Newton–Okounkov polytopes of symplectic flag varieties

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Convex polytopes in algebraic geometry and in representation theory

0. Toric geometry

Newton (or moment) polytopes

1. Representation theory

Gelfand–Zetlin polytopes and *string polytopes*
(Berenstein–Zelevinsky, Littelmann, 1998)

2. Algebraic geometry

Newton–Okounkov convex bodies
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1 & 2. Toric geometry on non-toric varieties

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Toric varieties

Theory of Newton polytopes

To every smooth projective toric variety X^n there corresponds a simple convex lattice polytope $\Delta(X) \subset \mathbb{R}^n$.

Geometry of $X \leftrightarrow$ combinatorics of $\Delta(X)$

Faces F of $\Delta(X)$ are in bijection with closures of torus orbits $\mathcal{O}_F$ in X.

Intersection theory

$$\mathcal{O}_F \cdot \mathcal{O}_E = \mathcal{O}_{F \cap E}$$

if F and E are transverse.

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To every projective variety X^n there corresponds a convex body $\Delta_v(X) \subset \mathbb{R}^n$ (it depends not only on X but also on a valuation v on $\mathbb{C}(X)$). In many cases of interest (e.g. for spherical varieties) it is a convex lattice polytope.

Main property of $\Delta_v(X)$

$$\deg X = n!\text{volume}(\Delta_v(X))$$

Question

Is there a useful relation between intersection theory on X and intersection of faces of $\Delta_v(X)$ (when $\Delta_v(X)$ is a polytope)?

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Motivating example: flag varieties

Definition

The *flag variety* X is the variety of complete flags in $\mathbb{C}^n$:

$$X = \{\{0\} = V^0 \subset V^1 \subset \ldots \subset V^{n-1} \subset V^n = \mathbb{C}^n \mid \dim V^i = i\}$$

Remark

Alternatively, $X = GL_n(\mathbb{C})/B$, where B denotes the group of upper-triangular matrices (*Borel subgroup*). In this form, the definition can be extended to arbitrary connected reductive groups.

Schubert varieties

$$X_w = \overline{BwB/B}, \; w \in S_n$$

give basis in $H^*(X, \mathbb{Z})$.

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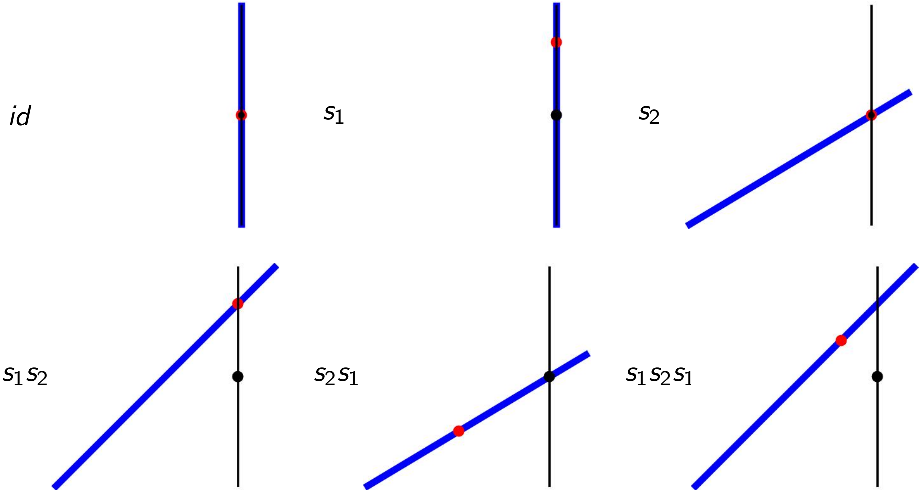
Schubert varieties for GL_3/B.



Gelfand–Zetlin polytopes

The Gelfand–Zetlin polytope Δ_λ is defined by inequalities:

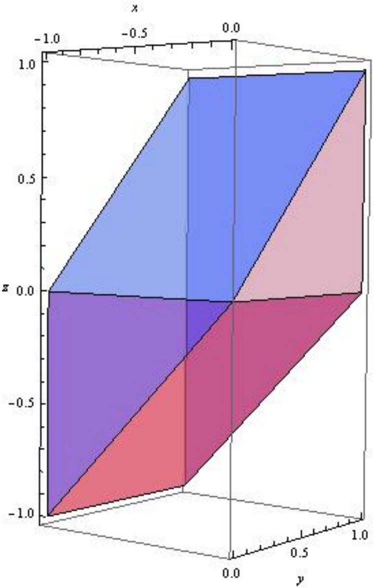
$$
\begin{array}{ccccccccc}
\lambda_1 & & \lambda_2 & & \lambda_3 & & \ldots & & \lambda_n \\
& x_1^1 & & x_2^1 & & \ldots & & x_{n-1}^1 & \\
& & x_1^2 & & \ldots & & x_{n-2}^2 & & \\
& & & \ddots & \ldots & & & & \\
& & & x_1^{n-2} & & x_2^{n-2} & & & \\
& & & & x_1^{n-1} & & & &
\end{array}
$$

where $(x_1^1, \ldots, x_{n-1}^1; \ldots; x_1^{n-1})$ are coordinates in $\mathbb{R}^d$, and the notation

$$
\begin{array}{ccc}
a & & b \\
& c &
\end{array}
$$

means $a \leq c \leq b$.

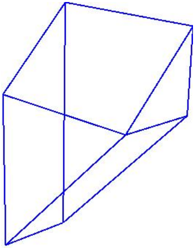
Gelfand–Zetlin polytopes



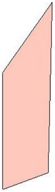
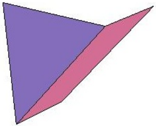
A Gelfand–Zetlin polytope for GL_3:

$$\begin{array}{ccccc} -1 & & 0 & & 1 \\ & x & & y & \\ & & z & & \end{array}$$

Schubert calculus and Gelfand–Zetlin polytopes



$[X_{s_1}] =$; $[X_{s_2}] =$ 

$[X_{s_1 s_2}] \cdot [X_{s_2 s_1}] =$  $\cdot$  $=$ $+$  $= [X_{s_1}] + [X_{s_2}]$

Results

- Relation between Schubert varieties and preimages of rc-faces of Δ_λ under the Guillemin–Sternberg moment map $X \to \Delta_\lambda$ (Kogan, 2000)
- Degenerations of Schubert varieties to (reducible) toric varieties given by (unions of) faces of Δ_λ (Kogan–E.Miller, Knutson–E.Miller, 2003)
- Description of $H^*(X, \mathbb{Z})$ using *volume polynomial* of Δ_λ (Kaveh, 2011)
- Schubert calculus: intersection product of Schubert cycles in $H^*(X, \mathbb{Z})$ = intersection of faces in Δ_λ (K.–Smirnov–Timorin, 2012)

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Generalized flag varieties

Let G be an arbitrary connected reductive group, and $X = G/B$ the complete flag variety.

Question

Which polytopes are best suited for Schubert calculus on G/B?

Polytopes

Generalizatons of Gelfand–Zetlin polytopes from GL_n to G include *string polytopes*, Newton–Okounkov polytopes of flag varieties, and polytopes constructed via convex-geometric divided difference operators (K., 2013).

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Newton–Okounkov polytopes

Valuation

Let $X^n \subset \mathbb{P}^N$ be a projective subvariety with coordinates $(x_1, \ldots, x_n)$ in a neighborhood of a smooth point $p \in X$. Define the valuation $v : \mathbb{C}(X) \to \mathbb{Z}^n$ by sending every polynomial $f(x_1, \ldots, x_n)$ to $(k_1, \ldots, k_n)$ where $x_1^{k_1} \cdots x_n^{k_n}$ is the lowest degree term in f (assuming that $x_1 \succ x_2 \succ \ldots \succ x_n$).

Vector space

Let $V \subset \mathbb{C}(X)$ be the vector space spanned by restrictions to $X \subset \mathbb{P}^N$ of linear functions on

Example

If $X = \nu_N(\mathbb{P}^1) = \{(y_0^N : y_1 y_0^{N-1} : \ldots : y_1^N)\} \subset \mathbb{P}^N$ and $x_1 = \frac{y_1}{y_0}$, then $v(f) =$ the order of zero (or pole) of f at p and $V = \langle 1, x_1, \ldots, x_1^N \rangle$.

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Naive definition

The Newton–Okounkov polytope $\Delta_v(X) \subset \mathbb{R}^n$ of X^n is the convex hull of $v(f)$ for all $f \in V$.

Example

$\Delta_v(\nu_N(\mathbb{P}^1)) = [0, N] \subset \mathbb{R}^1$

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A toric variety X^n has a natural system of coordinates $(x_1, \ldots, x_n)$ coming from $(\mathbb{C}^*)^n \subset X^n$. For a projective embedding $X^n \subset \mathbb{P}^N$, the space V is spanned by monomials in $x_1, \ldots, x_n$. Hence, the valuation v does not matter, and $\Delta_v(X^n)$ is always the Newton polytope of X.

Observation

If $n!\mathrm{volume}(\Delta_v(X)) = \deg(X)$, then the naive definition coincides with the correct definition.

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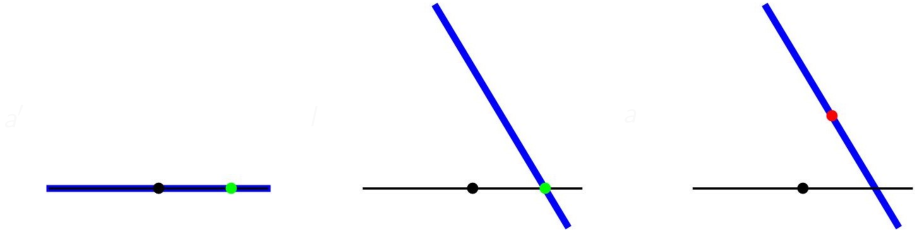
A Newton–Okounkov polytope of GL_3/B

Coordinates on the open Schubert cell

If the flag $(a \in l \subset \mathbb{P}^2)$ is in general position with a fixed flag $(a_0 \in l_0 \subset \mathbb{P}^2)$, then $l \cap l_0 = a' \neq a_0$ and $a \notin l_0$. Hence,

$$a' = (x : 1 : 0); \quad l = \langle a', (y : 0 : 1) \rangle; \quad a = (xz + y : z : 1)$$

are coordinates (assuming that $a_0 = (1 : 0 : 0)$, $l_0 = \{(\star : \star : 0)\}$).



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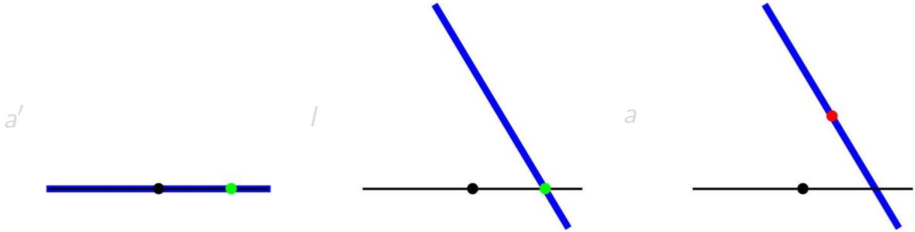

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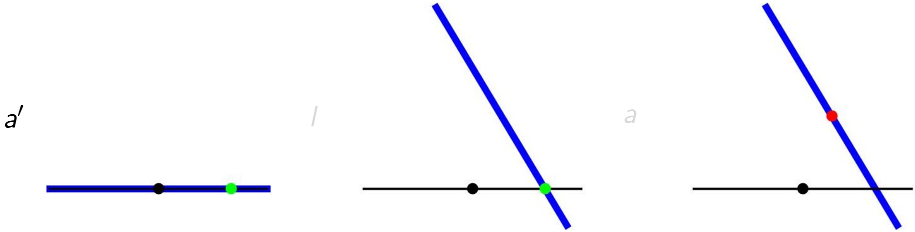

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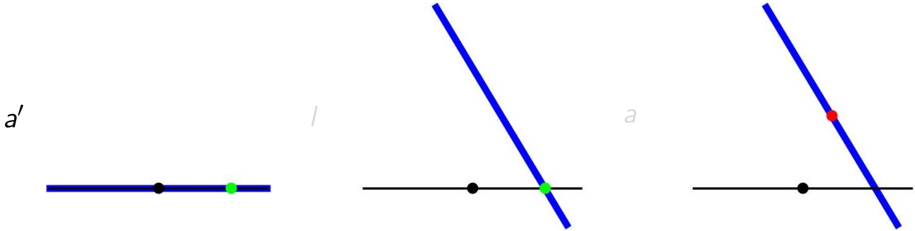

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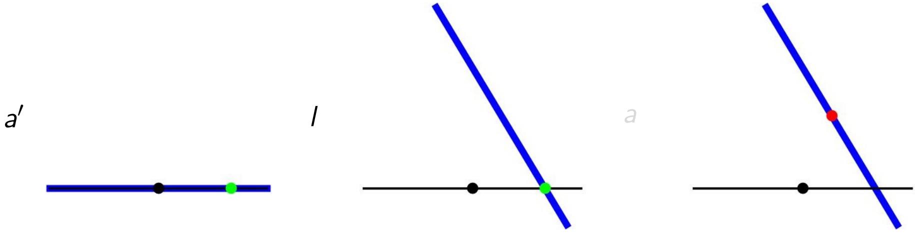

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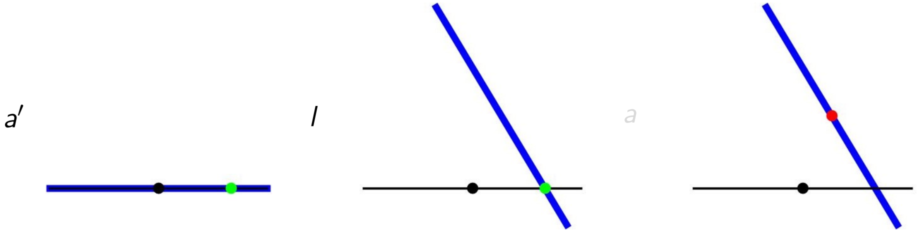

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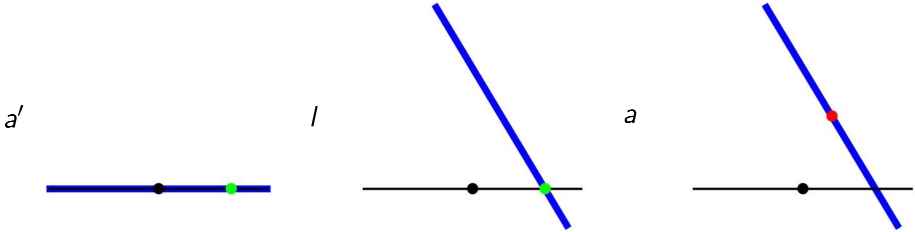

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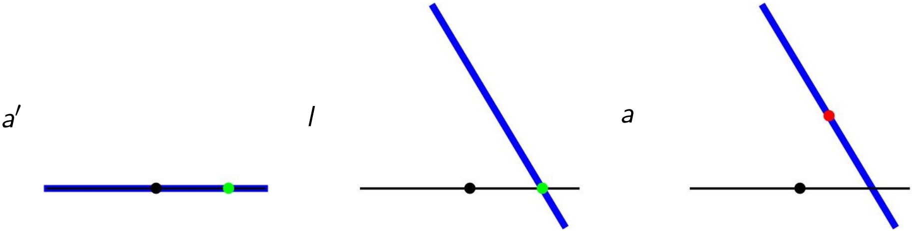
If the flag $(a \in l \subset \mathbb{P}^2)$ is in general position with a fixed flag $(a_0 \in l_0 \subset \mathbb{P}^2)$, then $l \cap l_0 = a' \neq a_0$ and $a \notin l_0$. Hence,

$$a' = (x : 1 : 0); \quad l = \langle a', (y : 0 : 1) \rangle; \quad a = (xz + y : z : 1)$$

are coordinates (assuming that $a_0 = (1 : 0 : 0)$, $l_0 = \{(\star : \star : 0)\}$).



A Newton–Okounkov polytope of GL_3/B

(Anderson, 2011)

Consider the embedding $p : GL_3/B \hookrightarrow \mathbb{P}^2 \times (\mathbb{P}^2)^* \hookrightarrow \mathbb{P}^8$;
$p : (a, l) \mapsto a \times l$. Then p takes the flag with coordinates (x, y, z) to

$$\begin{pmatrix} xz+y & z & 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -x \\ -y \end{pmatrix} = \begin{pmatrix} xz+y & -x^2z-xy & -xyz-y^2 \\ z & -xz & -yz \\ 1 & -x & -y \end{pmatrix}$$

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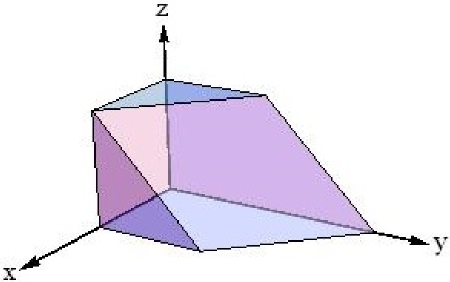

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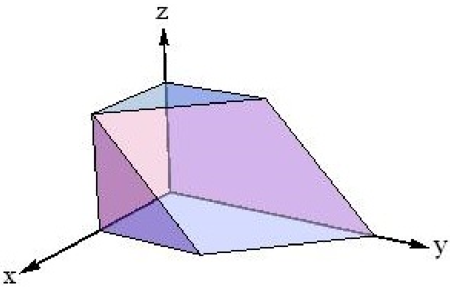

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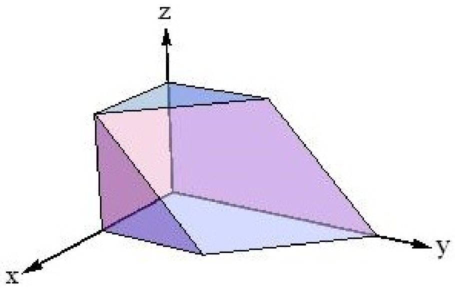

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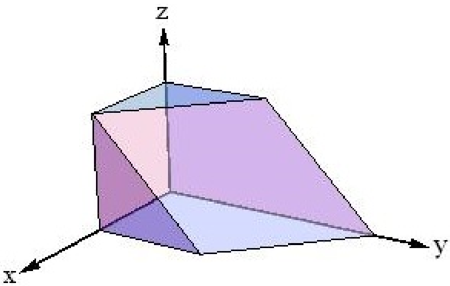

Valuations on $\mathbb{C}(G/B)$

Decomposition of w_0

Fix a reduced decomposition $\overline{w_0} = s_{i_1} \dots s_{i_d}$ of the longest element w_0 in the Weyl group of G.

Flag of Schubert varieties

Choose coordinates compatible with the flag $X_{id} \subset X_{s_{i_d}} \subset X_{s_{i_{d-1}} s_{i_d}} \subset \dots \subset X_{s_{i_2} \cdots s_{i_d}} \subset X$ (coordinates "at infinity").

Flag of translated Schubert varieties

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Generalized Gelfand–Zetlin polytopes

(Okounkov, 1998)

The *symplectic* Gelfand–Zetlin polytope coincides with the Newton–Okounkov polytopes of Sp_{2n}/B for the lowest order term valuation v associated with the flag of Schubert varieties for $\overline{w_0} = (s_1)(s_2 s_1 s_2) \dots (s_n s_{n-1} \dots s_2 s_1 s_2 \dots s_{n-1} s_n)$.

(Kaveh, 2013)

The string polytopes associated with $\overline{w_0}$ coincide with the Newton–Okounkov polytopes of X for the highest order term valuation v associated with the flag of Schubert varieties for $\overline{w_0}$.

Example

If $G = GL_n$ and $\overline{w_0} = s_1(s_2 s_1) \cdots (s_{n-1} \cdots s_1)$ then the corresponding string polytopes are exactly Gelfand–Zetlin polytopes.

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String polytopes

(J.Miller, 2014)

Newton–Okounkov polytopes of Schubert varieties can be represented by unions of faces of a given string polytope.

Remark

This is an existence result. Explicit descriptions of such faces are so far known in the case of GL_n, $\overline{w_0} = s_1(s_2s_1)\cdots(s_{n-1}\cdots s_1)$ (K.–Smirnov–Timorin, 2012) and Sp_4, $\overline{w_0} = s_1s_2s_1s_2$ (Ilyukhina, 2012).

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Mitosis

Combinatorial mitosis on pipe-dreams for GL_n

Pipe-dreams corresponding to permutation w can be obtained from pipe-dreams corresponding to permutation $s_i w$ (if $l(s_i w) < l(w)$) by an explicit combinatorial algorithm (Knutson–E.Miller, 2003).

Mitosis on parallelepipeds

Basic steps of mitosis on pipe-dreams admit a geometric realization (mitosis on parallelepipeds) compatible with the action of Demazure operators (K.–Smirnov-Timorin, 2012).

Geometric mitosis

If Gelfand–Zetlin polytope is replaced by a DDO polytope for another reductive group (e.g. for $Sp(2n)$) then mitosis on parallelepipeds still works and produces a new combinatorial algorithm (K., 2014).

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Geometric mitosis: type A

Gelfand–Zetlin polytope

$$
\begin{array}{ccccccccc}
\lambda_1 & & \lambda_2 & & \lambda_3 & & \dots & & \lambda_n \\
& x_1^1 & & x_2^1 & & \dots & & x_{n-1}^1 & \\
& & x_1^2 & & \dots & & x_{n-2}^2 & & \\
& & & \ddots & & \dots & & & \\
& & & x_1^{n-2} & & x_2^{n-2} & & & \\
& & & & x_1^{n-1} & & & &
\end{array}
$$

has $(n-1)$ different fibrations by coordinate parallelepipeds. Hence, there are $(n-1)$ different mitosis operations on its faces.

Geometric mitosis: type C

(K., 2013)

Take $\overline{w_0} = s_2s_1s_2s_1$. The corresponding DDO polytope Q_λ is given by inequalities

$$0 \le x \le \lambda_1, \quad z \le x + \lambda_2, \quad y \le 2z,$$
$$y \le z + \lambda_2, \quad 0 \le t \le \lambda_2, \quad t \le \frac{y}{2}.$$

(K., 2014)

The polytopes Q_λ coincide with the Newton–Okounkov polytopes of Sp_4/B for the lowest order term valuation v associated with the flag of subvarieties $w_0X_{id} \subset s_1s_2s_1X_{s_2} \subset s_1s_2X_{s_1s_2} \subset s_1X_{s_2s_1s_2} \subset X$.

Remark

The polytopes Q_λ have 11 vertices so they are not combinatorially equivalent to string polytopes (=symplectic Gelfand–Zetlin polytopes) associated with $s_2s_1s_2s_1$ or $s_1s_2s_1s_2$

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Geometric mitosis: type C

Skew pipe-dreams

Faces that contain the lowest vertex $a_\lambda = (0,0,0,0)$ can be encoded by the diagrams:

	$+ \iff 0 = t$
$+ \iff 0 = x$	$+ \iff t = \frac{y}{2}$
	$+ \iff y = 2z$

Parallelepipeds

The polytope Q_λ admits two different fibrations (by translates of xy- and zt-planes), hence, there are two mitosis operations M_1 and M_2 on faces of Q_λ.

Isotropic flags

$Sp_4/B = \{(V^1 \subset V^2 \subset V^3 \subset \mathbb{C}^4) \mid \omega|_{V^2} = 0, V^1 = V^{3\perp}\} =$
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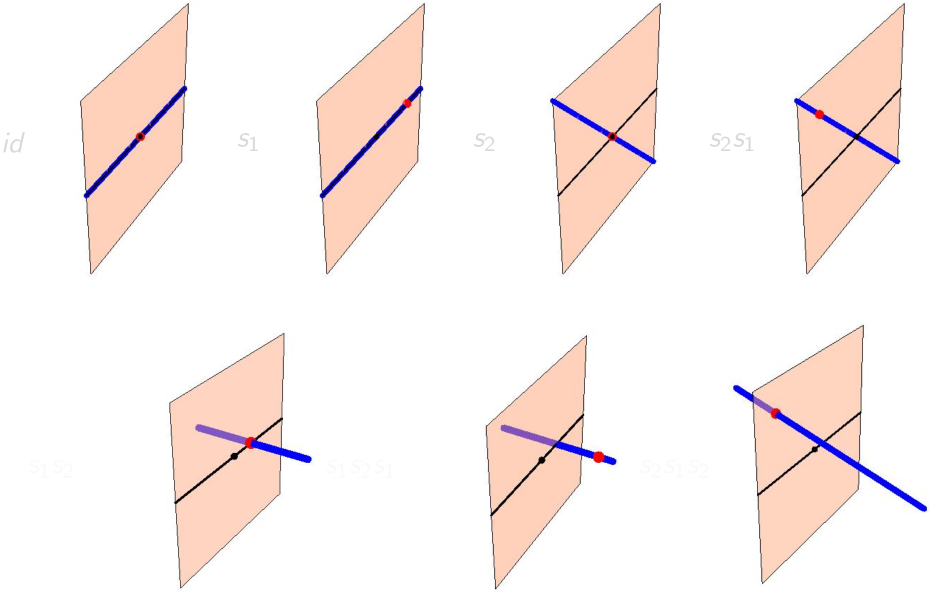

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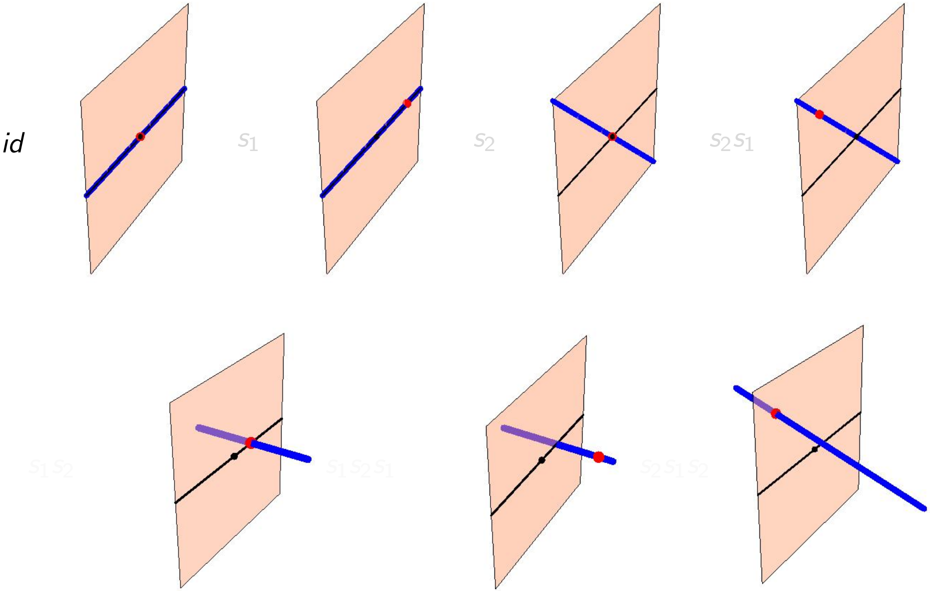

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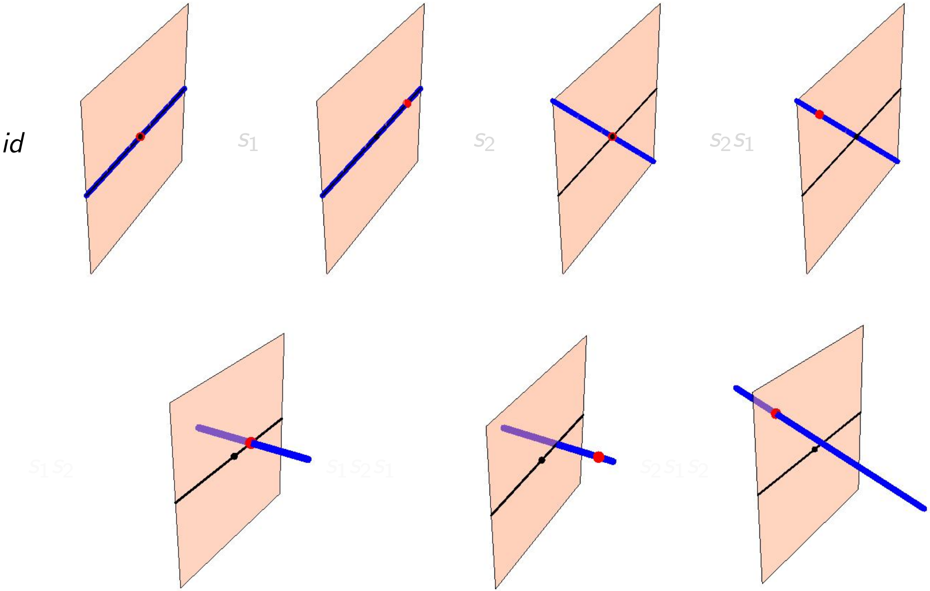

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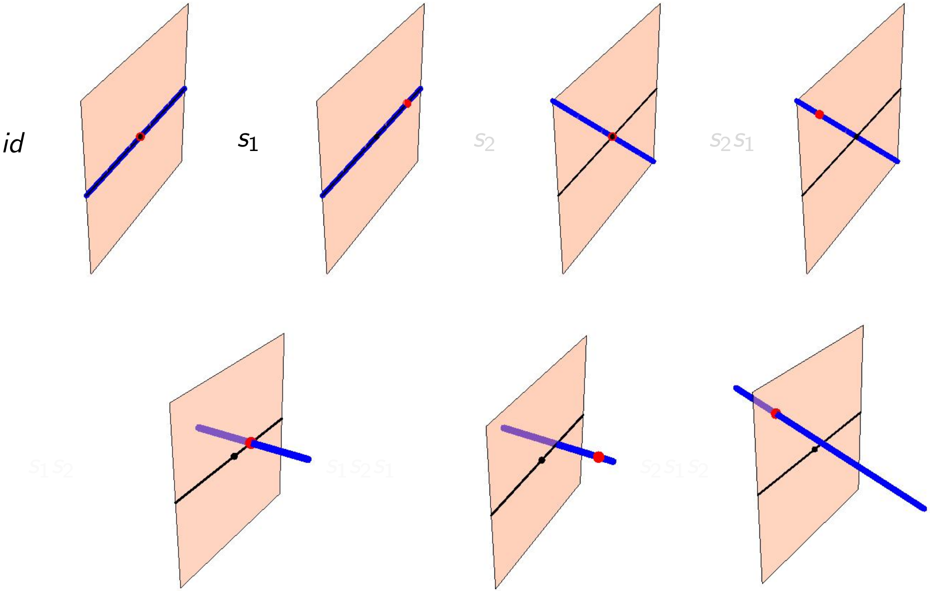

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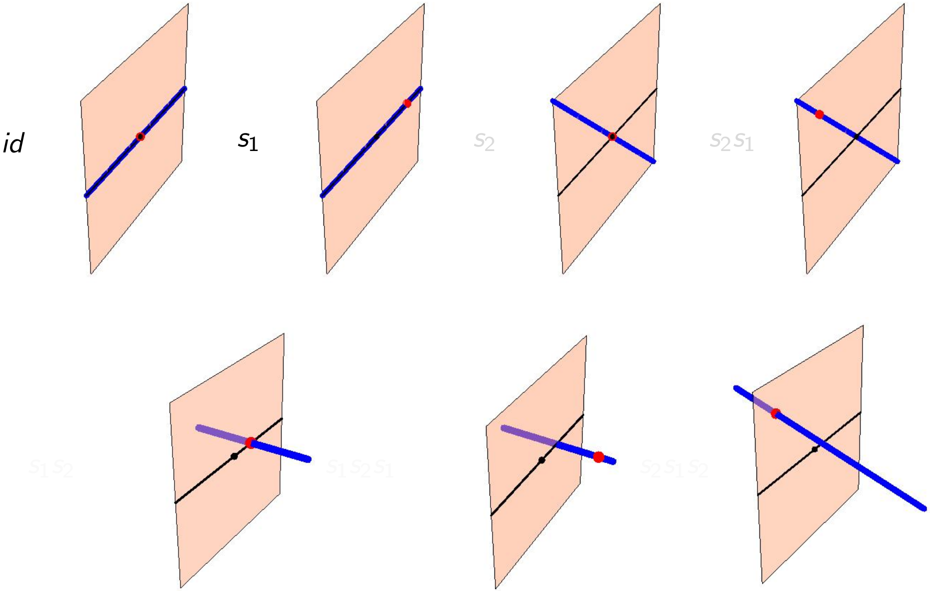

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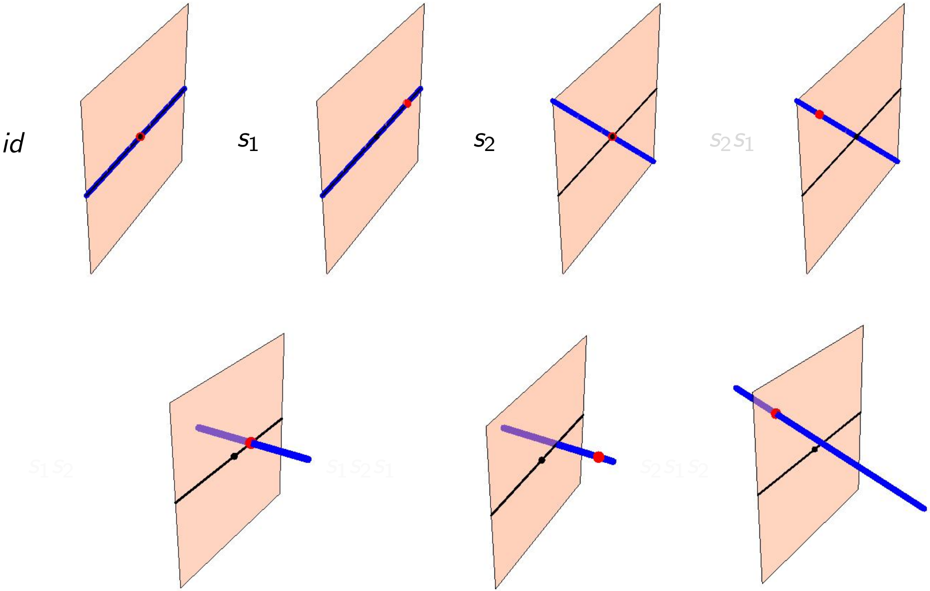

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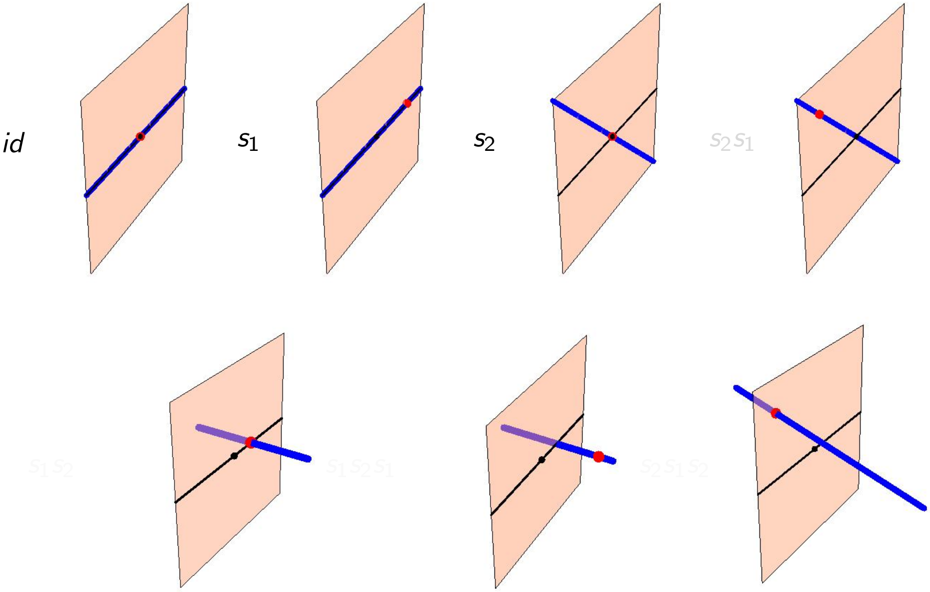

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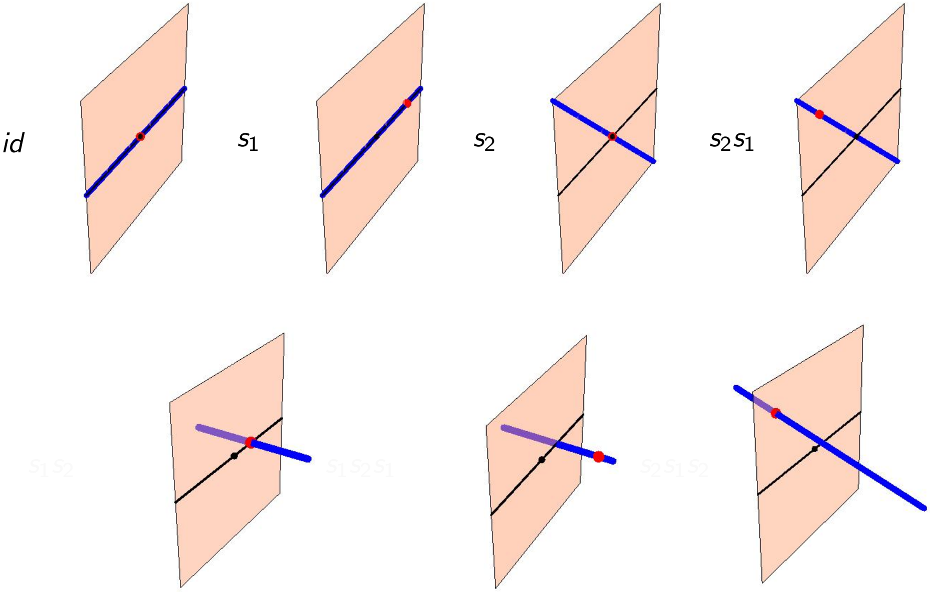

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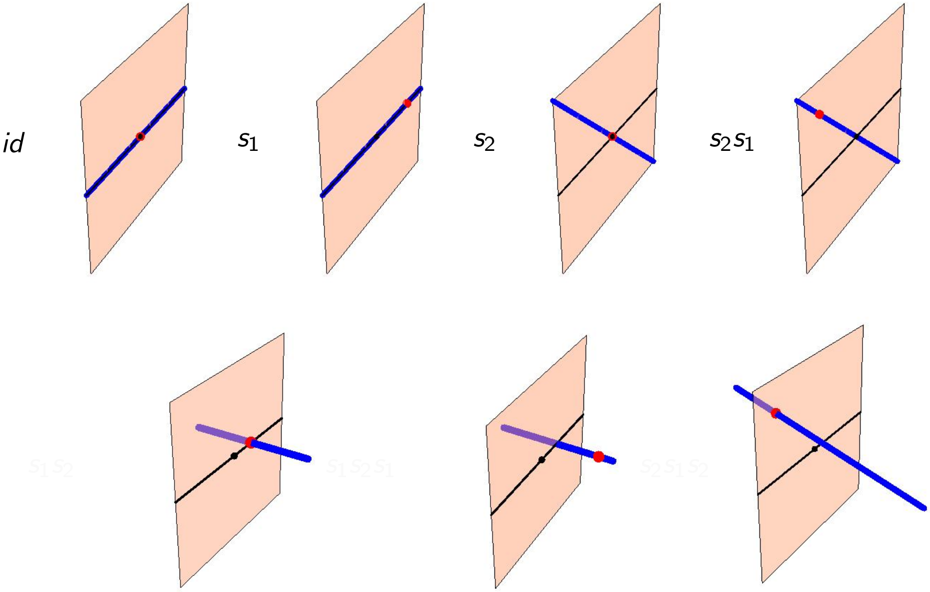

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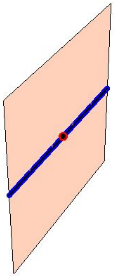
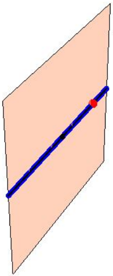
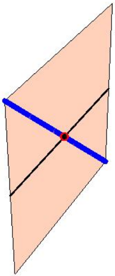
id



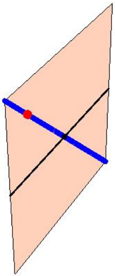
s_1



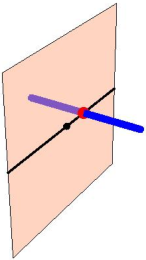
s_2



$s_2 s_1$



$s_1 s_2$



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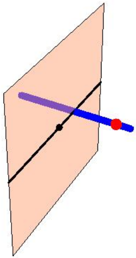

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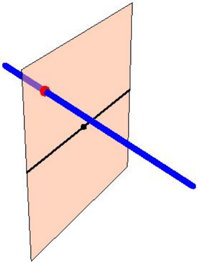

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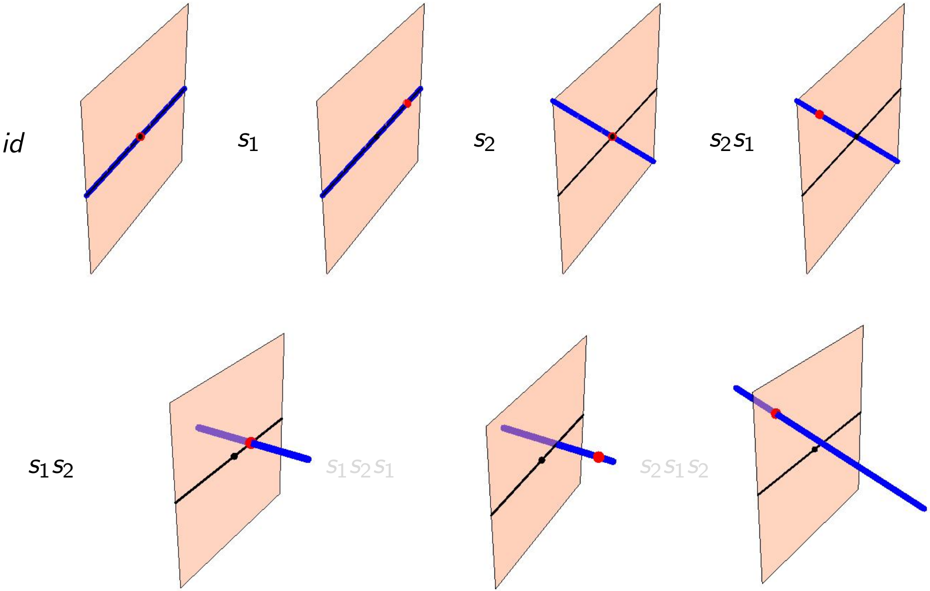

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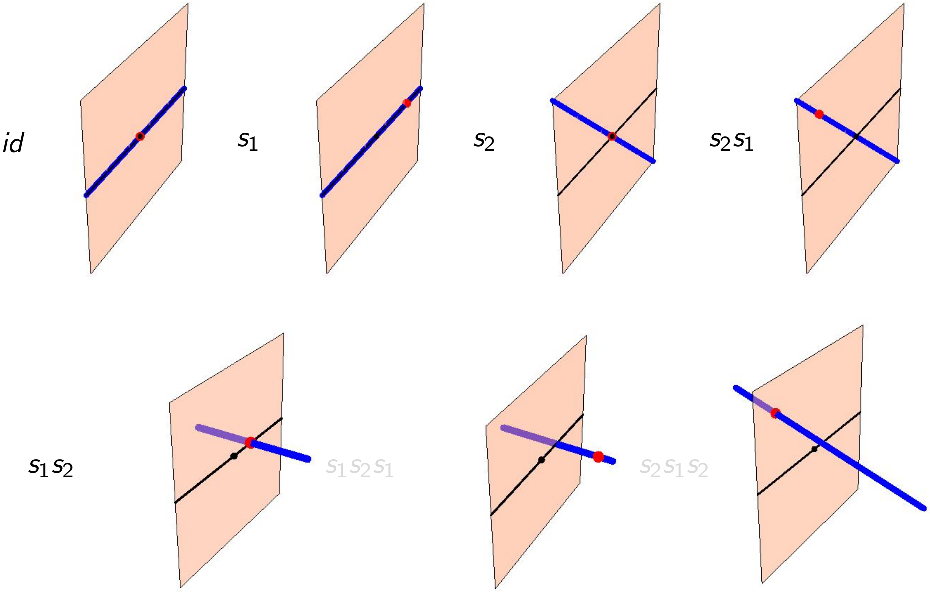
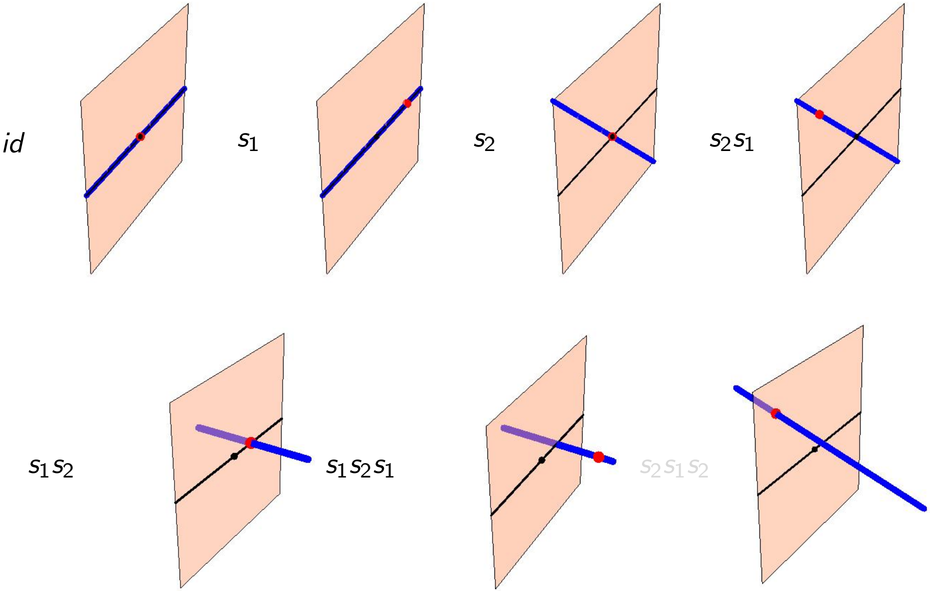
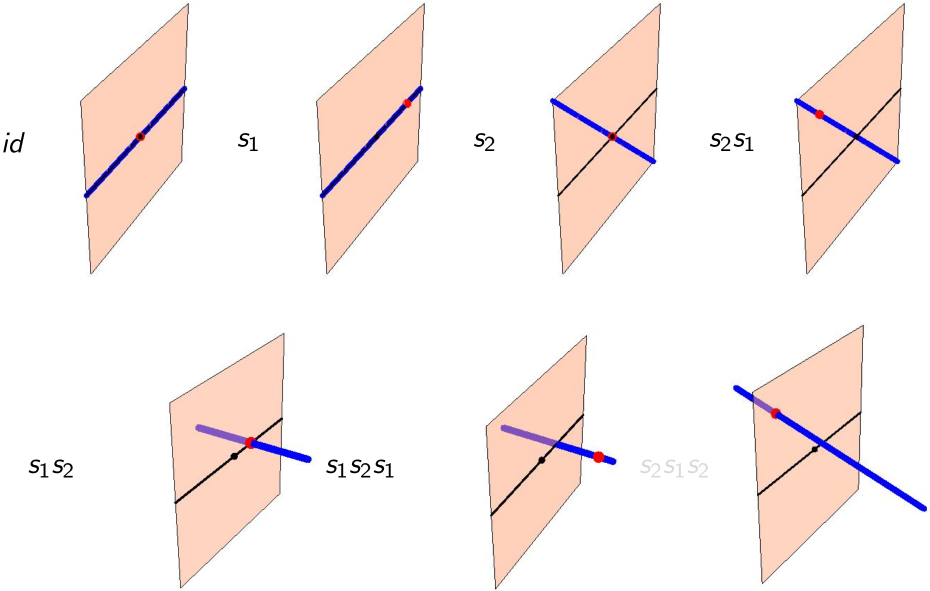


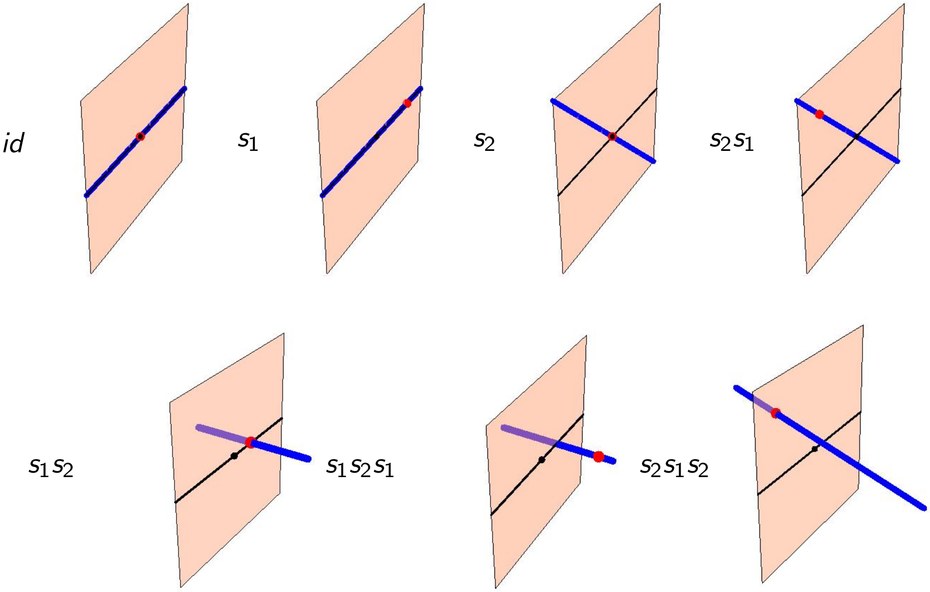
Schubert cycles for Sp_4



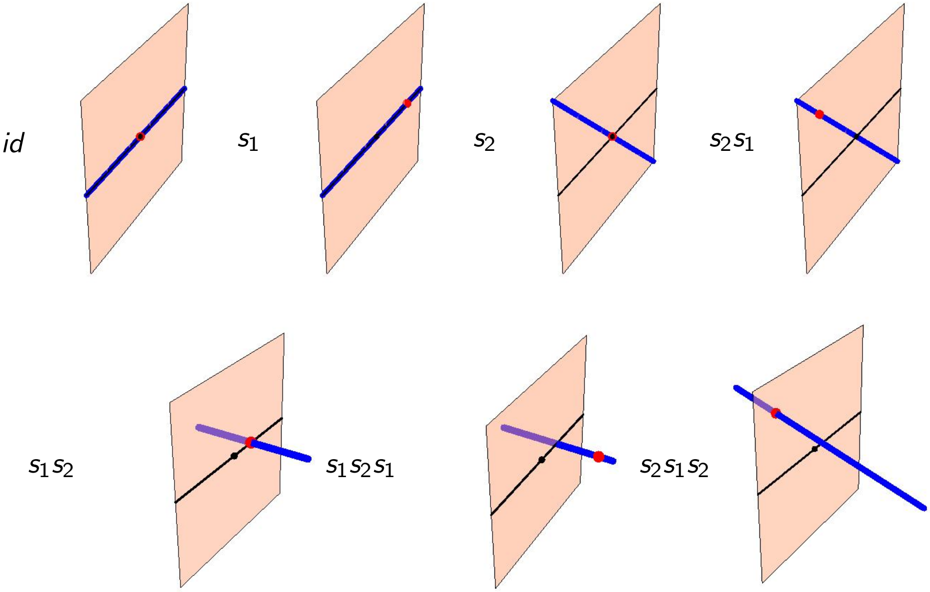
Schubert cycles for Sp_4



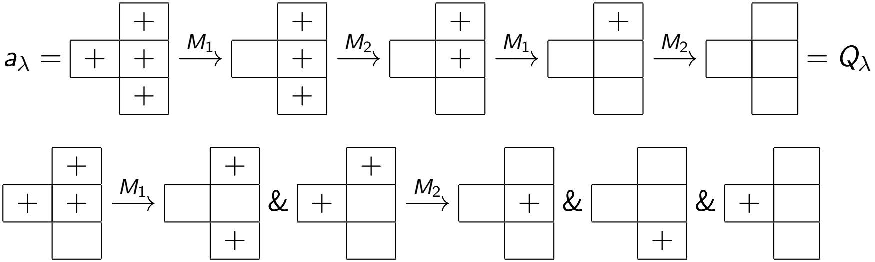
Schubert cycles for Sp_4



Schubert cycles for Sp_4



Geometric mitosis: type C



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