus in G. Denote by $[S_1], \ldots, [S_n]$ the Chern classes of the tangent bundle of G as elements of the ring of conditions, and denote by $S_1,\ldots, S_n$ subvarieties representing these classes. In the case of the tangent bundle, it turns out (see Lemma 3.6) that the the higher Chern classes $[S_{n-k+1}],\ldots, [S_n]$ vanish. E.g. if G is a torus, then all Chern classes $[S_i]$ vanish.

Let $H_1,\ldots, H_m$ be a *generic* collection of m hyperplane sections corresponding to faithful representations $\pi_1,\ldots, \pi_m$ of the group G (for the precise meaning of "generic" see Subsection 4.3). Then the following theorem holds.

Theorem 1.1. *The Euler characteristic of the complete intersection $H_1 \cap \ldots \cap H_m$ is equal to the term of degree n in the expansion of the following product:*

$$(1 + S_1 + \ldots + S_{n-k}) \cdot \prod_{i=1}^{m} H_i(1 + H_i)^{-1}.$$

The product in this formula is the intersection product in the ring of conditions.

This is very similar to the classical adjunction formula in compact setting.

In particular, the Euler characteristic of just one hyperplane section corresponding to a representation π is equal to the following alternating sum. Put $S_0 = G$. Then

$$\chi(\pi) = \sum_{i=0}^{n-k} (-1)^{n-i-1} \deg \pi(S_i).$$

The latter formula may have applications in the theory of generalized hypergeometric equations. In the torus case, I.Gelfand, M.Kapranov and A.Zelevinsky showed that the Euler characteristic $\chi(\pi)$ gives the number of integral solutions of the generalized hypergeometric system associated with the representation π [13]. A similar system can be associated with the representation π of any reductive group [15]. In the reductive case, the number of integral solutions of such a system is also likely to coincide with $\chi(\pi)$.

The proof of Theorem 1.1 is similar to the proof by Khovanskii [18] in the torus case. Namely, Theorem 1.1 follows from the adjunction formula applied to the closure of a complete intersection in a suitable regular compactification of G (see Subsection 4.3). The key ingredient is a description of the tangent bundles of regular compactifications due to Ehlers [11] and Brion [5]. This description is outlined in Subsection 4.2.

The remaining problem is to describe the Chern classes $[S_1], \ldots, [S_{n-k}]$ so that their intersection indices with hyperplane sections may be computed explicitly. So far there is such a description for the first and the last Chern classes (see Subsection 3.3). Namely, $[S_1]$ is the class of a generic hyperplane section corresponding to the irreducible representation with the highest weight 2ρ. Here ρ is the sum of all fundamental weights of G. This description follows from a result of A.Rittatore [25] concerning the first Chern class of reductive group compactifications. The last Chern class $[S_{n-k}]$ is up to a scalar multiple the class of a maximal torus in G. There is a hope that the intersection indices of other Chern classes S_i with hyperplane sections can also be computed using a formula similar to the Brion–Kazarnovskii formula.

If a complete intersection is a curve, i.e. $m = n - 1$, then the formula of Theorem 1.1 involves only the first Chern class $[S_1]$. In this case, the computation of $[S_1]$ together with the Brion–Kazarnovskii formula allows us to compute explicitly the Euler characteristic and the genus of a curve in G in terms of the weight polytopes of π_1,..., π_m (see Corollaries 4.8 and 4.9, Subsection 4.3). Note that these two numbers completely describe the topological type of a curve.

Most of the constructions and results of this paper can be extended without any change to the case of arbitrary spherical homogeneous spaces. This is discussed in Section 5.

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Part of the results of this paper were included into my PhD thesis at the University of Toronto [19].

Throughout this paper, whenever a group action is mentioned, it is always assumed that a complex algebraic group acts on a complex algebraic variety by algebraic automorphisms. In particular, by a homogeneous space for a group I will always mean the quotient of the group by some closed algebraic subgroup.

The following remarks concern notations. In this paper, the term equivariant (e.g. equivariant compactification, bundle, etc.) will always mean equivariant under the action of the doubled group $G \times G$, unless otherwise stated. The Lie algebra of G is denoted by $\mathfrak{g}$. I also fix an embedding $G \subset GL(W)$ for some vector space W. Then for $g \in G$ and $A \in \mathfrak{g}$, notation Ag and gA mean the product of linear operators in $\mathrm{End}(W)$.

2 Equivariant compactifications and the ring of conditions

This section contains some well-known notions and theorems, which will be used in the sequel. First, I define the notion of spherical action and describe equivariant compactifications of reductive groups following [9], [15] and [26]. Then I state Kleiman's transversality theorem [20] and recall the definition of the ring of conditions [10, 8].

Spherical action. Reductive groups are partial cases of more general *spherical* homogeneous spaces. They are defined as follows. Let G be a connected complex reductive group, and let M be a homogeneous space under G. The action of G on M is called *spherical*, if a Borel subgroup of G has an open dense orbit in M. In this case, the homogeneous space M is also called spherical. An important and very useful property, which characterizes a spherical homogeneous space M, is that any compactification of M equivariant under the action of G contains only a finite number of orbits [21].

There is a natural action of the group $G \times G$ on G by left and right multiplications. Namely, an element $(g_1, g_2) \in G \times G$ maps an element $g \in G$ to $g_1 g g_2^{-1}$. This action is spherical as follows from the Bruhat decomposition of G with respect to some Borel subgroup. Thus the group G can be considered as a spherical homogeneous space of the doubled group $G \times G$ with respect to this action. For any representation $\pi : G \to GL(V)$ this action can be extended straightforwardly to the action of $\pi(G) \times \pi(G)$ on the whole $\mathrm{End}(V)$ by left and right multiplications. I will call such actions *standard*.

Equivariant compactifications. With any representation π one can associate the following compactification of $\pi(G)$. Take the projectivization $\mathbb{P}(\pi(G))$ of $\pi(G)$ (i.e. the set of all lines in $\mathrm{End}(V)$ passing through a point of $\pi(G)$ and the origin), and then take its closure in $\mathbb{P}(\mathrm{End}(V))$. We obtain a projective variety $X_\pi \subset \mathbb{P}(\mathrm{End}(V))$ with a natural action of $G \times G$ coming from the standard action of $\pi(G) \times \pi(G)$ on $\mathrm{End}(V)$. Below I will list some important properties of this variety.

Assume that $\mathbb{P}(\pi(G))$ is isomorphic to G. Fix a maximal torus $T \subset G$. Let L_T be its character lattice. Consider all weights of the representation π, i.e. all characters of the maximal torus T occurring in π. Take their convex hull P_π in $L_T \otimes \mathbb{R}$. Then it is easy to see that P_π is a polytope invariant under the action of the Weyl group of G. It is called the *weight polytope* of the representation π. The polytope P_π contains information about the compactification X_π.

Theorem 2.1. *1)* ([26], Proposition 8) *The subvariety X_π consists of a finite number of $G\times G$-orbits. These orbits are in one-to-one correspondence with the orbits of the Weyl group acting on the faces of the polytope P_π.*

2) Let σ be another representation of G. The normalizations of subvarieties X_π and X_σ are isomorphic if and only if the normal fans corresponding to the polytopes X_π and X_σ coincide. If the first fan is a subdivision of the second, then there exists an equivariant map from the normalization of X_π to X_σ, and vice versa.

The second part of Theorem 2.1 follows from the general theory of spherical varieties (see [21], Theorem 5.1) combined with the description of compactifications X_π via colored fans (see [26], Sections 7, 8).

In particular, suppose that the group G is of adjoint type, i.e. the center of G is trivial. Let π be an irreducible representation of G with a strictly dominant highest weight. It is proved in [9] that the corresponding compactification X_π of the group G is always smooth and, hence, does not depend on the choice of a highest weight. Indeed, the normal fan of the weight polytope P_π coincides with the fan of the Weyl chambers and their faces, so the second part of Theorem 2.1 applies. This compactification is called the wonderful compactification and is denoted by X_{can}. It was introduced by De Concini and Procesi [9]. The boundary divisor $X_{can}\setminus G$ is a divisor with normal crossings. There are k orbits $\mathcal{O}_1,\ldots,\mathcal{O}_k$ of codimension one in X_{can}. The other orbits are obtained as the intersections of the closures $\overline{\mathcal{O}}_1,\ldots,\overline{\mathcal{O}}_k$. More precisely, to any subset $\{i_1,i_2,\ldots,i_m\}\subset\{1,\ldots,k\}$ there corresponds an orbit $\overline{\mathcal{O}}_{i_1}\cap\overline{\mathcal{O}}_{i_2}\cap\ldots\cap\overline{\mathcal{O}}_{i_m}$ of codimension m. So the number of orbits equals to 2^k. There is a unique closed orbit $\overline{\mathcal{O}}_1\cap\ldots\cap\overline{\mathcal{O}}_k$, which is isomorphic to the product of two flag varieties $G/B\times G/B$. Here B is a Borel subgroup of G.

Compactifications of a reductive group arising from its representations are examples of more general *equivariant compactifications* of the group. A compact complex algebraic variety with an action of $G\times G$ is called an *equivariant compactification* of G if it satisfies the following conditions. First, it contains an open dense orbit isomorphic to G. Second, the action of $G\times G$ on this open orbit coincides with the standard action by left and right multiplications.

The ring of conditions. The following theorem gives a tool to define the intersection index on a noncompact group, or more generally, on a homogeneous space. Recall that two irreducible algebraic subvarieties Y_1 and Y_2 of an algebraic variety X are said to have *proper* intersection if either their intersection $Y_1\cap Y_2$ is empty or all irreducible components of $Y_1\cap Y_2$ have dimension $\dim Y_1+\dim Y_2-\dim X$.

Theorem 2.2. (Kleiman's transversality theorem) [20] *Let H be a connected algebraic group, and let M be a homogeneous space under H. Take two algebraic subvarieties $X,Y\subset M$. Denote by gX the left translate of X by an element $g\in H$. There exists an open dense subset of H such that for all elements g from this subset the intersection $gX\cap Y$ is proper. If X and Y are smooth, then $gX\cap Y$ is transverse for general $g\in H$.*

In particular, if X and Y have complementary dimensions (but are not necessarily smooth), then for almost all g the translate gX intersects Y transversally at a finite number of points, and this number does not depend on g.

If X and Y have complementary dimensions, define the *intersection index* (X, Y) as the number $\#(gX \cap Y)$ of the intersection points for a generic $g \in H$. If one is interested in solving enumerative problems, then it is natural to consider algebraic subvarieties of M up to the following equivalence. Two subvarieties X_1, X_2 of the same dimension are equivalent if and only if for any subvariety Y of complementary dimension the intersection indices (X_1, Y) and (X_2, Y) coincide. This relation is similar to the numerical equivalence in algebraic geometry (see [12], Chapter 19). Consider all formal linear combinations of algebraic subvarieties of M modulo this equivalence relation. Then the resulting group $C^*(M)$ is called the *group of conditions* of M.

One can define an *intersection product* of two subvarieties $X, Y \subset M$ by setting $X \cdot Y = gX \cap Y$, where $g \in G$ is generic. However, the intersection product sometimes is not well-defined on the group of conditions (see [10] for a counterexample). A remarkable fact is that for spherical homogeneous spaces the intersection product is well-defined, i.e. if one takes different representatives of the same classes, then the class of their product will be the same [10, 8]. The corresponding ring $C^*(M)$ is called the *ring of conditions*.

In particular, the group of conditions $C^*(G)$ of a reductive group is a ring. De Concini and Procesi related the ring of conditions to the cohomology rings of equivariant compactifications as follows. Consider the set $\mathcal{S}$ of all smooth equivariant compactifications of the group G. This set has a natural partial order. Namely, a compactification X_σ is greater than X_π if X_σ *lies over* X_π, i.e. if there exists a map $X_\sigma \to X_\pi$ commuting with the action of $G \times G$. Clearly, such a map is unique, and it induces a map of cohomology rings $H^*(X_\pi) \to H^*(X_\sigma)$.

Theorem 2.3. [10, 8] *The ring of conditions $C^*(G)$ is isomorphic to the direct limit over the set $\mathcal{S}$ of the cohomology rings $H^*(X_\pi)$.*

De Concini and Procesi proved this theorem in [10] for symmetric spaces. In [8] De Concini noted that their arguments go verbatim for arbitrary spherical homogeneous spaces.

3 Chern classes of reductive groups

3.1 Preliminaries

Reminder about the classical Chern classes. In this paragraph, I will recall one of the classical definitions of the Chern classes, which I will use in the sequel. For more details see [14].

Let M be a compact complex manifold, and let E be a vector bundle of rank d over M. Consider d global sections $s_1, \ldots, s_d$ of E that are C^∞-smooth. Define their *i-th degeneracy locus* as the set of all points $x \in M$ such that the vectors $s_1(x), \ldots, s_{d-i+1}(x)$ are linearly dependent. The homology class of the i-th degeneracy locus is the same for all *generic* choices of the sections $s_1(x), \dots, s_d(x)$ [14]. It is called the *i-th Chern class* of E.

In what follows, I will only consider complex vector bundles that have plenty of algebraic global sections (so that in the definition of the Chern classes, it will be possible to take only algebraic global sections instead of C^∞–smooth ones).

In particular, there is the following way to choose generic global sections. Let $\Gamma(E)$ be a finite-dimensional subspace in the space of all global C^∞-smooth sections of the bundle E. Suppose that at each point $x \in M$ the sections of $\Gamma(E)$ span the fiber of E at the point x. Then there is an open dense subset U in $\Gamma(E)^d$ such that for any collection of global sections $(s_1, \dots, s_d) \subset U$ their i-th degeneracy locus is a representative of the i-th Chern class of E.

I will also use the following classical construction that associates with the subspace $\Gamma(E)$ a map from the variety M to a Grassmannian. Denote by N the dimension of $\Gamma(E)$. Let $\mathrm{G}(N-d, N)$ be the Grassmannian of subspaces of dimension $(N-d)$ in $\Gamma(E)$. One can map M to $\mathrm{G}(N-d, N)$ by assigning to each point $x \in M$ the subspace of all sections from $\Gamma(E)$ that vanish at x. By construction of the map the vector bundle E coincides with the pull-back of the tautological quotient vector bundle over the Grassmannian $\mathrm{G}(N-d, N)$. Recall that the tautological quotient vector bundle over $\mathrm{G}(N-d, N)$ is the quotient of two bundles. The first one is the trivial vector bundle whose fibers are isomorphic to $\Gamma(E)$, and the second is the tautological vector bundle whose fiber at a point $\Lambda \in \mathrm{G}(N-d, N)$ is isomorphic to the corresponding subspace Λ of dimension $N-d$ in $\Gamma(E)$.

Using the definition of the Chern classes given above, it is easy to check that the i-th Chern class of the tautological quotient vector bundle is the homology class of the following Schubert cycle. Let $\Lambda^1 \subset \dots \subset \Lambda^d \subset \Gamma(E)$ be a partial flag such that dim $\Lambda^j = j$. In the sequel, by a partial flag I will always mean a partial flag of this type. The *i-th Schubert cycle C_i* corresponding to such a flag consists of all points $\Lambda \in \mathrm{G}(N-d, N)$ such that the subspaces Λ and Λ^{d-i+1} have nonzero intersection.

The following proposition relates the Schubert cycles C_i to the Chern classes of E.

Proposition 3.1. [14] *Let $p : M \to \mathrm{G}(N-d, N)$ be the map constructed above, and let C_i be the i-th Schubert cycle corresponding to a generic partial flag in $\Gamma(E)$. Then the i-th Chern class of E coincides with the homology class of the inverse image of C_i under the map p:*

$$c_i(E) = [p^{-1}(C_i)].$$

In particular, this proposition allows to relate the definition of the Chern classes via degeneracy loci to other classical definitions.

In the sequel, the following statement will be used. For any algebraic subvariety $X \subset \mathrm{G}(N-d, N)$, a partial flag can be chosen in such a way that the corresponding Schubert cycle C_i has proper intersection with X. This follows from Kleiman's transversality theorem, since the Grassmannian $\mathrm{G}(N-d, N)$ can be regarded as a homogeneous space under the natural action of the group GL_N. Then any left translate of a Schubert cycle C_i is again a Schubert cycle of the same type.

Equivariant vector bundles. In this paragraph, I will recall the definition and some well-known properties of equivariant vector bundles.

Let E be a vector bundle of rank d over G. Denote by $V_g \subset E$ the fiber of E lying over an element $g \in G$. Assume that the standard action of $G \times G$ on G can be extended linearly to E. More precisely, there exists a homomorphism $A : G \times G \to \mathrm{Aut}(E)$ such that $A(g_1, g_2)$ restricted to the fiber V_g is a linear operator from V_g to $V_{g_1 g g_2^{-1}}$. If these conditions are satisfied, then the vector bundle E is said to be *equivariant* under the action of $G \times G$.

Two equivariant vector bundles E_1 and E_2 are equivalent if there exists an isomorphism between E_1 and E_2 that is compatible with the structure of fiber bundle and with the action of $G \times G$. The following simple and well-known proposition describes equivariant vector bundles on G up to this equivalence relation.

Proposition 3.2. *The classes of equivalent equivariant vector bundles of rank d are in one-to-one correspondence with the linear representations of G of dimension d.*

Indeed, with each representation $\pi : G \to V$ one can associate a bundle E isomorphic to $G \times V$ with the following action of $G \times G$:

$$A(g_1, g_2) : (g, v) \to (g_1 g g_2^{-1}, \pi(g_1)v).$$

Then $A(g, g^{-1})$ stabilizes the identity element $e \in G$ and acts on the fiber $V_e = V$ by means of the operator $\pi(g)$.

E.g. the adjoint representation of G on the Lie algebra $\mathfrak{g} = \mathrm{T}G_e$ corresponds to the tangent bundle $\mathrm{T}G$ on G. This example will be important in the sequel.

Among all algebraic global sections of an equivariant bundle E there are two distinguished subspaces, namely, the subspaces of left- and right-invariant sections. They consists of sections that are invariant under the action of the subgroups $G \times e$ and $e \times G$, respectively. Both spaces can be canonically identified with the vector space V. Indeed, any vector $X \in V$ defines a right-invariant section $v_r(g) = (g, X)$. Then it is easy to see that any left-invariant section v_l is given by the formula $v_l(g) = (g, \pi(g)Y)$ for $Y \in V$.

Denote by $\Gamma(E)$ the space of all global sections of E that are obtained as sums of left- and right-invariant sections. Let us find the dimension of the vector space $\Gamma(E)$. Clearly, if the representation π does not contain any trivial sub-representations, then $\Gamma(E)$ is canonically isomorphic to the direct sum of two copies of V. Otherwise, let $C \subset V$ be the maximal trivial sub-representation. Embed C to $V \oplus V$ diagonally, i.e. $v \in C$ goes to (v, v). It is easy to see that $\Gamma(E)$ as a G-module is isomorphic to the quotient space $(V \oplus V)/C$. Denote by c the dimension of C. Then the dimension of $\Gamma(E)$ is equal to $2d - c$.

3.2 Chern classes with values in the ring of conditions

In this subsection, I define Chern classes of equivariant vector bundles over G. These Chern classes are elements of the ring of conditions $C^*(G)$. Unlike the usual Chern classes in the compact situation, they measure the complexity of the action of $G \times G$ but not the topological complexity (topologically any $G \times G$–equivariant vector bundle over G is trivial). While the definition of these classes does not use any compactification it turns out that they are related to the usual Chern classes of certain vector bundles over equivariant compactifications of G.

Throughout this subsection, E denotes the equivariant vector bundle over G of rank d corresponding to a representation $\pi : G \to GL(V)$. In the subsequent sections, I will only use the Chern classes of the tangent bundle.

Definition of the Chern classes. An equivariant vector bundle E has a special class $\Gamma(E)$ of algebraic global sections. It consists of all global sections that can be represented as sums of left- and right-invariant sections.

Example 1. If $E = \mathrm{T}G$ is the tangent bundle, then $\Gamma(E)$ is a very natural class of global sections. It consists of all vector fields coming from the standard action of $G \times G$ on G. Namely, with any element $(X, Y) \in \mathfrak{g} \oplus \mathfrak{g}$ one can associate a vector field $v \in \Gamma(E)$ as follows:

$$v(x) = \frac{d}{dt}\Big|_{t=0} [e^{tX} x e^{-tY}] = Xx - xY.$$

This example suggests that one represent elements of $\Gamma(E)$ not as sums but as differences of left- and right-invariant sections.

The space $\Gamma(E)$ can be employed to define Chern classes of E as usual. Take d generic sections $v_1, \ldots, v_d \in \Gamma(E)$. Then the i-th Chern class is the i-th degeneracy locus of these sections. More precisely, the i-th Chern class $S_i(E) \subset G$ consists of all points $g \in G$ such that the first $d-i+1$ sections $v_1(g), \ldots, v_{d-i+1}(g)$ taken at g are linearly dependent. This definition almost repeats one of the classical definitions of the Chern classes in the compact setting (see Subsection 3.1). The only difference is that global sections used in this definition are not generic in the space of all sections. They are generic sections of the special subspace $\Gamma(E)$. If one drops this restriction and applies the same definition, then the result will be trivial, since the bundle E is topologically trivial. In some sense, the Chern classes will sit at infinity in this case (the precise meaning will become clear from the second part of this subsection). The purpose of my definition is to pull them back to the finite part.

Thus for each $i = 1, \ldots, d$ we get a family $\mathcal{S}_i(E)$ of algebraic subvarieties $S_i(E)$ parameterized by collections of $d - i + 1$ elements from $\Gamma(E)$. In the compact situation, all generic members of an analogous family represent the same class in the cohomology ring. The same is true here, if one uses the ring of conditions as an analog of the cohomology ring in the noncompact setting.

Lemma 3.3. *For all collections $v_1, \ldots, v_{d-i+1}$ belonging to some open dense subset of $(\Gamma(E))^{d-i+1}$ the class of the corresponding subvariety $S_i(E)$ in the ring of conditions $C^*(G)$ is the same.*

The lemma implies that the family $\mathcal{S}_i(E)$ parameterized by elements of $(\Gamma(E))^{d-i+1}$ provides a well-defined class $[S_i(E)]$ in the ring of conditions $C(G)$.

Definition 1. *The class* $[S_i(E)] \in C^*(G)$ *defined by the family* $\mathcal{S}_i(E)$ *is called the* i*-th Chern class of a vector bundle* E *with value in the ring of conditions.*

Before proving the lemma let me give another description of the Chern classes $[S_i(E)]$.

Maps to Grassmannians. In this paragraph, I apply the classical construction discussed in Subsection 3.1 to define a map from the group G to the Grassmannian $G(d-c, \Gamma(E))$ of subspaces of dimension $(d-c)$ in the space $\Gamma(E)$. Recall that c is the dimension of the maximal trivial sub-representation of V, and the dimension of $\Gamma(E)$ is $2d-c$ (see the end of Subsection 3.1).

Note that the global sections from the subspace $\Gamma(E)$ span the fiber of E at each point of G. Hence, one can define a map φ_E from G to the Grassmannian $G(d-c, \Gamma(E))$ as follows. A point $g \in G$ gets mapped to the subspace $\Lambda_g \subset \Gamma(E)$ spanned by all global sections that vanish at g. Clearly, the dimension of Λ_g equals to $(\dim \Gamma(E) - d) = (d-c)$ for all $g \in G$. We get the map

$$\varphi_E : G \to \mathrm{G}(d-c, \Gamma(E)); \quad \varphi_E : g \mapsto \Lambda_g.$$

The subspace Λ_g can be alternatively described using the graph of the operator $\pi(g)$ in $V \oplus V$. Namely, it is easy to check that $\Lambda_g = \{(X, \pi(g)X), X \in V\}/C$. Then φ_E comes from the natural map assigning to the operator $\pi(g)$ on V its graph in $V \oplus V$.

Clearly, the pull-back of the tautological quotient vector bundle over $\mathrm{G}(d, \Gamma(E))$ is isomorphic to E. Hence, the Chern class $S_i(E)$ constructed via elements $v_1, \ldots, v_d$ is the inverse image of the Schubert cycle C_i corresponding to the partial flag $\langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \ldots \subset \langle v_1, \ldots, v_d \rangle \subset \Gamma(E)$ (see Subsection 3.1). Here $\langle v_1, \ldots, v_i \rangle$ denotes the subspace of $\Gamma(E)$ spanned by the vectors $v_1, \ldots, v_i$.

Remark 3.4. This gives the following equivalent definition of $S_i(E)$. The Chern class $S_i(E)$ consists of all elements $g \in G$ such that the graph of the operator $\pi(g)$ in $V \oplus V$ has a nontrivial intersection with a generic subspace of dimension $d-i+1$ in $V \oplus V$.

In particular, if the representation $\pi : G \to GL(V)$ corresponding to a vector bundle E has a nontrivial kernel, then the $S_i(E)$ are invariant under left and right multiplications by the elements of the kernel (since this is already true for the preimage $\varphi_E^{-1}(\Lambda)$ of any point $\Lambda \in \varphi_E(G)$). E.g. the Chern classes $S_i(\mathrm{T}G)$ are invariant under multiplication by the elements of the center of G.

We can now relate the Chern classes $S_i(E)$ to the usual Chern classes of a vector bundle over a compact variety.

Denote by X_E the closure of $\varphi_E(G)$ in the Grassmannian $G(d-c, \Gamma(E))$, and denote by E_X the restriction of the tautological quotient vector bundle to X_E. We get a vector bundle on a compact variety. The i-th Chern class of E_X is the homology class of $C_i \cap X_E$ for a generic Schubert cycle C_i (see Proposition 3.1). By Kleiman's transversality theorem applied to the Grassmannian $\mathrm{G}(d-c, \Gamma(E))$ (see Subsection 3.1), a generic Schubert cycle C_i has a proper intersection with the boundary divisor $X_E \backslash \varphi_E(G)$. Hence, there is the following relation between the Chern classes of E_X and generic members of the family $\mathcal{S}_i(E)$.

Proposition 3.5. *For a generic $S_i(E)$ the homology class of the closure of $\varphi_E(S_i(E))$ in X_E coincides with the i-th Chern class of E_X.*

Thus the Chern classes $[S_i(E)]$ can be described via the usual Chern classes of the bundle E_X over the compactification X_E.

Let us study the variety X_E in more detail. It is a $G \times G$–equivariant compactification of the group $\varphi_E(G)$. Indeed, the action of $G \times G$ on $\varphi_E(G)$ can be extended to the Grassmannian $\mathrm{G}(d, \Gamma(E))$ as follows. Identify $\Gamma(E)$ with $(V \oplus V)/C$ (see the end of Subsection 3.1). The doubled group $G \times G$ acts on $V \oplus V$ by means of the representation $\pi \oplus \pi$, i.e. $(g_1, g_2)(v_1, v_2) = (g_1 v_1, g_2 v_2)$ for $g_1, g_2 \in G, v_1, v_2 \in V$. The subspace $C \subset V \oplus V$ is invariant under this action. Hence, the group $G \times G$ acts on $\Gamma(E)$. This action provides an action of $G \times G$ on the Grassmannian $\mathrm{G}(d - c, \Gamma(E))$. Clearly, the subvariety X_E is invariant under this action.

Example 1 (Demazure embedding). Let G be a group of adjoint type, and let π be its adjoint representation on the Lie algebra $\mathfrak{g}$. The corresponding vector bundle E coincides with the tangent bundle of G. The corresponding map $\varphi_E : G \to \mathrm{G}(n, \mathfrak{g} \oplus \mathfrak{g})$ coincides with the embedding constructed by Demazure [9]. The Demazure map takes an element $g \in G$ to the Lie subalgebra $\mathfrak{g}_g = \{(gXg^{-1}, X), X \in \mathfrak{g}\} \subset \mathfrak{g} \oplus \mathfrak{g}$. Clearly, the Demazure map provides an embedding of G into $\mathrm{G}(n, \mathfrak{g} \oplus \mathfrak{g})$.

It is easy to check that the Lie subalgebra $\mathfrak{g}_g$ is the Lie algebra of the stabilizer of an element $g \in G$ under the standard action of $G \times G$. Thus for any $A \in \mathfrak{g}_g$ the corresponding vector field vanishes at g, and the Demazure embedding coincides with φ_E. The compactification X_E in this case is isomorphic to the wonderful compactification X_{can} of the group G [9]. In particular, it is smooth.

Definition 2. *Let G and E be as in Example 1. The restriction of the tautological quotient vector bundle to $X_E \simeq X_{can}$ is called the Demazure bundle and is denoted by V_{can}.*

If E is the tangent bundle, then Proposition 3.5 implies that the Chern class $S_i(E)$ is the inverse image of the usual i-th Chern class of the Demazure bundle. The Demazure bundle is considered in [5], where it is related to the tangent bundles of regular compactifications of the group G.

Example 2. a) Let G be $GL(V)$ and let π be its tautological representation on the space V of dimension d. Then φ_E is an embedding of $GL(V)$ into the Grassmannian $\mathrm{G}(d, 2d)$. Notice that the dimensions of both varieties are the same. Hence, the compactification X_E coincides with $\mathrm{G}(d, 2d)$.

b) Take $SL(V)$ instead of $GL(V)$ in the previous example. Its compactification X_E is a hypersurface in the Grassmannian $\mathrm{G}(d, 2d)$ which can be described as a hyperplane section of the Grassmannian in the Plücker embedding. Consider the Plücker embedding $p : \mathrm{G}(d, 2d) \to \mathbb{P}(\Lambda^d(V_1 \oplus V_2))$, where V_1 and V_2 are two copies of V. Then $p(X_E)$ is a special hyperplane section of $p(\mathrm{G}(d,2d))$. Namely, the decomposition $V_1 \oplus V_2$ yields a decomposition of $\Lambda^d(V_1 \oplus V_2)$ into a direct sum. This sum contains two one-dimensional components $p(V_1)$ and $p(V_2)$ (which are considered as lines in $\Lambda^d(V_1 \oplus V_2)$). In particular, for any vector in $\Lambda^d(V_1 \oplus V_2)$ it makes sense to speak of its projections to $p(V_1)$ and $p(V_2)$. On V_1 and V_2 there are two special n-forms, preserved by $SL(V)$. These forms give rise to two 1-forms l_1 and l_2 on $p(V_1)$ and $p(V_2)$, respectively. Consider the hyperplane H in $\Lambda^d(V_1 \oplus V_2)$ consisting of all vectors v such that the functionals l_1 and l_2 take the same values on the projections of v to $p(V_1)$ and $p(V_2)$, respectively. Then it is easy to check that $p(X_E) = p(\mathrm{G}(d,2d)) \cap \mathbb{P}(H)$.

In the next section, I will be concerned with the case when $E = \mathrm{T}G$ is the tangent bundle. In this case, the vector bundle E_X is closely related to the tangent bundles of regular compactifications of the group G. Let us discuss this case in more detail.

Example 3. This example is a slightly more general version of Example 1. Let $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{c}$ be the decomposition of the Lie algebra $\mathfrak{g}$ into the direct sum of the semisimple and the central subalgebras, respectively. Denote by c the dimension of the center $\mathfrak{c}$. Let $E = \mathrm{T}G$ be the tangent bundle on G. Then φ_E maps G to the Grassmannian $G(n-c, (\mathfrak{g} \oplus \mathfrak{g})/\mathfrak{c})$. It is easy to show that the image of the map φ_E coincides with the adjoint group of G and the image contains only subspaces that belong to $(\mathfrak{g}' \oplus \mathfrak{g}') \subset (\mathfrak{g} \oplus \mathfrak{g})/\mathfrak{c}$. Comparing this with Example 1, one can easily see that X_E is isomorphic to the wonderful compactification X_{can} of the adjoint group of G.

In this case, the bundle E_X is the direct sum of the Demazure bundle and the trivial vector bundle of rank c. Indeed, for any subspace $\Lambda_x \in X_E \simeq X_{can} \subset \mathrm{G}(n-c, \Gamma(E))$ its intersection with the subspace $\mathfrak{c}^- = \{(c,-c), c \in \mathfrak{c}\} \subset \Gamma(E)$ is trivial. Hence, the quotient space $\Gamma(E)/\Lambda_x$ coincides with the direct sum $((\mathfrak{g}' \oplus \mathfrak{g}')/\Lambda_x) \oplus \mathfrak{c}^-$.

Proof of Lemma 3.3 The proof of Lemma 3.3 relies on the following fact. Let Y_1 and Y_2 be two subvarieties of codimension i in the group G. Using Kleiman's transversality theorem and continuity arguments, it is easy to show that Y_1 and Y_2 represent the same class in the ring of conditions $C^*(G)$ if there exists an equivariant compactification X of the group G such that the closures of Y_1, Y_2 in X have proper intersections with all $G \times G$–orbits (see [10] for the proof).

In particular, to prove Lemma 3.3 it is enough to produce an equivariant compactification X such that the closure of a generic $S_i(E)$ has proper intersections with all $G \times G$–orbits in X. I claim that the compactification X_E discussed in the previous paragraph (see Proposition 3.5) satisfies this condition.

Indeed, the closure of any $S_i(E)$ in X_E coincides with the intersection of X_E with the Schubert cycle C_i corresponding to a partial flag in $\Gamma(E)$. By Kleiman's transversality theorem applied to the Grassmannian $\mathrm{G}(d-c, \Gamma(E))$ (see Subsection 3.1), a partial flag can be chosen in such a way that the corresponding Schubert cycle has proper intersections with all $G \times G$–orbits in X_E. All partial flags with such property form an open dense subset in the space of all partial flags. Hence, for generic flags the corresponding subvarieties S_i represent the same class in the ring of conditions.

In the sequel, $S_i(E)$ will denote any subvariety of the family $\mathcal{S}_i(E)$ whose class in the ring of conditions coincides with the Chern class $[S_i(E)]$.

Remark. Recall that the ring of conditions $C^*(G)$ can be identified with the direct limit of cohomology rings of equivariant compactifications of G (see Theorem 2.3). It follows that under this identification the Chern class $[S_i(E)] \in C^*(G)$ corresponds to an element in the cohomology ring of the compactification X_E. In particular for an adjoint group G, the Chern class $[S_i(\mathrm{T}X)]$ of the tangent bundle corresponds to some cohomology class of the wonderful compactification of G.

Properties of the Chern classes of reductive groups. The next lemma computes the dimensions of the Chern classes. It also shows that if G acts on V without an open dense orbit, then the higher Chern classes automatically vanish.

For any representation $\pi : G \to GL(V)$, there exists an open dense G–invariant subset in V such that the stabilizers of any two elements from this subset are conjugate subgroups of G (see [24]). In particular, all elements from this subset have isomorphic G–orbits. Such orbits are called *principal*. Denote by $d(\pi)$ the dimension of a principal orbit of G in V. If G has an open dense orbit in V, then $d(\pi) = d$. In my main example, when π is the adjoint representation, $d(\pi) = n - k$.

Lemma 3.6. *If $i > d(\pi)$, then $S_i(E)$ is empty, and if $i \leq d(\pi)$ then the dimension of $S_i(E)$ is equal to $n - i$.*

Proof. Recall that $S_i(E)$ is the inverse image of C_i under the map $\varphi_E : G \to \mathrm{G}(d{-}c, \Gamma(E))$. Here C_i is the i-th Schubert cycle corresponding to a generic partial flag in $\Gamma(E)$. The codimension of C_i in the Grassmannian $\mathrm{G}(d - c, \Gamma(E))$ is equal to i. Hence, by Kleiman's transversality theorem applied to $\mathrm{G}(d - c, \Gamma(E))$ (see Subsection 3.1), the intersection $C_i \cap \varphi_E(G)$ is either empty or proper and has codimension i in $\varphi_E(G)$. Then $S_i(E) = \varphi_E^{-1}(C_i \cap \varphi_E(G))$ is either empty or has codimension i in G, because all fibers of the map φ_E are isomorphic to each other (each of them is isomorphic to the kernel of π). It remains to find out all i for which $S_i(E)$ is empty.

By Remark 3.4, the Chern class $S_i(E)$ consists of all elements $g \in G$ such that the graph $\Gamma_g = \{(v, \pi(g)v), v \in V\} \subset V \oplus V$ of $\pi(g)$ has a nontrivial intersection with a generic subspace Λ^{d-i+1} of dimension $d - i + 1$ in $V \oplus V$. For all $g \in S_i(E) \setminus S_{i+1}(E)$ the intersection $\Gamma_g \cap \Lambda^{d-i+1}$ has dimension 1. Indeed, if $\dim(\Gamma_g \cap \Lambda^{d-i+1}) \geq 2$, then $\dim(\Gamma_g \cap \Lambda^{d-i}) \geq 1$ (since the subspace $\Lambda^{d-i} \subset \Lambda^{d-i+1}$ has codimension one in Λ^{d-i+1}), and g belongs to $S_{i+1}(E)$. Hence, there is a well-defined map

$$p : S_i(E) \setminus S_{i+1}(E) \to \mathbb{P}(D \cap \Lambda^{d-i+1}); \quad p : g \mapsto \mathbb{P}(\Gamma_g \cap \Lambda^{d-i+1}).$$

Here $D \subset V \oplus V$ is the union of all graphs Γ_g for $g \in G$. In particular, the Chern class $S_i(E)$ is nonempty if and only if $\mathbb{P}(D \cap \Lambda^{d-i+1})$ is nonempty.

We now estimate the dimension of $D \cap \Lambda^{d-i+1}$. Since D is not a variety, it is more convenient to take its Zariski closure $\overline{D}$. The subvariety $\overline{D}$ is the closure of the image of the following morphism:

$$F : G \times V \to V \times V; \quad F : (g, v) \mapsto (v, \pi(g)v).$$

The source space $G \times V$ is an irreducible variety of dimension $n + d$, and the general fibers of F are isomorphic to the principal stabilizers, of dimension $n - d(\pi)$. Hence $\dim \overline{D} = d + d(\pi)$, that is, $\overline{D}$ has codimension $d - d(\pi)$.

Next, observe that D is a constructible set, invariant under scalar multiplication. Hence it contains a dense open subset (also invariant under scalar multiplication) of the irreducible variety $\overline{D}$. Thus a general vector space Λ^{d-i+1} satisfies $\dim(\overline{D} \cap \Lambda^{d-i+1}) = d(\pi) - i + 1$, if $i \leq d(\pi)$, and $D \cap \Lambda^{d-i+1}$ is dense in this intersection. In particular, if $i = d(\pi)$, then $D \cap \Lambda^{d-i+1}$ consists of several lines whose number is equal to the degree of $\overline{D}$. If $i > d(\pi)$, then $\overline{D} \cap \Lambda^{d-i+1}$ contains only the origin. It follows that if $i > d(\pi)$, then $S_i(E)$ is empty.

□

This proof also implies the following corollary. Denote by $H \subset G$ the stabilizer of an element in a principal orbit of G in V. The subgroup H is defined up to conjugation so its class in the ring of conditions is well-defined.

Corollary 3.7. *An open dense subset of the subvariety $S_i(E)$ admits* almost *a fibration whose fibers are translates of H. Here* almost *means that the intersection of different fibers always lies in $S_{i+1}(E) \subset S_i(E)$. In particular, the last Chern class $S_{d(\pi)}(E)$ admits a true fibration and coincides with the disjoint union of several translates of H. Their number equals to the degree of a generic principal orbit of G in V.*

The last statement follows from the fact that the degree of D in $V \oplus V$ (see the proof of Lemma 3.6) is equal to the degree of a generic principal orbit of G in V.

In particular, let E be the tangent bundle. Then the stabilizer of a generic element in $\mathfrak{g}$ is a maximal torus in G. Hence, the last Chern class $S_{n-k}(\mathrm{T}G)$ is the union of several translates of a maximal torus. The number of translates is the cardinality of the Weyl group (the degree of a general orbit in the adjoint representation).

3.3 The first and the last Chern classes

Throughout the rest of the paper, I will only consider the Chern classes $S_i = S_i(\mathrm{T}G)$ of the tangent bundle unless otherwise stated. Theorem 1.1 expresses the Euler characteristic of a complete intersection via the intersection indices of the Chern classes S_i with generic hyperplane sections. The question is how to compute these indices. If $[S_i]$ is a linear combination of complete intersections of *generic* hyperplane sections corresponding to some representations of G, then the answer to this question is given by the Brion–Kazarnovskii formula. A hyperplane section corresponding to the representation π is called *generic* if its closure in the compactification X_π has proper intersections with all $G \times G$–orbits in X_π.

In this subsection, I describe S_1 as a generic hyperplane section. The description follows from a result of Rittatore [25]. One can also compute the intersection indices with the last Chern class S_{n-k}, because S_{n-k} is the union of translates of a maximal torus (see Corollary 3.7). However, it seems that in general the Chern class S_i, for $i \neq 1$, is not a sum of complete intersections. E.g. I can show that for $G = SL_3(\mathbb{C})$ the Chern class $[S_3]$ does not lie in the subring of $C^*(G)$ generated by the classes of hypersurfaces.

Description of S_1. The result of Rittatore for the first Chern class of regular compactifications (see [25], Proposition 4) implies that the class $[S_1]$ in the ring of conditions can be represented by the doubled sum of the closures of all codimension one Bruhat cells in G. Below I will deduce this description directly from the definition of S_1.

It is easy to show that $S_1 \subset G$ is given by the equation $\det(\mathrm{Ad}(g) - A) = 0$ for a generic $A \in \mathrm{End}(\mathfrak{g})$. Indeed, the first Chern class $S_1(E)$ of any equivariant vector bundle E over G consists of elements $g \in G$ such that the graph of the operator $\pi(g)$ in $V \oplus V$ has a nontrivial intersection with a generic subspace of dimension n in $V \oplus V$ (see Remark 3.4). As a generic subspace, one can take the graph of a generic operator A on V. Then the graphs of operators $\pi(g)$ and A have a nonzero intersection if and only if the kernel of the operator $\pi(g) - A$ is nonzero.

The function $\det(\mathrm{Ad}(g) - A)$ is a linear combination of matrix coefficients corresponding to all exterior powers of the adjoint representation. Hence, the equation of S_1 is the equation of a hyperplane section corresponding to the sum of all exterior powers of the adjoint representation. Denote this representation by σ. It is easy to check that the weight polytope P_σ coincides with the weight polytope of the irreducible representation θ with the highest weight 2ρ (here ρ is the half sum of all positive roots, or equivalently the sum of all fundamental weights). It remains to prove that S_1 is generic, which means that the closure of S_1 in X_σ intersects all $G \times G$–orbits along subvarieties of codimension one. The proof of Lemma 3.3 implies that this is true for the wonderful compactification, and the normalization of X_σ is the wonderful compactification by Theorem 2.1 (since $P_\theta = P_\sigma$).

It is now easy to show that the doubled sum of the closures of all codimension one Bruhat cells in G is equivalent to S_1. This is because the closures of codimension one Bruhat cells are generic hyperplane sections corresponding to the irreducible representations with fundamental highest weights.

Description of S_{n-k}. By Corollary 3.7 the last Chern class S_{n-k} is the disjoint union of translates of a maximal torus. Their number is equal to the degree of a generic adjoint orbit in $\mathfrak{g}$. The latter is equal to the order of the Weyl group W. Denote by $[T]$ the class of a maximal torus in the ring of conditions $C^*(G)$. Then the following identity holds in $C^*(G)$:

$$[S_{n-k}] = |W|[T].$$

The degree of $\pi(T)$ can be computed using the formula of D.Bernstein, Khovanskii and Koushnirenko [18].

3.4 Examples

$\mathbf{G} = \mathbf{SL_2}(\mathbb{C})$. Consider the tautological embedding of G, namely, $G = \{(a,b,c,d) \in \mathbb{C}^4 : ad - bc = 1\}$. Since the dimension of G is 3 and the rank is 1, by Lemma 3.6 we get that there are only two nontrivial Chern classes: S_1 and S_2. Let us apply the results of the preceding subsection to find them. The first Chern class S_1 is a generic hyperplane section corresponding to the second symmetric power of the tautological representation, i.e. to the representation $\theta : SL_2(\mathbb{C}) \to SO_3(\mathbb{C})$. In other words, it is the intersection of $SL_2(\mathbb{C})$ with a generic quadric in $\mathbb{C}^4$. The second Chern class S_2 (which is also the last one in this case) is the union of two translates of a maximal torus (or the intersection of S_1 with a hyperplane in $\mathbb{C}^4$).

Let π be a faithful representation of $SL_2(\mathbb{C})$. It is a direct sum of irreducible representations. Any irreducible representation of $SL_2(\mathbb{C})$ is isomorphic to the i-th symmetric power of the tautological representation for some i. Its weight polytope is the line segment $[-i, i]$. Hence the weight polytope of π is the line segment $[-n, n]$ where n is the greatest exponent of symmetric powers occurring in π. Then the matrix coefficients of π are polynomials in a, b, c, d of degree n. In this case, it is easy to compute the degrees of subvarieties $\pi(G)$, $\pi(S_1)$ and $\pi(S_2)$ by the Bezout theorem. Then deg $\pi(G) = 2n^3$, deg $\pi(S_1) = 4n^2$, deg $\pi(S_2) = 4n$. Also, if one takes another faithful representation σ with the weight polytope $[-m, m]$, then the intersection index of S_1 with two generic hyperplane sections corresponding to π and σ, equals to $4mn$.

Since by Theorem 1.1 the Euler characteristic $\chi(\pi)$ of a generic hyperplane section is equal to deg $\pi(G) -$ deg $\pi(S_1) +$ deg $\pi(S_2)$, we get

$$\chi(\pi) = 2n^3 - 4n^2 + 4n.$$

This answer was first obtained by Kaveh who used different methods [16].

If π is not faithful, i.e. $\pi(SL_2(\mathbb{C})) = SO_3(\mathbb{C})$, consider π as a representation of $SO_3(\mathbb{C})$. Then $\chi(\pi)$ is two times smaller and equals to $n^3 - 2n^2 + 2n$.

Apply Theorem 1.1 to a curve C that is the complete intersection of two generic hyperplane sections corresponding to the representations π and σ. Then

$$\chi(C) = H_\pi \cdot H_\sigma \cdot H_\theta - H_\pi \cdot H_\sigma \cdot (H_\pi + H_\sigma) = -2mn(m + n - 2).$$

$\mathbf{G} = (\mathbb{C}^*)^\mathbf{n}$ is a complex torus. In this case, all left-invariant vector fields are also right-invariant since the group is commutative. Hence, they are linearly independent at any point of $G = (\mathbb{C}^*)^n$ as long as their values at the identity are linearly independent. It follows that all subvarieties S_i are empty, and all the Chern classes vanish. Then Theorem 1 coincides with a theorem of D.Bernstein and Khovanskii [18].

4 Chern classes of regular compactifications and proof of Theorem 1.1

4.1 Preliminaries

Chern classes of the tangent bundle. In this paragraph, I explain a method from [11], which in some cases allows to find the Chern classes of smooth varieties.

Let X be a smooth complex variety of dimension n, and let $D \subset X$ be a divisor. Suppose that D is the union of l smooth irreducible hypersurfaces $D_1, \ldots, D_l$ with normal crossings. One can relate the tangent bundle $\mathrm{T}X$ of X to the logarithmic tangent bundle, consisting of those vector fields that preserve the divisor D.

Let $L_X(D_1), \ldots, L_X(D_l)$ be the line bundles over X associated with the hypersurfaces $D_1, \ldots, D_l$, respectively. I.e. the first Chern class of the bundle $L_X(D_i)$ is the homology class of D_i. One can also associate with D the logarithmic tangent bundle $V_X(D)$. It is a holomorphic vector bundle over X of rank n that is uniquely defined by the following property. The holomorphic sections of $V_X(D)$ over an open subset $U \subset X$ consist of all holomorphic vector fields $v(x)$ on U such that $v(x)$ restricted to $U \cap D_i$ is tangent to the hypersurface D_i for any i. The precise definition is as follows. Cover X by local charts. If a chart intersects the divisors $D_{i_1}, \ldots, D_{i_k}$ choose local coordinates $x_1, \ldots, x_n$ such that the equation of D_{i_j} in these coordinates is $x_j = 0$. Then V_X is given by the collection of trivial vector bundles spanned by the vector fields $x_1\frac{\partial}{\partial x_1}, \ldots, x_k\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_{k+1}}, \ldots, \frac{\partial}{\partial x_n}$ over each chart with the natural transition operators.

For a vector bundle E, denote by $\mathcal{O}(E)$ the sheaf of its holomorphic sections.

Proposition 4.1. [11] *There is an exact sequence of coherent sheaves*

$$0 \to \mathcal{O}(V_X(D)) \to \mathcal{O}(\mathrm{T}X) \to \bigoplus_{i=1}^{l} \mathcal{O}(L_X(D_i)) \otimes_{\mathcal{O}_X} \mathcal{O}_{D_i} \to 0.$$

In particular, the tangent bundle $\mathrm{T}X$ *has the same Chern classes as the direct sum of the bundle* $V_X(D)$ *with* $L_X(D_1), \ldots, L_X(D_l)$.

Proposition 4.1 gives the answer for the Chern classes of X, when the Chern classes of $V_X(D)$ are known. In particular, this is the case when X is a smooth toric variety, and $D = X \setminus (\mathbb{C}^*)^n$ is the divisor at infinity. In this case, the vector bundle $V_X(D)$ is trivial, and the Chern classes of $\mathrm{T}X$ can be found explicitly. This was done by Ehlers [11]. A more general class of examples is given by regular compactifications of reductive groups (see the next paragraph for the definition) and, more generally, of arbitrary spherical homogeneous spaces (see Section 5). In this case, the vector bundle $V_X(D)$ is no longer trivial but still has a nice description, which is due to Brion [5]. I recall his result in Subsection 4.2 and use it to prove Theorem 1.1.

Regular compactifications. In this paragraph, I will define the notion of regular compactifications of reductive groups following [6]. Let X be a smooth $G\times G$–equivariant compactification of a connected reductive group G of dimension n. Denote by $\mathcal{O}_1,\ldots,\mathcal{O}_l$ the orbits of codimension one in X. Then the complement $X\setminus G$ to the open orbit is the union of the closures $\overline{\mathcal{O}}_1,\ldots,\overline{\mathcal{O}}_l$ of codimension one orbits.

Definition 3. *A smooth $G\times G$–equivariant compactification X is called regular if the following three conditions are satisfied.*

(1)The hypersurfaces $\overline{\mathcal{O}}_1,\ldots,\overline{\mathcal{O}}_l$ are smooth and intersect each other transversally.

(2)The closure of any $G\times G$–orbit in $X\setminus G$ coincides with the intersection of those hypersurfaces $\overline{\mathcal{O}}_1,\ldots,\overline{\mathcal{O}}_l$ that contain it.

(3) For any point $x\in X$ and its $G\times G$–orbit $\mathcal{O}_x\subset X$, the stabilizer $(G\times G)_x\subset G\times G$ acts with a dense orbit on the normal space $T_xX/T_x\mathcal{O}_x$ to the orbit.

This definition was introduced by E.Bifet, De Concini and Procesi in a more general setting ([2], see also Section 5).

If G is a complex torus, then the regularity of X is just equivalent to the smoothness. However, for other reductive groups, there exist compactifications that are smooth but not regular. In particular, it follows from Proposition 4.2 below that the compactification X_π associated with a representation $\pi: G\to GL(V)$ (see Section 2) is regular if and only it is smooth and none of the vertices of the weight polytope of π lies on the walls of the Weyl chambers.

Regular compactifications of reductive groups generalize smooth toric varieties and retain many nice properties of the latter. E.g. any regular compactification X can be covered by affine charts $X_\alpha\simeq\mathbb{C}^n$ in such a way that only k hypersurfaces $\overline{\mathcal{O}}_{i_1},\ldots,\overline{\mathcal{O}}_{i_k}$ intersect X_α, and intersections $\overline{\mathcal{O}}_{i_1}\cap X_\alpha,\ldots,\overline{\mathcal{O}}_{i_k}\cap X_\alpha$ are k coordinate hyperplanes in X_α [9, 6]. Here k denotes the rank of G. In particular, all $G\times G$-orbits in X have codimension at most k, and all closed orbits have codimension k.

If G is of adjoint type, then it has the wonderful compactification X_{can}, which is regular. This example is crucial for the study of the other regular compactifications.

For arbitrary reductive group G, denote by X_{can} the wonderful compactification of the adjoint group of G. There is the following criterion of regularity.

Proposition 4.2. [6] *Let X be a smooth $G\times G$–equivariant compactification of G. Then the condition that X is regular is equivalent to the existence of a $G\times G$–equivariant map from X to X_{can}.*

E.g. if G is a complex torus, then the latter condition is always satisfied because X_{can} is a point in this case.

Thus the set of regular compactifications of G consists of all smooth $G\times G$–equivariant compactifications lying over X_{can}. In particular, for reductive groups of adjoint type the wonderful compactification is the minimal regular compactification.

4.2 Demazure bundle and the Chern classes of regular compactifications

In this subsection, I state a formula for the Chern classes of regular compactifications of reductive groups. It follows from a more general result proved for arbitrary toroidal spherical varieties by Brion [5]. This formula gives a description of the Chern classes in terms of two different collections of subvarieties. The first collection is given by the Chern classes of G, which are independent of a compactification, and the second is given by the closures of codimension one orbits, which are easy to deal with (in particular, all their intersection indices with other divisors can be computed via the Brion–Kazarnovskii theorem).

Let X be a regular compactification of G, and let $\overline{\mathcal{O}}_1, \dots, \overline{\mathcal{O}}_l$ be the closures of the $G \times G$–orbits of codimension one in X. Then the tangent bundle $\mathrm{T}X$ of X can be described using the Demazure vector bundle V_{can} over the wonderful compactification X_{can} (see Example 1 from Subsection 3.2) and the line bundles corresponding to the hypersurfaces $\overline{\mathcal{O}}_i$.

Let $L(\overline{\mathcal{O}}_1), \dots, L(\overline{\mathcal{O}}_l)$ be the line bundles over X associated with the hypersurfaces $\overline{\mathcal{O}}_1, \dots, \overline{\mathcal{O}}_l$, respectively. Let $p : X \to X_{can}$ be the canonical map from Proposition 4.2, and let $p^*(V_{can})$ be the pull-back of the Demazure vector bundle to X. It turns out that $p^*(V_{can})$ coincides up to a trivial summand with the logarithmic tangent bundle corresponding to the boundary divisor $X \setminus G$.

Theorem 4.3. [5] *The tangent bundle* $\mathrm{T}X$ *has the same Chern classes as the direct sum of the pull-back* $p^*(V_{can})$ *with the line bundles* $L(\overline{\mathcal{O}}_1), \dots, L(\overline{\mathcal{O}}_l)$.

In the case when G is a complex torus, Theorem 4.3 was proved by Ehlers [11]. For arbitrary reductive groups, Theorem 4.3 follows from a more general result by Brion ([5], 1.6 Corollary 1).

This theorem implies the following formula for the Chern classes $c_1(X), \dots, c_n(X)$ of the tangent bundle of X. Let $S_i = S_i(TG) \subset G$ for $i = 1, \dots, n-k$ be the Chern classes of the tangent bundle of G defined in the previous section (see Definition 1). Denote by $\overline{S}_i$ the closure of S_i in X. Note that $\overline{S}_i$ has proper intersections with all $G \times G$-orbits in X (since this is already true for the wonderful compactification X_{can}, and X lies over X_{can}).

Corollary 4.4. *The total Chern class* $c(X) = 1 + c_1(X) + \ldots + c_n(X)$ *coincides with the following product:*

$$c(X) = (1 + \overline{S}_1 + \ldots + \overline{S}_{n-k}) \cdot \prod_{i=1}^{l} (1 + \overline{\mathcal{O}}_i).$$

The product in this formula is the intersection product in the (co)homology ring of X.

Below I sketch the proof of Theorem 4.3 following mostly the proofs by Ehlers and Brion. The goal is to explain the main idea of their proofs, which is very transparent, and motivate the definition of the Chern classes S_i. In the torus case, this idea can be extended to a complete elementary proof. For more details see [11] and [5].

Take n generic vector fields $v_1, \dots, v_n$ coming from the action of $G \times G$. It is not hard to show that $v_1, \dots, v_n$ are generic in the space of all C^∞–smooth vector fields on X (it is enough to prove it for each affine chart on X). Hence, their degeneracy loci give Chern classes of X. Note that these fields are not only C^∞–smooth but also algebraic so their degeneracy loci are algebraic subvarieties in X.

The picture is especially simple in the torus case, because in this case $v_1, \dots, v_{n-i+1}$ are linearly dependent precisely on all orbits of codimension greater than or equal to i (since they all belong to the tangent bundle of the orbit) and independent on the other orbits. Hence, the i-th Chern class of X consists of all orbits of codimension at least i.

In the reductive case, the situation is more complicated because the degeneracy loci of $v_1, \dots, v_n$ have nontrivial intersections with the open orbit $G \subset X$. These intersections are exactly the Chern classes S_1,..., S_{n-k} of G. So it seems more convenient to use the method described in Subsection 4.1 (see Proposition 4.1). Namely, consider the logarithmic tangent bundle $V_X = V_X(X \setminus G)$ corresponding to the boundary divisor $X \setminus G = \overline{\mathcal{O}}_1 \cup \ldots \cup \overline{\mathcal{O}}_l$. Recall that c denotes the dimension of the center of G.

Proposition 4.5. *The vector bundle V_X is isomorphic to the direct sum of the pull-back $p^*(V_{can})$ with the trivial vector bundle E^c of rank c.*

Proof. The vector fields coming from the action of $G \times G$ on X are global sections of the bundle V_X, since they are tangent to all $G \times G$–orbits in X. It follows easily from condition (3) in the definition of regular compactifications that these global sections span the fiber of V_X at any point of X. Hence, the map $\varphi_E : G \to G(n - c, (\mathfrak{g} \oplus \mathfrak{g})/\mathfrak{c})$ considered in Example 3 extends to a map $p : X \to G(n - c, (\mathfrak{g} \oplus \mathfrak{g})/\mathfrak{c})$. The rest follows from Example 3. □

Remark 4.6. There is also another construction of the map $p : X \to X_{can}$ by Brion (see [4]).

4.3 Applications

In this subsection, I prove Theorem 1.1 using the formula for the Chern classes of regular compactifications (Corollary 4.4). Then I apply it to compute the Euler characteristic and the genus of a curve in G.

Proof of Theorem 1.1. First, define the notion of *generic* collection of hyperplane sections used in the formulation of Theorem 1.1. A collection of m hyperplane sections $H_1, \dots, H_m$ corresponding to representations $\pi_1, \dots, \pi_m$, respectively, is called *generic*, if there exists a regular compactification X of G such that the closure $\overline{H}_i$ of any hyperplane section H_i is smooth, and all possible intersections of $\overline{H}_1, \dots, \overline{H}_m$ with the closures of $G \times G$–orbits in X are transverse. E.g. one can take the compactification X_π corresponding to the tensor product π of the representations $\pi_0, \pi_1, \dots, \pi_m$, where π_0 is any irreducible representation with a strictly dominant highest weight. Then it is not hard to show that the set of all generic collections (with respect to the compactification X_π) is an open dense subset in the space of all collections.

So the closure $Y = \overline{C}$ of $C = H_1 \cap \ldots \cap H_m$ in X is the transverse intersection of smooth hypersurfaces. In particular, Y is smooth, and its normal bundle N_Y in X is the direct sum of m line bundles corresponding to the hypersurfaces $\overline{H}_i$. The analogous statement is true for any subvariety of the form $Y \cap \overline{\mathcal{O}}_I$, where $I = \{i_1, \ldots, i_p\}$ is a subset of $\{1, \ldots, l\}$ and $\overline{\mathcal{O}}_I = \overline{\mathcal{O}}_{i_1} \cap \ldots \cap \overline{\mathcal{O}}_{i_p}$. Let us find the Euler characteristic of $Y \cap \overline{\mathcal{O}}_I$ using the classical adjunction formula. Denote by $J = \{1, \ldots, l\} \setminus I$ the complement to the subset I. We get that $\chi(Y \cap \overline{\mathcal{O}}_I)$ is the term of degree n in the decomposition of the following intersection product in X:

$$(1 + \overline{S}_1 + \ldots + \overline{S}_{n-k}) \cdot \prod_{s=1}^{m} H_s(1 + H_s)^{-1} \cdot \prod_{i \in I} \overline{\mathcal{O}}_i \cdot \prod_{j \in J}(1 + \overline{\mathcal{O}}_j). \qquad (*)$$

On the other hand, since the Euler characteristic is additive, and $C = Y \setminus (\overline{\mathcal{O}}_1 \cup \ldots \cup \overline{\mathcal{O}}_l)$, one can express the Euler characteristic $\chi(C)$ in terms of the Euler characteristics $\chi(Y \cap \overline{\mathcal{O}}_I)$ over all subsets $I \subset \{1, \ldots, l\}$:

$$\chi(C) = \sum_{I \subset \{1, \ldots, l\}} (-1)^{|I|} \chi(Y \cap \overline{\mathcal{O}}_I). \qquad (**)$$

Combining formulas $(*)$ and $(**)$, we get the formula of Theorem 1.1. Indeed, we have that $\chi(C)$ is the term of degree n in the decomposition of the following intersection product in X:

$$(1 + \overline{S}_1 + \ldots + \overline{S}_{n-k}) \cdot \prod_{s=1}^{m} H_s(1 + H_s)^{-1} \cdot \left(\sum_{I \sqcup J = \{1, \ldots, l\}} (-1)^{|I|} \prod_{i \in I} \overline{\mathcal{O}}_i \prod_{j \in J}(1 + \overline{\mathcal{O}}_j) \right).$$

The sum in the parentheses is equal to 1, since for any commuting variables x_1, x_2, $\ldots$, x_l we have the identity:

$$1 = \prod_{i=1}^{l}((1 + x_i) - x_i) = \sum_{I \sqcup J = \{1, \ldots, l\}} (-1)^{|I|} \prod_{i \in I} x_i \prod_{j \in J}(1 + x_j).$$

Computation for a curve. Apply Theorem 1.1 and the formula for the first Chern class S_1 to a curve in G. We get that if $C = H_1 \cap \ldots \cap H_{n-1}$ is a complete intersection of $n - 1$ generic hyperplane sections, then

$$\chi(C) = (S_1 - H_1 - \ldots - H_{n-1}) \cdot \prod_{i=1}^{n-1} H_i.$$

Since S_1 is also a generic hyperplane section, the computation of $\chi(C)$ reduces to the computation of the intersection indices of hyperplane sections.

Recall the Brion-Kazarnovskii formula for such intersection indices. Denote by R^+ the set of all positive roots of G. Recall that ρ denotes the half of the sum of all positive roots of G and L_T denotes the character lattice of a maximal torus $T \subset G$. Since G is reductive, we can assume that $\mathfrak{g}$ is embedded into $\mathfrak{gl}(W)$ so that the trace form $\mathrm{tr}(A, B) = \mathrm{tr}(AB)$ for $A, B \in \mathfrak{gl}(W)$ is nondegenerate on $\mathfrak{g}$. Then the inner product $(\cdot, \cdot)$ on $L_T \otimes \mathbb{R}$ used in Theorem 4.7 is given by the trace form on $\mathfrak{g}$. Choose a Weyl chamber $\mathcal{D} \subset L \otimes \mathbb{R}$.

Theorem 4.7. [4, 17] *If H_π is a hyperplane section corresponding to a representation π with the weight polytope $P_\pi \subset L_T \otimes \mathbb{R}$, then the self-intersection index of H_π in the ring of conditions is equal to*

$$n! \int\limits_{P_\pi \cap \mathcal{D}} \prod_{\alpha \in R^+} \frac{(x, \alpha)^2}{(\rho, \alpha)^2} dx.$$

The measure dx on $L_T \otimes \mathbb{R}$ is normalized so that the covolume of L_T is 1.

This theorem in particular implies that the self-intersection index H_π^n depends not on a representation but only on its weight polytope. Note also that the integrand is invariant under the action of the Weyl group.

Let $H_1, \ldots, H_n$ be n generic hyperplane sections corresponding to different representations $\pi_1, \ldots, \pi_n$. To compute their intersection index one needs to take the *polarization* of H_π^n. Namely, the formula of Theorem 4.7 gives a polynomial function $D(P)$ of degree n on the space of all virtual polytopes $P \subset L_T \otimes \mathbb{R}$ (the addition in this space is the Minkowski sum). The *polarization* D_{pol} is the unique symmetric n-linear form on this space such that $D_{pol}(P_\pi, \ldots, P_\pi) = D(P_\pi)$. Then $D_{pol}(P_{\pi_1}, \ldots, P_{\pi_n})$ is the intersection index $H_1 \cdot \ldots \cdot H_n$. For instance, it can be found by applying the differential operator $\frac{1}{n!}\frac{\partial^n}{\partial t_1 \ldots \partial t_n}$ to the function $F(t_1, \ldots, t_n) = D(t_1 P_{\pi_1} + \ldots + t_n P_{\pi_n})$. E.g. if $P_{\pi_2} = \ldots = P_{\pi_n}$, then the computation of $D_{pol}(P_{\pi_1}, \ldots, P_{\pi_n}) = \frac{1}{n}\frac{\partial}{\partial t}\big|_{t=0} D(tP_{\pi_1} + P_{\pi_2})$ reduces to the integration over the facets of P_{π_2}.

Thus we get the following answer for $\chi(C)$. For simplicity, the answer is given in the case when $\pi_1 = \ldots = \pi_{n-1} = \pi$. Then its polarization provides the answer in the general case. Denote by $P_{2\rho}$ the weight polytope of the irreducible representation of G with the highest weight 2ρ.

Corollary 4.8. *Let C be a curve obtained as the transverse intersection of a generic collection of $n - 1$ hyperplane sections corresponding to the representation π. Then*

$$\chi(C) = D_{pol}(P_{2\rho}, P_\pi, \ldots, P_\pi) - (n-1)D(P_\pi)$$

A similar answer can be obtained for the genus of C since it is equal to the genus of the compactified curve $\overline{C} \subset X_\pi$. Hence, $g(C) = g(\overline{C}) = 1 - \chi(\overline{C})/2$. To compute the Euler characteristic of $\overline{C}$ we need to sum up $\chi(C)$ and the number of points in $\overline{C} \setminus C$. The latter is the intersection index of H_π^{n-1} with the codimension one orbits in X_π and can be again computed by the Brion-Kazarnovskii formula. Choose l facets $F_1, \ldots, F_l$ of P_π so that they parameterize the codimension one orbits in X_π. This means that each orbit of the Weyl group acting on the facets of P_π contains exactly one F_i (see Theorem 2.1).

Corollary 4.9. *The genus $g(C)$ of C is given by the following formula:*

$$g(C) = 1 - \frac{1}{2}\left(\chi(C) + (n-1)!\sum_{i=1}^{l} \int\limits_{F_i\cap\mathcal{D}} \prod_{\alpha\in R^+} \frac{(x,\alpha)^2}{(\rho,\alpha)^2}dx\right)$$

The measure dx on a facet F_i is normalized as follows. Let $H \subset L\otimes\mathbb{R}$ be the hyperplane containing F_i. Then the covolume of the sublattice $L\cap H$ in H is equal to 1.

In the above answer, one can rewrite the polarization $D_{pol}(P_{2\rho}, P_\pi, \ldots, P_\pi)$ in terms of the integrals over the facets of P_π. E.g. in the case when π is the irreducible representation with a strictly dominant highest weight λ, the answer takes the following form. Let $2\rho = \sum_{i=1}^k a_i\alpha_i$ be the decomposition of 2ρ in the basis of simple roots $\alpha_1, \ldots, \alpha_k$.

$$\chi(C) = n!\left(\frac{1}{n}\sum_{i=1}^{k}[a_i \int\limits_{F_i\cap\mathcal{D}} \prod_{\alpha\in R^+} \frac{(x,\alpha)^2}{(\rho,\alpha)^2}dx] - (n-1)\int\limits_{P_\pi\cap\mathcal{D}} \prod_{\alpha\in R^+} \frac{(x,\alpha)^2}{(\rho,\alpha)^2}dx\right).$$

$$g(C) = 1 - \frac{n!}{2}\left(\frac{1}{n}\sum_{i=1}^{k}[(a_i+1) \int\limits_{F_i\cap\mathcal{D}} \prod_{\alpha\in R^+} \frac{(x,\alpha)^2}{(\rho,\alpha)^2}dx] - (n-1)\int\limits_{P_\pi\cap\mathcal{D}} \prod_{\alpha\in R^+} \frac{(x,\alpha)^2}{(\rho,\alpha)^2}dx\right)$$

5 The case of regular spherical varieties

The results of this paper concerning the Chern classes of the tangent bundle can be generalized straightforwardly to the case of arbitrary spherical homogeneous space. In this section, I briefly outline how this can be done.

Let G be a connected complex reductive group of dimension r, and let H be a closed algebraic subgroup of G . Suppose that the homogeneous space G/H is spherical, i.e. the action of G on the homogeneous space G/H by left multiplication is spherical. In the preceding sections, we considered a particular case of such homogeneous spaces, namely, the space $(G\times G)/G \simeq G$.

The definition of the Chern classes S_i of the tangent bundle $\mathrm{T}(G/H)$ can be repeated verbatim for G/H. Denote the dimension of G/H by n. There is a space of vector fields on G/H coming from the action of G. Take n arbitrary vector fields $v_1, \ldots, v_n$ of this type. Define the subvariety $S_i \subset G/H$ as the set of all points $x \in G/H$ such that the vectors $v_1(x), \ldots, v_{n-i+1}(x)$ are linearly dependent.

Denote by $\mathfrak{h} \subset \mathfrak{g}$ the Lie algebra of the subgroup H. Again, there is the Demazure map $p : G/H \to \mathrm{G}(r-n, \mathfrak{g})$, which takes $g \in G/H$ to the Lie subalgebra $g\mathfrak{h}g^{-1}$. Denote by X_{can} the closure of $p(X)$ in the Grassmannian $\mathrm{G}(r-n, \mathfrak{g})$. This is a compactification of a spherical homogeneous space $G/N(\mathfrak{h})$, where $N(\mathfrak{h}) \subset G$ is the normalizer of $\mathfrak{h}$. Brion conjectured that if H coincides with $N(H)$, then the compactification X_{can} is smooth, and hence, regular [5]. F. Knop proved that under the same assumption the normalization of X_{can} is smooth [22]. The conjecture has been proved for semisimple Lie algebras of type A by D. Luna [23], and in type D by P. Bravi and G. Pezzini [3]. In the general case, one can still define the Demazure bundle over X_{can} as the restriction of the tautological quotient vector bundle over $\mathrm{G}(r-n,\mathfrak{g})$.

Since we have not used the regularity of X_{can} in the proof of Lemma 3.3 the same arguments imply two facts. First, for a generic choice of vector fields $v_1,\ldots,v_n$, the resulting subvariety S_i belongs to a fixed class $[S_i]$ in the ring of conditions. Second, for any compactification X of G/H lying over X_{can} the closure of a generic S_i in X intersects properly any orbit of X. Repeating the proof of Lemma 3.6 one can also show that S_i is empty unless $i \le n-k$. Here k is the difference between the ranks of G and of H. Therefore, we have $n-k$ well-defined classes $[S_1],\ldots,[S_{n-k}]$ in the ring of conditions $C^*(G/H)$. Recently, M. Brion and I. Kausz proved that the *G–equivariant* Chern classes of the Demazure bundle also vanish for $i > n-k$ [7].

To extend Theorem 1.1 to an arbitrary spherical homogeneous space one can use the same description of the Chern classes of its regular compactifications. The definition of regular compactifications repeats Definition 3.

Theorem 5.1. *Let X be a regular compactification of G/H. Then the total Chern class of X equals to*

$$(1+\overline{S}_1+\ldots+\overline{S}_{n-k})\cdot\prod_{i=1}^{l}(1+\overline{\mathcal{O}}_i).$$

This description also follows from Subsection 4.1. The proof uses the methods mentioned in Subsection 4.2. In fact, regular compactifications of spherical homogenous spaces arise naturally when one try to apply these methods to a wider class of varieties with a group action. Namely, suppose that a connected complex affine group G of dimension r acts on a compact smooth irreducible complex variety X with a finite number of orbits. Then there is a unique open orbit in X isomorphic to G/H for some subgroup $H \subset G$, so X can be regarded as a compactification of G/H. Denote by $\mathcal{O}_1,\ldots,\mathcal{O}_l$ the orbits of codimension one in X. Then one can describe the tangent bundle of X exactly by the methods mentioned in Subsection 4.2 if the following conditions hold. First, the hypersurfaces $\overline{\mathcal{O}}_1,\ldots,\overline{\mathcal{O}}_l$ are smooth and intersect each other transversally (this allows to apply Ehlers' method to the divisor $X\backslash(G/H)=\overline{\mathcal{O}}_1\cup\ldots\cup\overline{\mathcal{O}}_l$). Second, the vector bundle V_X (defined as in Subsection 4.2) is generated by its global sections $v_1,\ldots,v_r$, where $v_1,\ldots,v_r$ are infinitesimal generators of the action of G on X (this allows to give a uniform description of V_X for all compactifications of G/H satisfying these conditions). It is not hard to check that these two conditions are equivalent to the definition of regular compactifications.

It turns out that a homogeneous space G/H admits a regular compactification if and only if G/H is spherical [1]. Regular compactifications of arbitrary spherical homogeneous spaces are exactly their smooth *toroidal* compactifications [1]. A compactification X of the spherical homogeneous space G/H is called *toroidal* if for any codimension one orbit of a Borel subgroup of G acting on G/H, its closure in X does not contain any G-orbit in X.

The proof of Theorem 1.1 goes without any change for complete intersections in arbitrary spherical homogeneous space G/H. Let $\tilde{H}_1,\ldots,\tilde{H}_m$ be smooth hypersurfaces in some regular compactification of G/H. Suppose that all possible intersections of $\tilde{H}_i$ with the closures of G–orbits are transverse.

Theorem 5.2. *Let $H_1, \ldots, H_m \subset G/H$ be the hypersurfaces $\tilde{H}_i \cap (G/H)$, and let $C = H_1 \cap \ldots \cap H_m$ be their intersection. Then the Euler characteristic of C equals to the term of degree n in the decomposition of*

$$(1 + S_1 + \ldots + S_{n-k}) \cdot \prod_{i=1}^{m} H_i(1 + H_i)^{-1}.$$

The products are taken in the ring of conditions $C(G/H)$.

For instance, if G/H is compact, then the S_i become the usual Chern classes and the above formula coincides with the classical adjunction formula. However, if G/H is noncompact then the Chern classes in the usual sense (as degeneracy loci of *generic* vector fields on G/H) do not usually yield the adjunction formula (although they do for $G = (\mathbb{C}^*)^n$). Indeed, when the homogeneous space is a noncommutative reductive group, all usual Chern classes are trivial but as we have seen $\chi(H) \neq (-1)^n H^n$ even for one smooth hypersurface H. Theorem 5.2 shows that the adjunction formula still holds for noncompact spherical homogeneous spaces, if one replaces the usual Chern classes with the refined Chern classes S_i that are defined as the degeneracy loci of the vector fields coming from the action of G.

References

[1] Frédéric Bien and Michel Brion. Automorphisms and local rigidity of regular varieties. *Compositio Math.*, 104(1):1–26, 1996.

[2] E. Bifet, C. De Concini, and C. Procesi. Cohomology of regular embeddings. *Adv. in Math.*, 82(1):1–34, 1990.

[3] P. Bravi and G. Pezzini. Wonderful varieties of type D. arXiv.org/math.AG/0410472.

[4] Michel Brion. Groupe de Picard et nombres caracteristiques des varietes spheriques. *Duke Math J.*, 58(2):397–424, 1989.

[5] Michel Brion. Vers une généralisation des espaces symétriques. *J. Algebra*, 134(1):115–143, 1990.

[6] Michel Brion. The behaviour at infinity of the Bruhat decomposition. *Comment. Math. Helv.*, 73(1):137–174, 1998.

[7] Michel Brion and Ivan Kausz. Vanishing of top equivariant Chern classes of regular embeddings. preprint arxiv.org/math.AG/0503196.

[8] C. De Concini. Equivariant embeddings of homogeneous spaces. In *Proceedings of the International Congress of Mathematicians (Berkeley, California, USA)*, volume 1,2, pages 369–377, Providence, RI, 1986. Amer. Math. Soc.

[9] C. De Concini and C. Procesi. Complete symmetric varieties I. In *Invariant theory (Montecatini, 1982)*, volume 996 of *Lect. Notes in Math.*, pages 1–44, Berlin, 1983. Springer.

[10] C. De Concini and C. Procesi. Complete symmetric varieties II Intersection theory. In *Algebraic groups and related topics (Kyoto/Nagoya, 1983)*, volume 6 of *Adv. Stud. Pure Math.*, pages 481–513, Amsterdam, 1985. North-Holland.

[11] F. Ehlers. Eine Klasse komplexer Mannigfaltigkeiten und die Auflösung einiger isolierter Singularitäten. *Math. Ann.*, 218(2):127–157, 1975.

[12] W. Fulton. *Intersection theory*. Springer, Berlin, 1984.

[13] I.M. Gelfand, M.M. Kapranov, and A.V. Zelevinsky. Generalized Euler integrals and A-hypergeometric functions. *Adv. Math.*, 84(2):255–271, 1990.

[14] P. Griffiths and J. Harris. *Principles of algebraic geometry*. Pure and Applied Mathematics. Wiley-Interscience [John Wiley & Sons], New York, 1978.

[15] M. Kapranov. Hypergeometric functions on reductive groups. In *Integrable systems and algebraic geometry (Kobe/Kyoto, 1997)*, pages 236–281, River Edge, NJ, 1998. World Sci. Publishing.

[16] Kiumars Kaveh. Morse theory and Euler characteristic of sections of spherical varieties. *Transformation Groups*, 9(1):47–63, 2004.

[17] B.Ya. Kazarnovskii. Newton polyhedra and the Bezout formula for matrix-valued functions of finite-dimensional representations. *Funct. Anal. Appl.*, 21(4):319–321, 1987.

[18] A.G. Khovanskii. Newton polyhedra, and the genus of complete intersections. *Funct. Anal. Appl.*, 12(1):38–46, 1978.

[19] Valentina Kiritchenko. *A Gauss-Bonnet theorem, Chern classes and an adjunction formula for reductive groups*. PhD thesis, University of Toronto, Toronto, Ontario, 2004.

[20] S.L. Kleiman. The transversality of a general translate. *Compositio Mathematica*, 28(3):287–297, 1974.

[21] Friedrich Knop. The Luna-Vust theory of spherical embeddings. In *Proceedings of the Hyderabad Conference on Algebraic Groups (Hyderabad, 1989)*, pages 225–249, Madras, 1991. Manoj Prakashan.

[22] Friedrich Knop. Automorphisms, root systems, and compactifications of homogeneous varieties. *J. Amer. Math. Soc.*, 9(1):153–174, 1996.

[23] D. Luna. Sur les plongements de Demazure. *J. Algebra*, 258(1):205–215, 2002.

[24] R.W. Richardson. Principal orbit types for algebraic transformation spaces in characteristic zero. *Invent. Math.*, 16:6–14, 1972.

[25] Alvaro Rittatore. Reductive embeddings are Cohen-Macaulay. *Proc. Amer. Math. Soc.*, 131(3):675–684, 2003.

[26] D. Timashev. Equivariant compactifications of reductive groups. *Sb. Math.*, 194(3–4):589–616, 2003.