Intersection Theory course notes

Valentina Kiritchenko

1 Introduction

Goals. Let X be an algebraic variety over an algebraically closed field k, and M and N two algebraic subvarieties in X of complementary dimensions (i.e. $\dim M + \dim N = \dim X$). In all our examples X will be an affine or projective variety over the field $\mathbb{C}$ of complex numbers. Our first goal is to define the *intersection index* $M \cdot N$ of M and N. We will assign to each pair (M, N) an integer number $M \cdot N$ satisfying the "conservation of number principle", that is, if we move subvarieties M and N inside X in a certain way then their intersection index does not change. We will formulate this principle explicitly for some interesting examples and see why it is useful.

Let us first consider a naive definition of the intersection index, namely, set $M \cdot N$ to be the number of points $|M \cap N|$ in the intersection of M and N. The following example illustrates what is wrong with this definition and how it can be improved.

Example. Take an affine plane $X = \mathbb{C}^2$ with coordinates x and y, and let $M = \{f(x, y) = 0\}$ and $N = \{g(x, y) = 0\}$ be two curves in X. Consider four cases. In the first three cases, M is a fixed parabola and N is a line. Let us translate and rotate N continuously and see how the number $|M \cap N|$ changes.

1. $f = x^2 - y$, $g = y - 2$; then $|M \cap N| = 2$
2. $f = x^2 - y$, $g = y$; then $|M \cap N| = 1$ so the conservation of number principle fails. However, the intersection point $(0, 0)$ is a point of tangency of curves M and N, so this point should be counted with multiplicity two.
3. $f = x^2 - y$, $g = x$; then $|M \cap N| = 1$, because the other intersection point went to infinity. So to preserve the intersection index we need to find a way to count intersection points at infinity or consider only compact X.
4. $f = x^2 - 1$, $g = x - 1$; then $|M \cap N| = \infty$ since $M \cap N = \{x - 1 = 0\}$ is a line. However, if we rotate N a little bit we again get exactly two intersection points.

The last case suggests to replace subvarieties M and N with the families $\{M_t\}$ of $\{N_t\}$ of subvarieties parameterized by a parameter t so that $|M_t \cap N_t|$ is the same for generic t. We will give a definition of the intersection index using this approach.

Motivation. Intersection theory had been developed mainly in order to give a rigorous foundation for methods of enumerative geometry. Here is a typical question considered in enumerative geometry.

How many lines in 3-space intersect 4 given lines in general position?

Here is Schubert's solution. Choose 4 lines l_1, l_2, l_3, l_4, so that l_1 and l_2 lie in the same plane, and so do l_3 and l_4. It is easy to check that in this case there are exactly two lines intersecting all 4 lines, namely, the line passing through the intersection points $l_1 \cap l_2$ and $l_3 \cap l_4$ and the intersection line of two planes containing l_1, l_2 and l_3, l_4. Then by "conservation of number principle" the number of solutions in general case is also two.

To solve problems in enumerative geometry, Schubert developed calculus of *conditions* (the original German word for condition is "Bedingung"). It is now called Schubert calculus. An example of *condition* is the condition that a line in 3-space intersects a given line. Two conditions can be added and multiplied. E.g. denote by σ_i the condition that a line intersects a given line l_i. Then $\sigma_1 + \sigma_2$ is the condition that a line intersects either l_1 or l_2 , and $\sigma_1 \cdot \sigma_2$ is the condition that a line intersects both l_1 and l_2. Then the above question can be reformulated as follows: find the product $\sigma_1 \cdot \sigma_2 \cdot \sigma_3 \cdot \sigma_4$ of four conditions.

We will discuss interpretation of Schubert calculus via intersection theory on Grassmannians. For instance, we will see that each condition σ_i defines a hypersurface in the variety of lines in $\mathbb{P}^3$, and $\sigma_1 \cdot \sigma_2 \cdot \sigma_3 \cdot \sigma_4$ is the intersection index of four hypersurfaces.

2 Conservation of number principle and its applications

We will now consider several examples in low dimension where the conservation of number principle arises naturally. In each case, an appropriate version of this principle will be formulated explicitly and then used to obtain some numeric results. In particular, we will find the number of zeroes of a polynomial in one variable (Fundamental Theorem of Algebra), the genus of a generic plane curve and the number of common zeroes of two polynomials in two variables (Bezout Theorem).

Fundamental Theorem of Algebra. Let f be a complex polynomial of degree n. The Fundamental Theorem of Algebra asserts that a *generic* f has n distinct complex roots. We will call a polynomial *generic* if it does not have multiple roots (i.e. all roots are simple). The space of all monic polynomials of degree n can be identified with $\mathbb{C}^n$ (the polynomial $x^n + a_1x^{n-1} + \ldots + a_n$ goes to the point $(a_1, \ldots, a_n)$). Then it is easy to show that generic polynomials form a Zariski open dense subset in $\mathbb{C}^n$.

Remark. We will repeatedly use the notion of *generic* object. In each case, there will be a family of objects parameterized by the points of an algebraic variety, and *generic* objects will correspond to the points in some Zariski open dense subset of this variety. In particular, almost any object in the family is generic. In each case the subset of generic objects will be defined by an explicit condition (like the one above) and it will be left as an exercise to check that all generic objects indeed form a Zariski open dense subset.

To extend the Fundamental Theorem of Algebra to non-generic polynomials we need the notion of the *multiplicity* of a root. There are two equivalent definitions.

Algebraic definition of multiplicity. A root a of f has multiplicity k iff

$$f(a) = f'(a) = \ldots = f^{(k-1)}(a) = 0, \text{and} f^{(k)}(a) \neq 0.$$

Geometric definition of multiplicity. A root a of f has multiplicity k iff there is a neighborhood of a (that does not contain other roots of f) such that all generic polynomials close enough to f have exactly k roots in this neighborhood.

In the second definition one needs to check that all generic polynomials close enough to a have the same number of roots in some neighborhood of a. This follows easily from the Implicit Function Theorem.

Example. If $f = x^k$, then 0 is a root of multiplicity k. To check this using the geometric definition one can consider a generic polynomial $x^k - t$, which has k distinct roots for all $t \neq 0$.

We will now prove the Fundamental Theorem of Algebra in the following form.

Theorem 2.1 *Any complex polynomial f of degree n has exactly n complex roots counted with multiplicities.*

Note that this theorem holds over any algebraically closed field and can be proved in a purely algebraic way (by factoring out one root of f). However, the fact that $\mathbb{C}$ is algebraically closed is analytic and its proof must use some geometric arguments.

First, we will show that all generic polynomials have the same number of roots. Indeed, each generic polynomial has a neighborhood such that all polynomials in this neighborhood are generic and have the same number of roots (this again follows the Implicit Function Theorem). Note that this is also true over real numbers. The crucial observation is that any two generic polynomials (identified with the points in $\mathbb{C}^n$) can be connected by a path avoiding all non-generic polynomials (this is exactly what fails over real numbers). This follows from the simple but very important fact stated below.

Lemma 2.2 *Let X be an irreducible complex algebraic variety, and $Y \subset X$ a subvariety of codimension one. Then the complement $X \setminus Y$ is connected.*

Hence, we proved that all generic polynomials have the same number of roots. To actually find this number we can consider a specific polynomial, say, $x(x-1)\cdots(x-n+1)$, which obviously has n roots. The statement of the theorem for non-generic polynomials follows easily from the geometric definition of multiplicity.

There is the following generalization of the Fundamental Theorem of Algebra, which we will use in the sequel. Recall that a function f has pole of order k at a point a if the function $1/f$ has zero of multiplicity k at the point a.

Lemma 2.3 *Let C be a compact smooth curve over $\mathbb{C}$, and $f : C \to \mathbb{CP}^1$ a non-constant meromorphic function on C. Then the number of zeroes of f counted with multiplicities is equal to the number of poles of f counted with orders.*

This lemma can also be proved using a conservation of number principle. Namely, using the Implicit Function Theorem and Lemma 2.2 one can show that all non-critical values of f have the same number of preimages.

Genus of plane curve. Let C be a curve in $\mathbb{CP}^2$ given as the zero set of a homogeneous polynomial of degree d. We say that C is a *generic* plane curve of degree d if C is smooth. Recall that topologically each compact smooth complex curve is a 2-dimensional sphere with several handles. The number of handles is called the *genus* of a curve. E.g. $\mathbb{CP}^1$ is homeomorphic to a sphere, so its genus is zero. A curve of genus one is homeomorphic to a 2-dimensional tori (i.e. the direct product of two circles).

Theorem 2.4 *The genus of a generic plane curve of degree d is equal to*

$$\frac{(d-1)(d-2)}{2}.$$

In particular, generic conic has genus zero and generic cubic curve has genus one. Note also that according to this theorem a generic plane curve can not have genus two.

To prove the theorem one can show that all generic curves of degree d are homeomorphic and hence, have the same genus. Then it is enough to compute the genus of the easiest possible generic curve. At first glance, there is no particularly easy curve but one can do the following trick. Consider a non-generic curve which is just the union of d lines, i.e. it is given by the equation $l_1 \cdot \ldots \cdot l_d = 0$ for some linear functions $l_1, \ldots, l_n$. Topologically it looks like d spheres such that every two have one common point. Then we can perturb a little bit the coefficients of the equation $l_1 \cdot \ldots \cdot l_d = 0$ so that the curve becomes generic. Then it is easy to check that each common point of two spheres gets replaced by a tube. So the whole curve becomes the union of d spheres such that every two are connected by a tube. The genus of such curve is exactly $\frac{(d-1)(d-2)}{2}$.

Bezout Theorem. Let f and g be two homogeneous polynomials on $\mathbb{CP}^2$. We say that the pair (f, g) is *generic* if the intersection of the curves $\{f = 0\}$ and $\{g = 0\}$ in $\mathbb{CP}^2$ is transverse.

Theorem 2.5 *Two generic polynomials of degrees m and n on $\mathbb{CP}^2$ have exactly mn common zeroes.*

Again one shows that for all generic pairs of polynomials the number of common zeroes is the same and then finds this number for, say, polynomials $f(x, y) = x(x-1)\ldots(x-m+1)$ and $g(x, y) = y(y-1)\ldots(y-n+1)$.

We now return to the definition of intersection indices in the case of two curves $M = \{f = 0\}$ and $N = \{g = 0\}$ in $\mathbb{CP}^2$. Bezout Theorem tells us that if the pair (f, g) is generic then the number of intersection points $|M \cap N|$ depends only on the degrees of f and of g. Hence, it is natural to require that the intersection index be preserved when we move each curve in the family of curves defined by the equation of the same degree. This gives the following definition of the intersection index. If the pair (f, g) is generic then put $M \cdot N = |M \cap N|$. Otherwise, we perturb the coefficients of f and g so that they become generic and define $M \cdot N$ as the number of intersection points of the perturbed curves. The Bezout Theorem is then equivalent to the statement that the intersection index of M and N is always equal to the product of degrees of f and g.

3 Divisors and their intersection indices

We will now define the intersection index of n hypersurfaces in an n-dimensional variety. It is easier to do this not just for hypersurfaces but for all formal linear combinations of hypersurfaces, i.e. for *divisors*. To be able to move divisors we will need the notion of *linear equivalence* of divisors. We will discuss all these notions below. For more details see [2], Section *Divisors and line bundles*.

Divisors. Let X be a smooth algebraic variety. A *divisor* D on X is a formal finite linear combination

$$\sum_i k_i H_i,$$

where H_i is an irreducible algebraic hypersurface in X, and k_i is an integer. The hypersurface $|D| = \cup_i H_i$ is called the *support* of D.

Let f be a rational function on X, and $H \subset X$ a hypersurface locally defined by the equation $g = 0$, where g is a regular function in the neighborhood of a point $x \in H$. Define the *order* $\mathrm{ord}_H f$ of f along the hypersurface H to be the maximal integer k such that there is a decomposition $f = g^k h$ for some function h that is regular near x. It is easy to check that the order does not depend on the choice of the point $x \in H$. Define the *divisor* (f) of the function f by the formula

$$(f) = \sum_H \mathrm{ord}_H f,$$

where the sum is taken over hypersurfaces for which $\mathrm{ord}_H f \neq 0$ (there are only finitely many of them). Such divisors are called *principal divisors*.

All divisors form an Abelian group, and principal divisors form a subgroup in this group. Define the *Picard group* $\mathrm{Pic}(X)$ of X as the quotient group of all divisors modulo principal divisors. Two divisors are *linearly* (or *rationally*) equivalent if their difference is a principal divisor, i.e. they represent the same class in the Picard group.

Remark. Note that each divisor on X (when X is smooth) is locally principal, i.e. can be represented locally as the divisor of a rational function. In other words, a *Weil divisor* (defined above) is always a *Cartier divisor*. For non-smooth varieties these two notions may be different. E.g. if $X = \{xy = z^2\} \subset \mathbb{C}^3$ is a cone, then the Weil divisor $D = \{x = z = 0\} \subset X$ is not the divisor of any rational function in the neighborhood of the origin (though $2D$ is). In general, Picard group is defined as the quotient group of Cartier divisors modulo principal divisors.

Examples.

1. Let X be a compact smooth curve. Then a divisor D is a linear combination of points in X:

$$D = \sum_{a_i \in X} k_i a_i.$$

Define the *degree* $\mathrm{deg}D$ of D as the sum $\sum_i k_i$. The divisor of a function f on X is the sum of all zeroes of f counted with multiplicities minus the sum of all poles of f counted with their orders:

$$(f) = \sum_{a_i \in f^{-1}(0)} (\mathrm{mult}_{a_i} f) a_i - \sum_{b_i \in f^{-1}(\infty)} (\mathrm{ord}_{b_i} f) b_i.$$

Note that the degree of a principal divisor is always zero. This follows from Lemma 2.3.

- If $X = \mathbb{P}^1$ then every degree zero divisor is principal. Indeed, every point $a \in \mathbb{P}^1$ is equivalent to any other point, because $a - b$ is the divisor of a fractional linear function $(x - a)/(x - b)$. Hence, $\mathrm{Pic}(\mathbb{P}^1)$ is isomorphic to $\mathbb{Z}$. The isomorphism sends a divisor to the degree of the divisor.
- If $X = E$ is an elliptic curve, i.e. a curve of genus one (one can think of a generic cubic plane curve). Then not all degree zero divisors are principal. E.g. for any two distinct points a and b in E, the divisor $a - b$ is not principal. Indeed, if it were principal, i.e. $a - b = (f)$ for some $f : E \to \mathbb{P}^1$ then f would be a one-to-one holomorphic map (this again follows from Lemma 2.3). But elliptic curve is not homeomorphic to projective line since their genera are different.

  What is true for elliptic curve is that for every three points a, b and O there exists a unique point c such that the divisor $a + b - c - O$ is principal (it is easy to show this using that each elliptic curve is isomorphic to some cubic plane curve in $\mathbb{P}^2$). This allows to define the addition on E by fixing O (zero element) and putting $a + b = c$. This turns E into an Abelian group. In fact, as a complex manifold E is isomorphic to $\mathbb{C}/\mathbb{Z}^2$ for some integral lattice $\mathbb{Z}^2 \subset \mathbb{C}$, and this isomorphism is also a group isomorphism.

  It follows that E is isomorphic to the subgroup $\mathrm{Pic}^0(E)$ of degree zero divisors. The isomorphism sends $a \in E$ to the divisor $a - O$. Then $\mathrm{Pic}(E)$ is isomorphic to $E \oplus \mathbb{Z}$.

2. If $X = \mathbb{C}^n$ is an affine space, then every divisor is principal and $\text{Pic}(\mathbb{C}^n) = 0$. This shows that the notion of Picard group is more suited for study of the intersection indices on compact varieties. For instance, to study intersection indices of hypersurfaces in $\mathbb{C}^n$ one can consider everything in the compactification $\mathbb{CP}^n$. There are also other ways to define intersection theory on non-compact varieties, in particular in $\mathbb{C}^n$. We will discuss them later.

3. If $X = \mathbb{P}^n$ is a projective space then $\text{Pic}(X) = \mathbb{Z}$. Indeed, if D is a hypersurface given by the equation $\{f = 0\}$, then it is linearly equivalent to the degree of f times the class of hyperplane $H = \{x_0 = 0\}$ in $\mathbb{P}^n$ (since $(D - \deg f \cdot H)$ is the divisor of the rational function $f/x_0^{\deg f}$).

4. Let X be a projective variety, i.e. there is an embedding $X \subset \mathbb{P}^d$ for some d. Then a generic hyperplane $\mathbb{P}^{d-1}$ in $\mathbb{P}^d$ intersects X by a hypersurface. The divisor $X \cap \mathbb{P}^{d-1}$ is called a *divisor of hyperplane section.* It is easy to see that for generic hyperplanes in $\mathbb{P}^d$, the corresponding divisors on X are linearly equivalent.

Intersection indices of divisors. We now define the self-intersection index D^n of a divisor D on X using algebraic approach. For more details see [3], Chapter *Intersection indices.* Recall that n denotes the dimension of X.

If $n = 1$, then D^1 is the degree of D.

Consider now the case $n > 1$. Choose n divisors $D_1,\dots,D_n$ linearly equivalent to D and such that the intersection $|D_1|\cap\dots\cap|D_n|$ of their supports consists of a finite number of points. It is not hard to show that such divisors always exist (see [3]). However, it is not true that if D is an honest hypersurface then one can find hypersurfaces $D_1,\dots, D_n$ such that $D_1 \cap\dots\cap D_n$ is finite (see the example with the blow-up of a projective space below). This is the point where we really need to consider divisors and not just hypersurfaces.

Define first the intersection indices $H_1 \cdot \ldots \cdot H_n$ where H_i is an irreducible hypersurface in the support of D_i. Let x be one of the intersection points in $H_1 \cap \dots \cap H_n$. Define the *intersection multiplicity* $\text{mult}(x)$ of the point x as follows.

Definition of intersection multiplicity. Let $f_1,\dots,f_n$ be the local equations of $H_1,\dots,H_n$, respectively (i.e. $H_i = \{f_i = 0\}$ near x). Define the *intersection multiplicity* by the formula

$$\text{mult}(x) = \dim\left(\mathcal{O}_x/(f_1,\dots,f_n)\right),$$

where $\mathcal{O}_x$ is the local ring of x and $(f_1,\dots,f_n)$ is the ideal in $\mathcal{O}_x$ generated by $f_1,\dots, f_n$.

Remark. Note that if $H_1,\dots,H_n$ intersect transversally at the point x , then $\text{mult}(x) = 1$. In the next paragraph we will show that if D is a divisor of hyperplane section then it is always possible to choose hypersurfaces $H_1,\dots,H_n$ linearly equivalent to D so that they have transverse intersection.

Examples Take $X = \mathbb{C}^2$ and compute the intersection multiplicity at the origin of two curves given by equations $f(x, y) = 0$ and $g(x, y) = 0$. Take $f = x^2 - y$.

1. $g = x$; $\mathcal{O}_{(0,0)}/(f, g) = \mathbb{C}[[x, y]]/(x, x^2 - y) = \langle 1\rangle = \mathbb{C}$, so the multiplicity is one. This agrees with the fact that the intersection is transverse.

2. $g = y$; $\mathbb{C}[[x, y]]/(y, x^2 - y) = \langle 1, x\rangle = \mathbb{C}^2$, so the multiplicity is two. The multiplicity rises because curves are tangent at the origin.

3. $g = xy$; $\mathbb{C}[[x, y]]/(xy, x^2 - y) = \langle 1, x, y\rangle = \mathbb{C}^3$, so the multiplicity is three.

Define the intersection index $H_1 \cdot \ldots \cdot H_n$ as follows:

$$H_1 \cdot \ldots \cdot H_n = \sum_{x \in H_1 \cap \ldots \cap H_n} \mathrm{mult}(x).$$

The intersection index $D^n = D_1 \cdot \ldots \cdot D_n$ is defined by linearity, i.e. using that

$$H_1 \cdot \ldots \cdot (aH_i + bH_i') \cdot \ldots \cdot H_n = aH_1 \cdot \ldots \cdot H_i \cdot \ldots \cdot H_n + bH_1 \cdot \ldots \cdot H_i' \cdot \ldots \cdot H_n.$$

The definition of the intersection index $D_1 \cdot \ldots \cdot D_n$ of n different divisors $D_1, \ldots, D_n$ is completely analogous. One needs to check, of course, that the intersection index and multiplicities are well-defined (see [3] for the proof).

Remark. Note that the map

$$(\mathrm{Pic}(X))^n \to \mathbb{Z}, \quad (D_1, \ldots, D_n) \to D_1 \cdot \ldots \cdot D_n$$

is symmetric and n-linear. In particular, it is uniquely defined by the restriction to the diagonal $\{(D, \ldots, D), D \in \mathrm{Pic}(X)\}$. So to compute intersection indices of any divisors on X it is enough to know the self-intersection index of each divisor.

Examples.

1. **Projective spaces.** Let D be the divisor of a hyperplane in $\mathbb{P}^n$. Then to define D^n one can take n hyperplanes $H_1, \ldots, H_n$ in $\mathbb{P}^n$ such that their intersection $H_1 \cap \ldots \cap H_n$ is finite. E.g. take n coordinate hyperplanes $H_i = \{x_i = 0\}$. Then the intersection index $D^n = H_1 \cdot \ldots \cdot H_n = 1$.

2. **Blow-up of the projective plane.** Consider the blow up X of $\mathbb{P}^2$ at the point with homogeneous coordinates $(1 : 0 : 0)$. Let us compute D^2 for the exceptional divisor D. Recall that for the blow-up there is a map

$$p : X \to \mathbb{P}^2$$

such that p is one-to-one on $\mathbb{P}^2 \setminus \{(1:0:0)\}$, and $D = p^{-1}((1:0:0))$. Consider the function $f = x_1/x_0$ on $\mathbb{P}^2$. Then the composition fp is a function on X. It is easy to see that the divisor of fp is equal to $D + H_1 - H_0$, where H_0 and H_1 are the pull-backs to X of the hyperplanes $\{x_0 = 0\}$ and $\{x_1 = 0\}$, respectively (i.e. H_i is an irreducible hypersurface in X such that $p(H_i) = \{x_i = 0\}$). Hence, D is linearly equivalent to $H_0 - H_1$. Then

$$D^2 = D(H_0 - H_1) = -1$$

since D and H_0 do not intersect and D and H_1 intersect transversally at one point. We see that the self-intersection index of a hypersurface might be negative. In particular, it is impossible to find an honest hypersurface H in X such that the intersection $H \cap D$ is finite. We will necessarily have a negative component as well.

Degree of a subvariety. We now discuss a more geometric way to define the self-intersection index of a divisor. For this we will need the notion of the *degree* of a subvariety in projective space.

Let $X^n \subset \mathbb{P}^d$ be a subvariety of dimension n in a projective space. We say that a subspace $\mathbb{P}^{d-n}$ of codimension d is generic with respect to X if it intersects X transversally. By counting dimensions it is easy to show that generic subspaces always exist and form a dense open set in the space of all subspaces. The *degree* of X is the number of intersection points with a generic subspace:

$$\deg X = |X \cap \mathbb{P}^{d-n}|.$$

Note that an analogous definition makes sense for subvarieties of an affine space $\mathbb{A}^d \subset \mathbb{P}^d$, since for almost all generic subspaces all intersection points $X \cap \mathbb{P}^{d-n}$ lie in $\mathbb{A}^d$.

The notion of degree can be used to give more geometric definition for the self-intersection index of a divisor of hyperplane section. Namely, if D is the divisor of hyperplane section corresponding to some embedding $F : X^n \to \mathbb{P}^d$, then $D^n = \deg F(X)$. It is clear that this definition is equivalent to the one we had before. In fact, each projective variety X has a large collection of divisors that can be represented as the divisors of hyperplane section for some projective embeddings of X (such divisors are called *very ample*). Of course, different divisors correspond to different embeddings. The classes of very ample divisors form a semigroup in the Picard group of X (i.e. the sum of two very ample divisors is also very ample), and this semigroup generates $\mathrm{Pic}(X)$ (see [2], Subsection 1.4, Corollary from Kodaira Embedding Theorem). In particular, one can define the self-intersection index of any divisor on X using only the notion of degree and n-linearity of the intersection index of n divisors.

Examples.

1. Let $X = \{f = 0\}$ be a zero set of a homogeneous polynomial f in $\mathbb{P}^k$. Then $\deg X = \deg f$.

2. **Veronese embedding of $\mathbb{P}_1$.** Embed $\mathbb{P}^1$ into $\mathbb{P}^n$ by sending a point $(x_0 : x_1)$ to the collection $(x_0^d : x_0^{d-1}x_1 : \ldots : x_1^d)$ of all monomials of degree d in x_0 and x_1. Let $X \subset \mathbb{P}^d$

be the image of $\mathbb{P}^1$ under this embedding. Then $\deg X = d$ by the Fundamental Theorem of Algebra.

Note that Veronese embedding realizes a divisor D of degree n on $\mathbb{P}^1$ as a divisor of hyperplane section. The "functions" x_0^n, $x_0^{n-1}x_1$,..., x_1^n form the basis in the space of *holomorphic sections* of the *line bundle* associated with D (see [2] Subsection 1.1 for details on the relation between divisors and line bundles). Holomorphic sections define a projective embedding of X precisely when D is very ample. E.g. for $\mathbb{P}^1$ very ample divisors are divisors with nonnegative degrees.

3. **Veronese embedding of $\mathbb{P}_2$.** Embed $\mathbb{P}^2$ to $\mathbb{P}^5$ by mapping a point $(x_0 : x_1 : x_2)$ to the collection $(x_0^2 : x_1^2 : x_2^2 : x_0x_1 : x_1x_2 : x_0x_2)$ of all monomials of degree 2 in x_0, x_1 and x_2. Let $X \subset \mathbb{P}^5$ be the image of $\mathbb{P}^2$ under this embedding. Then the degree of X is equal to the number of common zeroes of two generic quadratic polynomials in $\mathbb{P}^2$. Hence, $\deg(X) = 4$ by Bezout Theorem.

4 Computation of intersection indices: examples

An interesting problems in intersection theory is the actual computation of the intersection indices of divisors on a given algebraic variety. More precisely, let X be an algebraic variety of dimension n, and D_1,...,D_n are n divisors on X. How to compute the intersection index $D_1 \cdot \ldots \cdot D_n$ for all possible collections of n divisors on X?

One way to approach this problem is to choose a basis E_1,..., E_l in the Picard group of X and compute intersection indices $E_1^{i_1} \ldots E_l^{i_l}$ for all collections of nonnegative integers $i_1, \ldots, i_l$ such that $i_1 + \ldots, i_l = n$. Then by n-linearity of the intersection index we get explicit formulas for the intersection index $D_1 \ldots D_n$ in terms of the coefficients in the decomposition of each divisor D_i in the basis E_1,..., E_l. There are several interesting examples where this program can be carried out successfully and leads to beautiful explicit answers. Below we consider some of these examples.

Projective space $\mathbb{P}^n$. The Picard group of $\mathbb{P}^n$ is spanned by the divisor H of hyperplane section. Its self-intersection index is obviously one. Hence, for any divisors $D_1 = d_1H$,..., $D_n = d_nH$ we get that $D_1 \cdot \ldots \cdot D_n = d_1 \cdot \ldots \cdot d_n$. This implies the Bezout theorem:

Theorem 4.1 *The number of common zeroes of n generic homogeneous polynomials of degrees d_1,...,d_n in $n+1$ variables is equal to the product $d_1 \cdot \ldots \cdot d_n$ of the degrees.*

Here generic means that the zero divisors of the polynomials intersect transversally. When all polynomials have the same degree, i.e. $d_1 = \ldots = d_n = d$, the Bezout theorem gives that the degree of $\mathbb{P}^n$ in the Veronese embedding $\mathbb{P}^n \to \mathbb{P}^{\binom{n+k}{n-1}}$ (given by all monomials of degree d) is equal to d^n.

Product of projective spaces $\mathbb{P}^n \times \mathbb{P}^m$. It is easy to show that for two varieties X and Y the Picard group $\mathrm{Pic}(X \times Y)$ is isomorphic to $\mathrm{Pic}(X) \times \mathrm{Pic}(Y)$. Hence, $\mathrm{Pic}(\mathbb{P}^n \times \mathbb{P}^m) = \mathbb{Z} \oplus \mathbb{Z}$, and the basis is given by the divisors of hyperplane section H_1 and H_2 corresponding to the projections of $\mathbb{P}^n \times \mathbb{P}^m$ onto the first and second factors, respectively. A straightforward calculation gives that $H_1^i H_2^j = 0$ unless $i = n$ and $j = m$, and $H_1^n H_2^m = 1$. We get another Bezout theorem.

Theorem 4.2 *Consider $m + n$ generic bihomogeneous polynomials in $m + n + 2$ variables $(x_0, \dots, x_n; y_0, \dots, y_m)$ (i.e. they are homogeneous with respect to both x_i's and y_i's of bidegrees (d_1, e_1),...,(d_{m+n}, e_{m+n}). Then the number of their common zeroes is equal to*

$$h_n(d_1, \dots, d_{m+n}) h_m(e_1, \dots, e_{m+n}).$$

Here h_i denotes the polynomial in $m+n$ variables $z_1, \dots, z_{m+n}$ equal to the sum of all monomials of degree i in z_1,..., z_{m+n}.

In particular, we get that the self-intersection index of the divisor $D = dH_1 + eH_2$ is equal to $\binom{m+n}{n} d^n e^m$. Note that bihomogeneous polynomials in $m + n + 2$ variables of bidegree (d, e) are not generic in the space of all homogeneous polynomials in the same variables of degree $d + e$ so the Bezout theorem for projective spaces is not applicable to bihomogeneous polynomials.

Toric varieties. A more general version of Bezout theorem (which generalizes the previous two examples) can be obtained using *toric varieties*. I will first formulate the theorem in completely elementary terms. Namely, I state an explicit formula for the number of common zeroes of n homogeneous polynomials in n variables assuming that these polynomials are generic inside some subspace V of the space of all homogeneous polynomials in n variables. The formula will, of course, depend on V. The formula is valid and will be formulated not just for polynomials but also for *Laurent polynomials*.

First, recall few definitions. A *Laurent polynomial* in n variables x_1,..., x_n is a finite linear combination of *Laurent monomials* $x_1^{k_1} x_2^{k_2} \dots x_n^{k_n}$ where k_1,..., k_n are (possibly negative) integers. Assign to each Laurent monomial $x_1^{k_1} x_2^{k_2} \dots x_n^{k_n}$ its exponent $(k_1, \dots, k_n)$ that can be regarded as a point in the integral lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. For a Laurent polynomial f, define its *Newton polytope* $P_f \subset \mathbb{R}^n$ as the convex hull of all the exponents of the Laurent monomials occuring in f.

Example. Take $n = 2$. Consider the Laurent polynomial $f = x_1 + x_2 + x_1^{-1} x_2^{-1}$. Its Newton polytope is the triangle with the vertices $(1, 0)$, $(0, 1)$ and $(-1, -1)$.

Note that the values of Laurent polynomials are defined when x_1,...,x_n do not vanish. So each Laurent polynomial is a regular function on the *complex torus* $(\mathbb{C}^*)^n = \{(x_1, \dots, x_n) | x_i \neq 0\} \subset \mathbb{C}^n$.

Theorem 4.3 *Fix a polytope $P \subset \mathbb{R}^n$ with integral vertices. Consider n generic Laurent polynomials whose Newton polytope is P. Then the number of their common zeroes lying inside $(\mathbb{C}^*)^n$ is equal to $n!$ times the volume of P.*

Example for n=2.

- Let P be the triangle with the vertices $(0,0)$, $(n,0)$ and $(0,n)$. Its area is equal to $n^2/2$. A generic Laurent polynomial with the Newton polygon P is just a generic usual polynomial in two variables (since there are no negative exponents) of degree n. By homogenizing we get a generic polynomial on $\mathbb{CP}^2$. Note that the common zeroes in $\mathbb{CP}^2$ of two such polynomials will all lie in $(\mathbb{C}^*)^2 \subset \mathbb{CP}^2$. Thus we get the Bezout theorem for the projective plane $\mathbb{CP}^2$.

- Let P be the square with the vertices $(0,0)$, $(n,0)$, $(0,m)$ and (n,m). Its area is equal to nm. A generic Laurent with the Newton polygon P is a generic bihomogeneous polynomial in two variables of bidegree (n,m). Thus we get the Bezout theorem for the product of projective lines $\mathbb{CP}^1 \times \mathbb{CP}^1$.

- Let P be the trapezium with the vertices $(n,0)$, $(m,0)$, $(0,n)$ and $(0,m)$ (assume that $m > n$). Its area is equal to $(m^2 - n^2)/2$. Consider the surface X obtained by blowing up the point $(1:0:0)$ in $\mathbb{CP}^2$. It is easy to check that the Picard group of X is a free Abelian group with two generators E and H, where E is the exceptional divisor and H is the pull-back of the divisor of hyperplane section in $\mathbb{CP}^2$. We already computed that $E^2 = -1$, $H^2 = 1$ and $EH = 0$. Then the self-intersection index of the divisor $nE + mH$ is equal to $m^2 - n^2$, i.e. to twice the area of the trapezium P. We have got the Bezout theorem for the blow-up of the projective plane.

These examples show that for some varieties there is a natural correspondence between very ample divisors on the variety and polytopes of fixed shape. Moreover, the self-intersection index of a divisor can be computed in terms of the corresponding polytope. A big class of such varieties consists of all smooth compact *toric varieties*. A *toric variety* of dimension n is an algebraic variety with an action of a complex torus $(\mathbb{C}^*)^n$ having an open dense orbit (isomorphic to $(\mathbb{C}^*)^n$). In particular, a compact toric variety can be viewed as a compactification of a complex torus. For each Newton polytope P one can construct a toric variety X_P such that the very ample divisors in the Picard group of X_P can be identified with the polytopes *analogous* to P. Then the self-intersection indices of the divisors can be computed according to Theorem 4.3.

5 Schubert calculus

We will now study geometry of Grassmannians: decomposition into Schubert cells and Plücker embedding. In the end of this section a solution to the problem about 4 lines in a 3-space will be given. For more details see [2], Section *Grassmannians* and [5], Chapter *Schubert varieties*.

Schubert cells. The *Grassmannian* $\mathrm{G}(k,n)$ is the set of all k-dimensional subspaces in an n-dimensional vector space. The Grassmannian can be turned into an algebraic variety using the following affine charts. For any complete flag of subspaces $F = \{\mathbb{C}^1 \subset \mathbb{C}^2 \subset \ldots \subset \mathbb{C}^n\}$

consider the set C_F of all k-dimensional subspaces *generic* with respect to this flag. We say that a k-dimensional subspace V^k is *generic* with respect to a given flag F if the intersection $V^k \cap \mathbb{C}^i$ is transverse for each subspace $\mathbb{C}^i$ in the flag F, i.e. $\dim(V^k \cap \mathbb{C}^i) = i + k - n$. Then it is easy to show that C_F can be identified with an affine space of dimension $k(n-k)$. Indeed, choose a basis $e_1,\ldots,e_n$ in $\mathbb{C}^n$ such that $e_{n-i+1},\ldots,e_n$ span $\mathbb{C}^i$ (i.e. $\mathbb{C}^i$ is given by the equations $x_1 = \ldots = x_{n-i} = 0$). Then each subspace V^k generic with respect to F has a unique basis $v_1,\ldots,v_k$ such that $v_i = e_i + a_{i,1}e_{k+1} + \ldots + a_{i,n-k}e_n$. This allows to identify all subspaces in C_F with the set of all $k \times (n-k)$-matrices (send V^k to the matrix $(a_{i,j})$). Note that for any point V^k of the Grassmannian there exists a flag F for which V^k is generic. Hence, we get a covering of the Grassmannian by affine spaces.

Example. The simplest example of Grassmannian is the projective space $\mathbb{P}^n = \mathrm{G}(1, n+1)$, i.e. the space of all lines in the vector space $\mathbb{C}^{n+1}$. Then the chart C_F consists of all lines that do not lie in a given hyperplane $\mathbb{C}^{n-1}$. Each line from C_F has a unique vector with coordinates $(1, a_1, \ldots, a_n)$. This gives a one-to-one correspondence between C_F and the affine space $\mathbb{C}^n$.

Remark. Note that the Grassmannians $\mathrm{G}(k, n)$ and $\mathrm{G}(n-k, n)$ are isomorphic since there is a canonical one-to-one correspondence between subspaces of dimension k in $\mathbb{C}^n$ and subspaces of codimension k in the dual space $(\mathbb{C}^n)^*$. In particular, $\mathrm{G}(n, n+1) = \mathbb{P}^n$.

We just showed that the Grassmannian can be made into a smooth algebraic variety that has a covering by affine charts. It turns out that Grassmannian is also projective. Namely, there is an embedding

$$P : \mathrm{G}(k, n) \to \mathbb{P}^{\binom{n}{k}-1} = \mathbb{P}(\Lambda^k \mathbb{C}^n);$$
$$P : V^k \mapsto \mathbb{P}(\Lambda^k V^k),$$

called Plücker embedding. If $k = 1$ then this is just the identity map $P : \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$.

To write the Plücker map in coordinates choose a basis $e_1,\ldots,e_n$ in $\mathbb{C}^n$ and a basis $v_1,\ldots,v_k$ in V^k. Denote by $A(V^k)$ the $k \times n$ matrix, whose i-th row consists of coordinates of v_i in the basis $e_1,\ldots,e_n$. Then the coordinates of $\mathbb{P}(\Lambda^k V^k) = \mathbb{P}(v_1 \wedge \ldots \wedge v_k)$ in the basis $\{e_{i_1} \wedge \ldots \wedge e_{i_k}\}_{1 \le i_1 < \ldots < i_k \le n}$ are exactly the $k \times k$ minors of $A(V^k)$.

Exercise. The simplest example of Grassmannian for which the Plücker embedding is not tautological is $G(2, 4)$, the Grassmannian of planes in a 4-dimensional space. Since the Plücker embedding maps $G(2, 4)$ to $\mathbb{P}^5$ the image is a hypersurface in $\mathbb{P}^5$. Show that this hypersurface is given by the equation

$$x_{12}x_{34} + x_{14}x_{23} - x_{13}x_{24} = 0,$$

where x_{ij} is the coordinate with the basis vector $e_i \wedge e_j$. This hypersurface is called the *Plücker quadric* and is isomorphic to a generic quadric in $\mathbb{P}^5$.

We now construct an algebraic cell decomposition of the Grassmannian $\mathrm{G}(k, n)$. Fix a complete flag $F = \{\mathbb{C}^1 \subset \mathbb{C}^2 \subset \ldots \subset \mathbb{C}^n\}$. A cell will consist of all subspaces that have prescribed intersection dimensions with the subspaces of the flag F.

Definition of Schubert cells. Let d_1, d_2, $\dots d_k$ be integers such that $1 \le d_1 < \dots < d_k \le n$. The *Schubert cell* $C_F(d_1, \dots, d_k)$ constructed using the flag F is a subset of the Grassmannian $\mathrm{G}(k,n)$ consisting of all subspaces V^k such that

$$\dim(V^k \cap \mathbb{C}^j) = i \text{ iff } d_i \le j < d_{i+1}.$$

I.e. the intersection dimensions $\dim(V^k \cap \mathbb{C}^j)$ form the sequence

$$(\underbrace{0,\dots,0}_{d_1}, \underbrace{1,\dots,1}_{d_2-d_1}, \dots, \underbrace{k,\dots,k}_{d_k-d_{k-1}}).$$

Sometimes this sequence is used instead of the sequence d_1,..., d_k in order to label the Schubert cell.

Example. We choose coordinates in $\mathbb{C}^n$ so that the flag F consists of coordinate subspaces $\mathbb{C}^i = \{x_1 = \dots = x_{n-i+1} = 0\}$.

- The cell $C_F(n-k+1, \dots, n)$ is exactly the affine chart C_F considered above, so it is isomorphic to the affine space of dimension $k(n-k)$.
- The cell $C_F(1, \dots, k)$ consists of a single point, which is the space $\mathbb{C}^k$.
- The projective space $\mathbb{P}^n = \mathrm{G}(1, n+1)$ has $n+1$ Schubert cells $C_F(1)$,...,$C_F(n)$. The cell $C_F(i)$ consists of all lines that lie in $\mathbb{C}^i$ but do not lie in $\mathbb{C}^{i-1} \subset \mathbb{C}^i$. Each such line has a unique basis vector with coordinates $(0, \dots, 0, 1, a_1, \dots, a_i)$. Hence, $C_F(i)$ is isomorphic to the affine space of dimension i and we get the following cell decomposition for $\mathbb{P}^n$:

$$\mathbb{P}^n = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \dots \sqcup \mathbb{A}^0$$

- The Grassmannian $\mathrm{G}(2,4)$ has 6 Schubert cells: $C_F(3,4) = \mathbb{A}^4$, $C_F(2,4) = \mathbb{A}^3$, $C_F(1,4) = \mathbb{A}^2$, $C_F(2,3) = \mathbb{A}^2$, $C_F(1,3) = \mathbb{A}^1$, $C_F(1,2) = \mathbb{A}^0$. Each cell can be identified with an affine space in the same way as for projective spaces (by choosing a special basis in each plane in the cell). E.g. each plane in $C_F(2,4)$ has a unique basis of the form $\{(1, a_1, 0, a_2), (0, 0, 1, a_3)\}$ and hence is isomorphic to $\mathbb{A}^3$.

Exercise. Show that each Schubert cell $C_F(d_1, \dots, d_k)$ is isomorphic to an affine space of dimension $(d_1 - 1) + (d_2 - 2) + \dots + (d_k - k)$.

In what follows, we will mostly deal with *Schubert cycles* rather than with Schubert cells. The *Schubert cycle* $Z_F(d_1, \dots, d_k)$ is the (Zariski) closure of the Schubert cell $C_F(d_1, \dots, d_k)$ in the Grassmannian $G(k,n)$. Each Schubert cycle is a closed subvariety of the Grassmannian. E.g. the Schubert cycles in $\mathbb{P}^n$ are projective subspaces of dimensions 0, 1,..., n. In general, Schubert cycles are not smooth.

Solution to the problem about 4 lines in a 3-space. The Grassmannian $\mathrm{G}(2,4)$ can also be regarded as the set of all lines in $\mathbb{P}^3$. Choose a complete flag $F = \{a \subset l \subset p \subset \mathbb{P}^3\}$ (i.e. a is a point, l is a line and p is a plane). Then the cell $C_F(2,4)$ consists of all lines not passing through the point a, not lying in the plane p but intersecting the line l. In particular, the Schubert cycle $Z_F(2,4)$ consists of all lines in $\mathbb{P}^3$ intersecting the line l. Note that $Z_F(2,4)$ is a hypersurface in $G(2,4)$. It is easy to check that this hypersurface can be realized as a hyperplane section corresponding to the Plücker embedding.

The problem about 4 lines in $\mathbb{P}^3$ reduces to the computation of the self-intersection index of the hypersurface $Z_F(2,4)$. Indeed, take 4 lines l_1, l_2, l_3 and l_4 in general position. Let σ_i be the hypersurface in $\mathrm{G}(2,4)$ consisting of all lines intersecting l_i. Then σ_1, σ_2, σ_3 and σ_4 are hyperplane sections in general position corresponding to the Plücker embedding. Hence, their intersection index is equal to the number of intersection points, which is exactly the number of lines intersecting 4 given lines. It remains to notice that the self-intersection index $Z_F(2,4)^4$ is the same as the degree of $G(2,4)$ in the Plücker embedding. The latter is equal to 2, since the image is a quadric.

This solution does not use Schubert's idea of degenerating 4 lines to some special position. A solution based on Schubert's idea requires a little more of intersection theory and will be given in the next section.

6 Intersection product and Chow ring

So far we only dealt with intersection indices of divisors. This is a partial case of *intersection product* of subvarieties of arbitrary dimension (in particular, their intersection is not necessarily a finite number of points). In this section, we briefly discuss how to define such a product in general and then consider an important example: the *intersection product* of two Schubert cycles in the Grassmannian $\mathrm{G}(k,n)$. In the end of this section, we will justify Schubert's solution to the problem about 4 lines in a 3-space. For more details see [7], [6] and [5].

Intersection product. Let X be an algebraic variety of dimension n, and M and N two subvarieties of dimensions k_1 and k_2, respectively. We no longer require that $k_1 + k_2$ be equal to n. We want to define the *intersection product* $M \cdot N$ as a subvariety in X of dimension $k_1 + k_2 - n$. As in the definition of the intersection index of divisors we put $M \cdot N = M \cap N$ if M and N have transverse intersection. It remains to deal with the case when M and N do not intersect transversally.

As with divisors we can replace the subvariety M with another subvariety M' of the same dimension, which is *equivalent* to M. There are different notions of *equivalence* leading to different definitions of intersection product. Some of them are listed below. For simplicity we assume that X is smooth and M is irreducible.

- Rational equivalence: the subvarieties M and M' are *rationally* equivalent if there exists a subvariety N of dimension $(\dim M + 1)$ containing both M and M' such that M

and M' are rationally equivalent divisors in N.

- Homological equivalence: the subvarieties M and M' are *homologically* equivalent if their homology classes $[M]$ and $[M']$ in the homology group of X are the same. This definition depends on a choice of homology theory for X. When X is a compact smooth complex variety we take its De Rham (or singular, which is the same in this case) homology.

- Numerical equivalence: the subvarieties M and M' are *numerically* equivalent if for any subvariety N of complementary dimension the intersection indices $M \cdot N$ and $M' \cdot N$ are the same. In this definition, it is assumed that we already have some way to define intersection indices of subvarieties of complementary dimension. In particular different definitions of the intersection indices lead to different notions of numerical equivalence. We will discuss some examples below.

Note that the rational equivalence is finer than the homological equivalence, i.e. any two rationally equivalent subvarieties are also homologous (which is easy to show using the definition of rational equivalence).

Given an equivalence relation $\sim$ one can define the groups $A_i^{\sim}(X, \mathbb{Z})$ as the formal linear combinations of subvarieties of dimension i in X quotient by the equivalence relation. In the case of rational equivalence, the group $A_i^{rat}(X, \mathbb{Z})$ is called the i-th *Chow group* of X and is sometimes denoted by $CH_i(X)$. In particular, the Chow group $A^{rat}_{(\dim X-1)}(X, \mathbb{Z})$ coincides with the Picard group of X.

Examples.

- Let X be the projective space $\mathbb{CP}^n$. Then $A_i^{rat}(X) = A_i^{hom}(X) = \mathbb{Z}$ if $0 \leq i \leq n$.

- Let X be an elliptic curve over $\mathbb{C}$. Then $A_0^{rat}(X) = \mathrm{Pic}(X) = X \oplus \mathbb{Z}$, while $A_0^{hom} = \mathbb{Z}$ since any two points on X are homologous.

In what follows we will use the notion of homological equivalence. Note that for the Grassmannians the notions of rational and homological equivalence coincide, i.e. the resulting groups $A_*(\mathrm{G}(d, n))$ of cycles are isomorphic. This is a corollary from a more general result about varieties with an algebraic cell decomposition (see [1]).

We now define the intersection product of two Schubert cycles $Z(\mathbf{d})$ and $Z(\mathbf{d}')$ in the homology $H_*(\mathrm{G}(d, n))$. Here $\mathbf{d}$ and $\mathbf{d}'$ denote the collections $(d_1, \ldots, d_k)$ and $(d'_1, \ldots, d'_k)$, respectively. There is a natural family of deformations for each Schubert cycle $Z_F(\mathbf{d})$ given by the action of $GL_n(\mathbb{C})$ on $\mathrm{G}(d, n)$. Namely, replace the flag F (used to construct $Z(\mathbf{d})$) with any other complete flag E. This yields a Schubert cycle $Z_E(\mathbf{d})$ that is homologous to the initial Schubert cycle $Z_F(\mathbf{d})$. Take now two complete flags F and E in $\mathbb{C}^n$. Take the Schubert cycles $Z_F(\mathbf{d})$ and $Z_E(\mathbf{d}')$. Then it is easy to show that for generic pairs of flags (F, E) the intersection

$Z_F(\mathbf{d}) \cap Z_E(\mathbf{d}')$ is transverse, and hence its homology class does not depend on the choice of F and E.

We are now ready to formalize Schubert's idea of degenerating 4 lines to some special position. To compute the intersection index $Z(2,4)^4$ we first find the intersection product $Z(2,4)^2$ as follows. Choose two complete flags $F = \{a_1 \subset l_1 \subset p\}$ and $E = \{a_2 \subset l_2 \subset p_2\}$ in $\mathbb{P}^3$ with the property that the lines l_1 and l_2 intersect. Then set-theoretically $Z_F(2,4) \cap Z_E(2,4)$ is the union of the Schubert cycle $Z(2,3)$ (all lines in the plane of l_1 and l_2) and $Z(1,4)$ (all lines passing through the intersection point $l_1 \cap l_2$). It turns out that that the intersection in this case is multiplicity free (although the flags F and E are not generic with respect to each other). This is a manifestation of a more general principle: Schubert calculus is multiplicity free (see [4] for precise statements and proofs). We get that

$$Z_(2,4)^2 = Z_(1,4) + Z(2,3).$$

It is easy to compute that

$$Z_(1,4)^2 = 1, \quad Z(2,3)^2 = 1, \quad Z(1,4)Z(2,3) = 0,$$

thus getting the desired intersection index $Z(2,4)^4 = 2$.

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