Geometry of spherical varieties and Newton–Okounkov polytopes

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Main results

Euler characteristic of complete intersections in reductive groups

How to extend Brion–Kazarnovskii formula to subvarieties that are not complete intersections?

Convex geometric models for Schubert calculus

How to extend results of K.–Smirnov–Timorin to Schubert cycles on complete flag varieties in any type?

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Notation

Let G be a complex connected reductive group of dimension d and rank r. Let $T \subset G$ be a maximal torus (that is, $\dim T = r$).

Examples

- $G = (\mathbb{C}^*)^n$ — complex torus; $d = r = n$;
- $SL_n(\mathbb{C})$ — special linear group; $d = n^2 - 1$; $r = n - 1$;
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More notation

Let $\pi : G \to GL(V)$ be a faithful finite-dimensional complex representation of G.

Definition

A generic *hyperplane section* $H_\pi \subset G$ is the preimage $\pi^{-1}(H)$ of a generic affine hyperplane $H \subset \mathrm{End(V)}$.

Definition

The *weight polytope* $P_\pi \subset L_T \otimes \mathbb{R}$ is the convex hull of all weights of T that occur in π.

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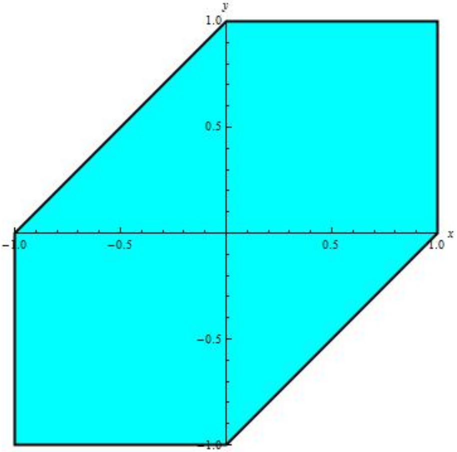
Example

Weight polytope of the adjoint representation of $SL_3(\mathbb{C})$:

$$V = \mathrm{End}(\mathbb{C}^3) \ni X;$$

$$\mathrm{Ad}(g) : X \mapsto gXg^{-1}.$$



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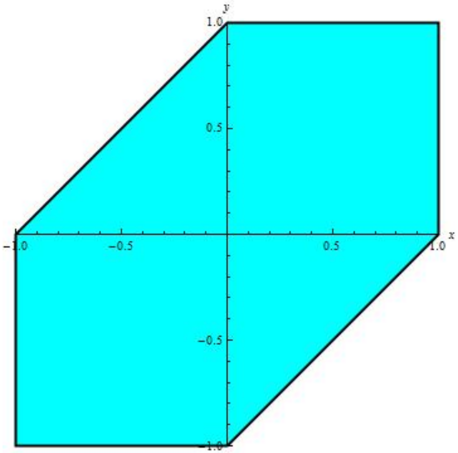

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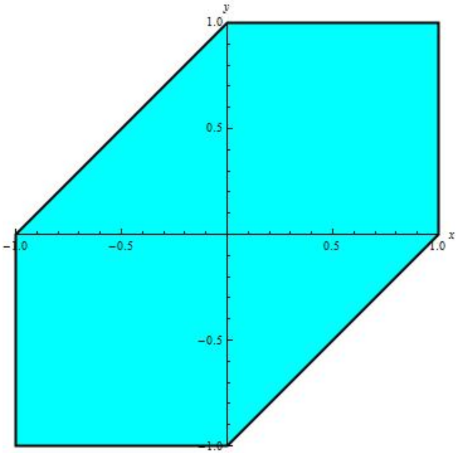

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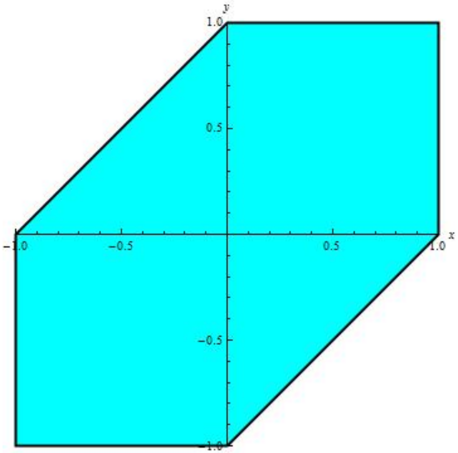

Euler characteristic of complete intersections in reductive groups

Theorem (D.Bernstein, Khovanskii, 1978)

Let $G = (\mathbb{C}^*)^n$. The topological Euler characteristic of a generic hyperplane section H_π can be computed as follows:

$$\chi(H_\pi) = (-1)^{d-1} d!\mathrm{Volume}(P_\pi).$$

Remark

In the torus case, the weight polytope P_π coincides with the Newton polytope of a Laurent polynomial f such that $H_\pi = \{f = 0\}$.

Outline of the proof

First show that $\chi(H_\pi) = (-1)^{d-1} H_\pi^d$, then apply the Kouchnirenko theorem.

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Theorem (Brion 1989, Kazarnovskii 1987)

Let $\mathcal{D} \subset L_T \otimes \mathbb{R}$ be a dominant Weyl chamber, R^+ the set of positive roots of G, and ρ the half of the sum of all positive roots of G.

$$H_\pi^d = d! \int_{P_\pi \cap \mathcal{D}} \prod_{\alpha \in R^+} \frac{(x, \alpha)^2}{(\rho, \alpha)^2} dx.$$

The measure dx on $L_T \otimes \mathbb{R}$ is normalized so that the covolume of L_T is 1.

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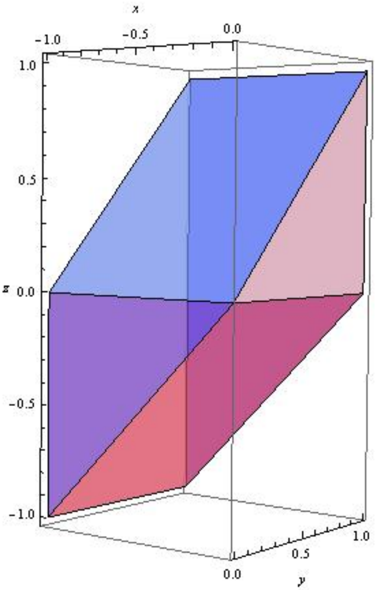
Gelfand–Zetlin polytope for SL_3

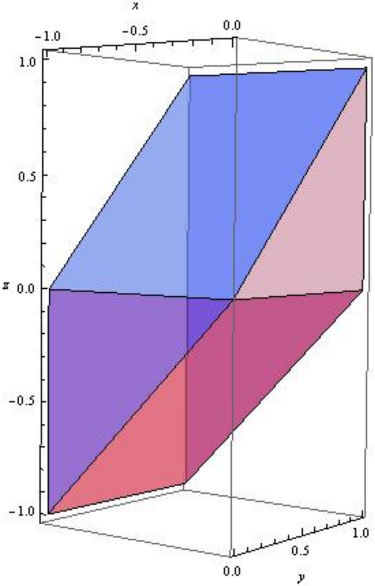


The Gelfand–Zetlin polytopes $GZ(\lambda)$ for SL_3:

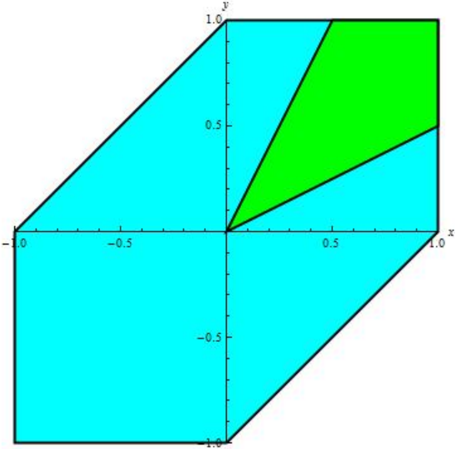
$$\begin{array}{ccccc} \lambda_1 & & \lambda_2 & & \lambda_3 \\ & x & & y & \\ & & z & & \end{array}$$

On the picture, $(\lambda_1, \lambda_2, \lambda 3) = (-1, 0, 1)$.

Brion–Kazarnovskii formula for SL_3



Take the polytope that projects to $P_\pi \cap \mathcal{D}$ and whose fiber at λ is $GZ(\lambda) \times GZ(\lambda)$



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Non-torus example (Kaveh, 2001)

Let $G = SL_2(\mathbb{C})$. If π is an irreducible representation of $SL_2(\mathbb{C})$ with the highest weight $n\omega_1$, then

$$\chi(H_\pi) = 2n^3 - 4n^2 + 4n.$$

Counterexample

The identity

$$\chi(H_\pi) = (-1)^d H_\pi^d$$

does not hold already for $SL_2(\mathbb{C})$.

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Theorem (K., 2004)

There exist elements $S_1, \ldots, S_{d-r}$ (*Chern classes*) in the ring of conditions of G (regarded as $G \times G$-space) such that

$$\chi(H_\pi) = (-1)^{d-1} H_\pi^d + \sum_{i=1}^{d-r} (-1)^{d-i-1} S_i H_\pi^{d-i}.$$

Example

If $G = SL_2(\mathbb{C})$, then $S_1 = [H_{\mathrm{Ad}}]$, $S_2 = 2[T]$. Hence,

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Theorem (K., 2006)

The Euler characteristic of the complete intersection $H_1 \cap \ldots \cap H_m$ is equal to the term of degree d in the expansion of the following product:

$$(1 + S_1 + \ldots + S_{d-r}) \cdot \prod_{i=1}^{m} H_i(1 + H_i)^{-1}.$$

The product in this formula is the intersection product in the ring of conditions.

Euler characteristic of complete intersections in reductive groups

Theorem (K., 2007)

Define the polynomial $F_i(x, y)$ on $(L_T \oplus L_T) \otimes \mathbb{R}$ by extending:

$$F_i(\lambda_1, \lambda_2) := c_i(G/B \times G/B) D^{d-r-i}(\lambda_1, \lambda_2)$$

Then

$$S_i H_\pi^{d-i} = \frac{(d-i)!}{(d-r-i)!} \int\limits_{P_\pi \cap \mathcal{D}} F_i(x, x) dx.$$

Remark

For $i = 0$ and $S_0 = G$, this formula becomes the Brion–Kazarnovskii formula for G.

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Let $G = SL_3(\mathbb{C})$. If π is an irreducible representation of $SL_3(\mathbb{C})$ with the highest weight $m\omega_1 + n\omega_2$, then $\chi(H_\pi)$ is equal to

$$-3(m^8+16m^7n+112m^6n^2+448m^5n^3+700m^4n^4+448m^3n^5+112m^2n^6+$$

$$16mn^7+n^8+18(m^6+12m^5n+50m^4n^2+80m^3n^3+50m^2n^4+12mn^5+n^6)+$$

$$+6(5m^4+40m^3n+72m^2n^2+40mn^3+5n^4)+6(m^2+4mn+n^2)-$$

$$-6(m+n)(m^6+13m^5n+71m^4n^2+139m^3n^3+71m^2n^4+13mn^5+n^6+$$

$$+5(m^4+9m^3n+19m^2n^2+9mn^3+n^4)+3(m^2+5mn+n^2))).$$

Convex geometric models for Schubert calculus

Let $X = G/B$ be the complete flag variety.

Question

How to represent Newton–Okounkov polytopes of Schubert cycles by unions of faces of a single polytope?

Polytopes

Generalizatons of Gelfand–Zetlin polytopes from GL_n to G include *string polytopes*, Newton–Okounkov polytopes of flag varieties, and polytopes constructed via convex-geometric divided difference operators (K., 2016).

Main tool

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Motivating example: flag varieties in type A

Definition

The *flag variety* X is the variety of complete flags in $\mathbb{C}^n$:

$$X = \{\{0\} = V^0 \subset V^1 \subset \ldots \subset V^{n-1} \subset V^n = \mathbb{C}^n \mid \dim V^i = i\}$$

Remark

Alternatively, $X = GL_n(\mathbb{C})/B$, where B denotes the group of upper-triangular matrices (*Borel subgroup*). In this form, the definition can be extended to arbitrary connected reductive groups.

Schubert varieties

$$X_w = \overline{BwB/B}, \; w \in S_n$$

give basis in $H^*(X, \mathbb{Z})$.

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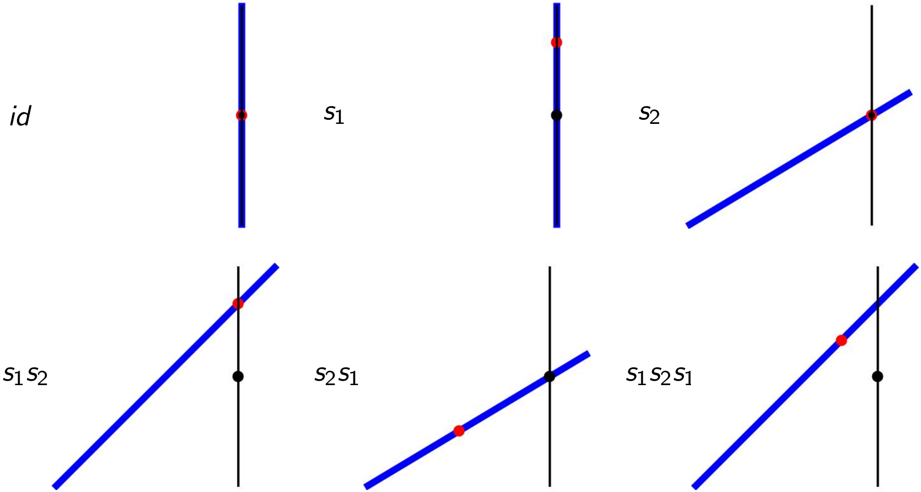
Schubert varieties for GL_3/B.



Gelfand–Zetlin polytopes

The Gelfand–Zetlin polytope Δ_λ is defined by inequalities:

$$
\begin{array}{ccccccccccc}
\lambda_1 & & \lambda_2 & & \lambda_3 & & \ldots & & \lambda_n \\
& x_1^1 & & x_2^1 & & \ldots & & x_{n-1}^1 & \\
& & x_1^2 & & \ldots & & x_{n-2}^2 & & \\
& & & \ddots & \ldots & & & & \\
& & & x_1^{n-2} & & x_2^{n-2} & & & \\
& & & & x_1^{n-1} & & & &
\end{array}
$$

where $(x_1^1, \ldots, x_{n-1}^1; \ldots; x_1^{n-1})$ are coordinates in $\mathbb{R}^d$, and the notation

$$
\begin{array}{ccc}
a & & b \\
& c &
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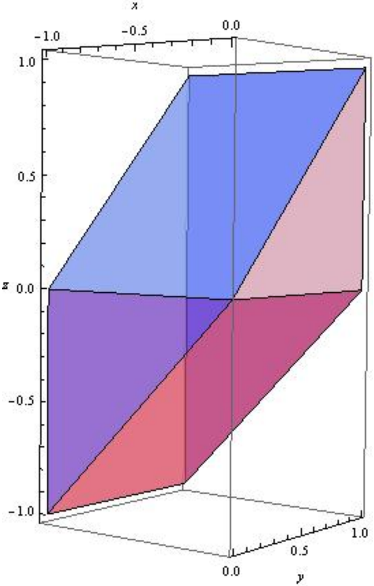
means $a \leq c \leq b$.

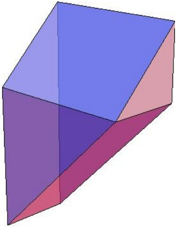
Gelfand–Zetlin polytopes

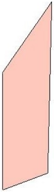


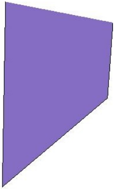
A Gelfand–Zetlin polytope for GL_3:

$$\begin{array}{ccccc} -1 & & 0 & & 1 \\ & x & & y & \\ & & z & & \end{array}$$

Schubert calculus and Gelfand–Zetlin polytopes

$[X_{s_1 s_2 s_1}] =$ 

$[X_{s_1 s_2}] =$ 

$[X_{s_2 s_1}] =$ 

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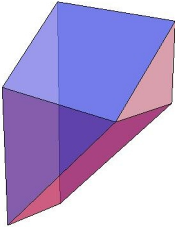
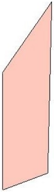
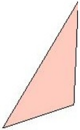
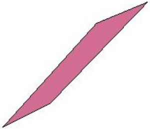
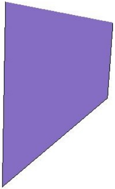
$[X_{s_1 s_2 s_1}] =$ 

$[X_{s_1 s_2}] =$  $=$  $+$ 

$[X_{s_2 s_1}] =$ 

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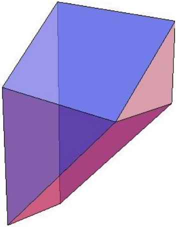
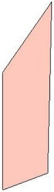
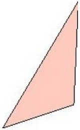
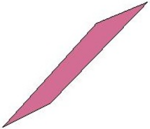
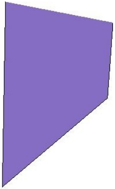
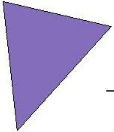
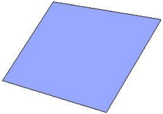
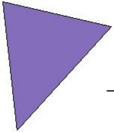
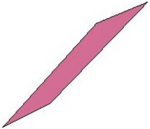
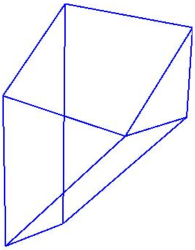
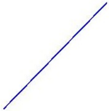
$[X_{s_1 s_2 s_1}] =$ 

$[X_{s_1 s_2}] =$  $=$  $+$ 

$[X_{s_2 s_1}] =$  $=$  $+$  $=$  $+$ 

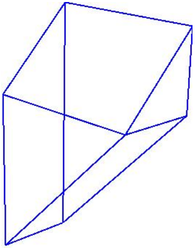
Schubert calculus and Gelfand–Zetlin polytopes

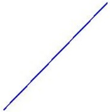


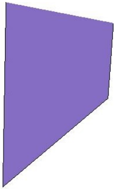
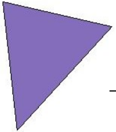
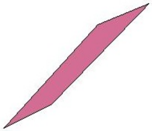
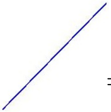
$[X_{s_1}] =$ $=$  $[X_{s_2}] =$  $=$

 $= [X_{s_1}] + [X_{s_2}]$

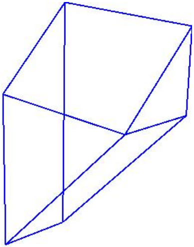
Schubert calculus and Gelfand–Zetlin polytopes



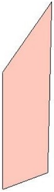
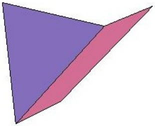
$[X_{s_1}] =$ $=$  $[X_{s_2}] =$  $=$

$[X_{s_2 s_1}]^2 =$  $\cdot \Big($  $+$  $\Big) =$  $= [X_{s_1}]$

Schubert calculus and Gelfand–Zetlin polytopes



$[X_{s_1}] =$; $[X_{s_2}] =$ 

$[X_{s_1 s_2}] \cdot [X_{s_2 s_1}] =$  $\cdot$  $=$ $+$  $= [X_{s_1}] + [X_{s_2}]$

Flag varieties and Gelfand–Zetlin polytopes

Results

- Relation between Schubert varieties and preimages of rc-faces of P_λ under the Guillemin–Sternberg moment map $X \to P_\lambda$ (Kogan, 2000)
- Degenerations of Schubert varieties to (reducible) toric varieties given by (unions of) faces of P_λ (Kogan–E.Miller, Knutson–E.Miller, 2003)
- Description of $H^*(X, \mathbb{Z})$ using *volume polynomial* of P_λ (Kaveh, 2011)
- Schubert calculus: intersection product of Schubert cycles in $H^*(X, \mathbb{Z})$ = intersection of faces in P_λ (K.–Smirnov–Timorin, 2012)

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String polytopes

(J.Miller, 2014)

Newton–Okounkov polytopes of Schubert varieties can be represented by unions of faces of a given string polytope.

Remark

This is an existence result. Explicit descriptions of such faces are so far known in the case of GL_n, $\overline{w_0} = s_1(s_2s_1)\cdots(s_{n-1}\cdots s_1)$ (K.–Smirnov–Timorin, 2012) and Sp_4, $\overline{w_0} = s_1s_2s_1s_2$ (Ilyukhina, 2012).

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Find an efficient algorithm for representing Schubert cycles explicitly by unions of faces.

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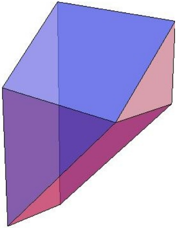
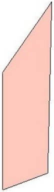
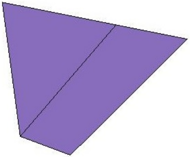
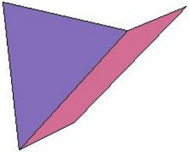
Geometric mitosis

$[X]$ $=$ 

$[X_{s_1 s_2}]$ $=$ 

$[X_{s_2 s_1}]$ $=$  $\sim$ 

Mitosis on parallelepipeds

Coordinate parallelepipeds

Let $\Pi := \Pi(\mu, \nu) \subset \mathbb{R}^n$ be given by inequalities $\mu_i \leq x_i \leq \nu_i$ for $i = 1, \ldots, n$.

Essential edges

An edge of Π is *essential* if it is given by equations

$$x_1 = \mu_1, \ldots, x_{i-1} = \mu_{i-1}; \quad x_{i+1} = \nu_{i+1}, \ldots, x_n = \nu_n$$

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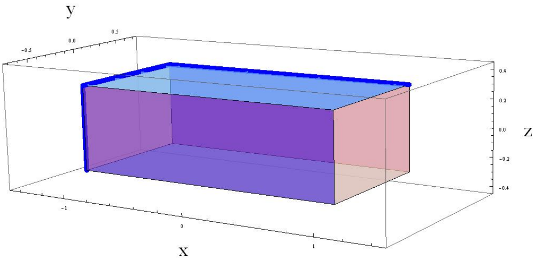

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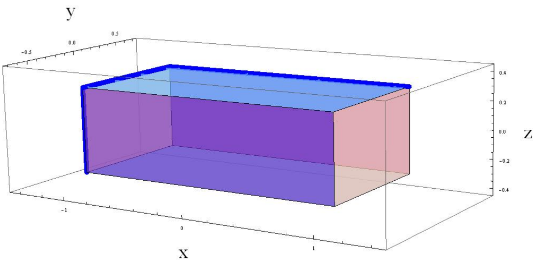

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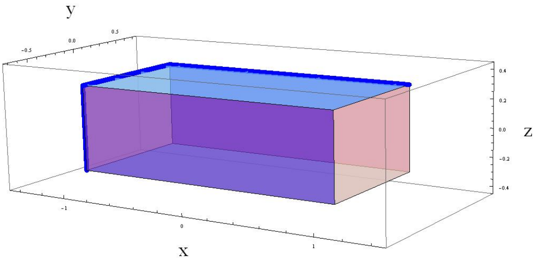

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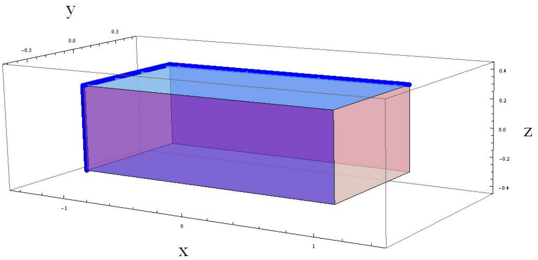

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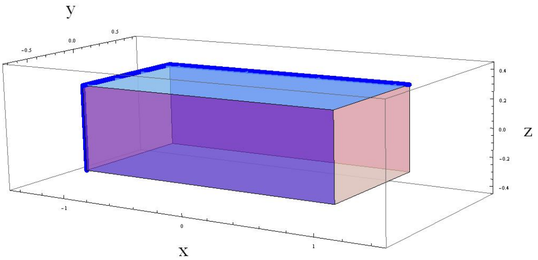

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For every face $\Gamma \subset \Pi$, we now define a collection of faces $M(\Gamma)$

1. Let k be the minimal number such that $\Gamma \subseteq \{x_i = \mu_i\}$ for all $i > k$ (in particular, $\Gamma \nsubseteq \{x_k = \mu_k\}$) and $\nu_i \neq \mu_i$ for at least one $i > k$. If no such k exists then $M(\Gamma) = \emptyset$.
2. Under the isomorphism $\mathbb{R}^n \simeq \mathbb{R}^k \times \mathbb{R}^{n-k}$; $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_k) \times (x_{k+1}, \ldots, x_n)$ we have

$$\Pi \simeq \Pi' \times \Pi''; \quad \Gamma \simeq \Gamma' \times v$$

   where $v = (\mu_{k+1}, \ldots, \mu_n) \in \Pi''$ and $\Gamma' \subset \Pi'$.
3. The set $M(\Gamma)$ consists of all faces $\Gamma' \times E$ such that E is an essential edge of Π''.

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If Γ is the vertex $(\mu_1, \ldots, \mu_n)$, then $M(\Gamma)$ is the set of essential edges of Π.

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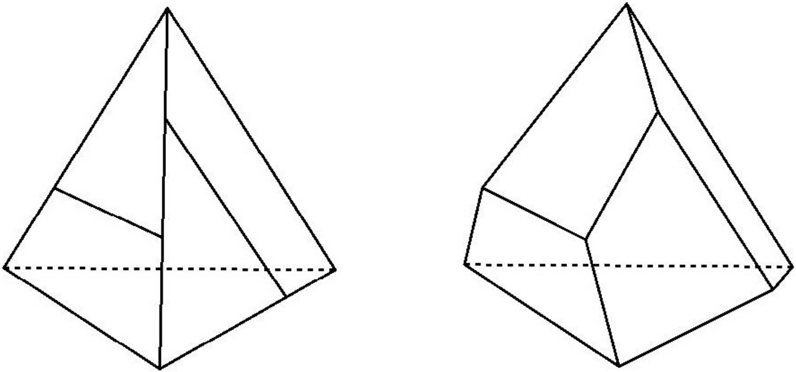

The subdivision of a tetrahedron by two extra edges yields a combinatorial cube. Essential edges of the cube form a single edge of the tetrahedron.

Mitosis on parallelepipeds and pipe-dreams

Faces can be encoded by $2 \times n$ tables

$+ \Leftrightarrow x_1 = \mu_1$	$\dots$	$+ \Leftrightarrow x_n = \mu_n$
$+ \Leftrightarrow x_1 = \nu_1$	$\dots$	$+ \Leftrightarrow x_n = \nu_n$

Example

If $\Pi(\mu, \nu) \subset \mathbb{R}^4$, where $\mu = (1,1,1,1)$ and $\nu = (2,2,1,2)$ (that is, $\mu_3 = \nu_3$), then the vertex $\Gamma = \{x_1 = \nu_1,\ x_2 = \mu_2,\ x_4 = \mu_4\}$ is

	$+$	$+$	$+$
$+$		$+$	

The set $M(\Gamma)$ consists of two edges represented by the tables

	$+$	$+$	
$+$		$+$	

&

		$+$	
$+$		$+$	$+$

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	$+$	$+$	
$+$		$+$	

&

		$+$	
$+$		$+$	$+$

Geometric mitosis: type A

Gelfand–Zetlin polytope

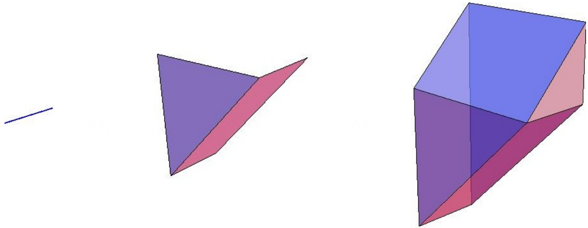
$$
\begin{array}{ccccccccc}
\lambda_1 & & \lambda_2 & & \lambda_3 & & \dots & & \lambda_n \\
& x_1^1 & & x_2^1 & & \dots & & x_{n-1}^1 & \\
& & x_1^2 & & \dots & & x_{n-2}^2 & & \\
& & & \ddots & \dots & & & & \\
& & & x_1^{n-2} & & x_2^{n-2} & & & \\
& & & & x_1^{n-1} & & & &
\end{array}
$$

has $(n-1)$ different fibrations by coordinate parallelepipeds. Hence, there are $(n-1)$ different mitosis operations on its faces.

Geometric mitosis: type A

Example GL_3

$$\begin{array}{|c|c|}\hline + & + \\ \hline + & \\ \hline \end{array} \xrightarrow{M_2} \begin{array}{|c|c|}\hline + & + \\ \hline & \\ \hline \end{array} \xrightarrow{M_1} \begin{array}{|c|c|}\hline & + \\ \hline & \\ \hline \end{array} \;\&\; \begin{array}{|c|c|}\hline & \\ \hline + & \\ \hline \end{array} \xrightarrow{M_2} \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array}$$



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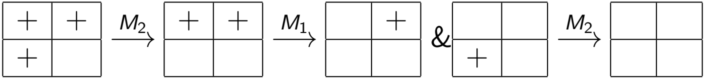

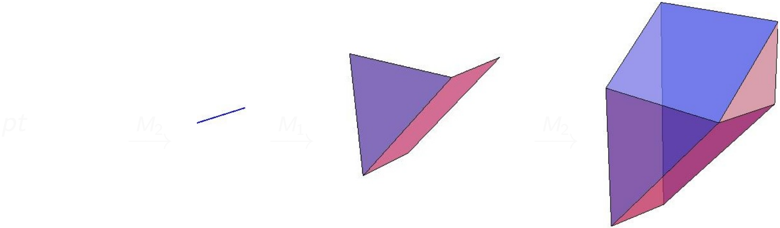

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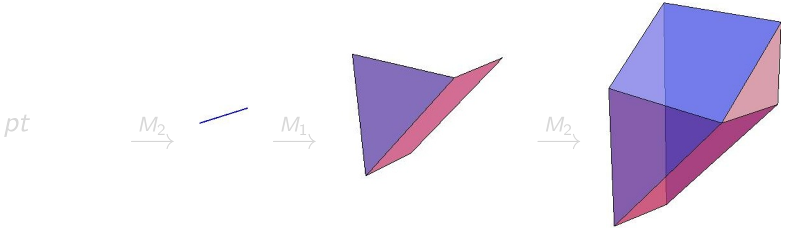

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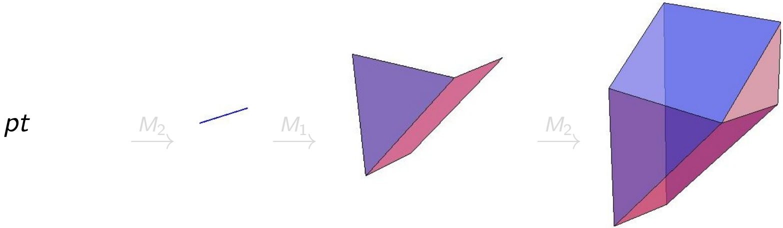

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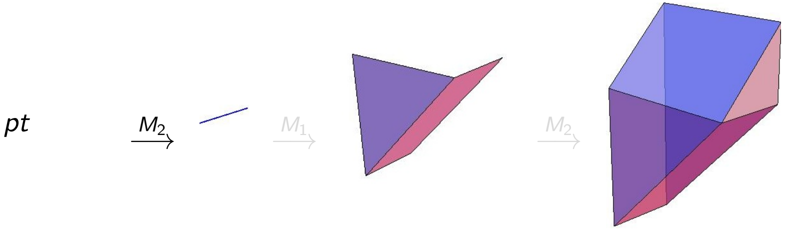

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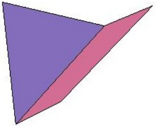
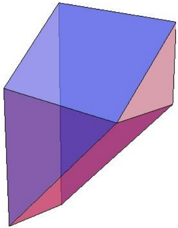
$pt \xrightarrow{M_2}$  $\xrightarrow{M_1}$  $\xrightarrow{M_2}$ 

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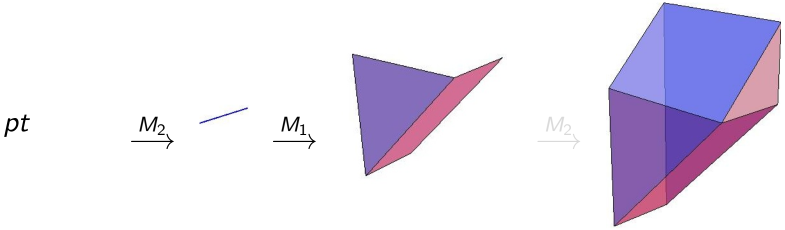

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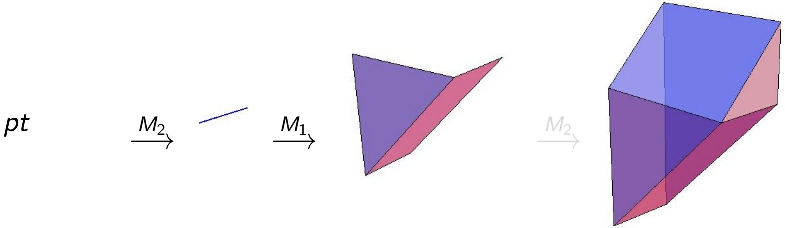

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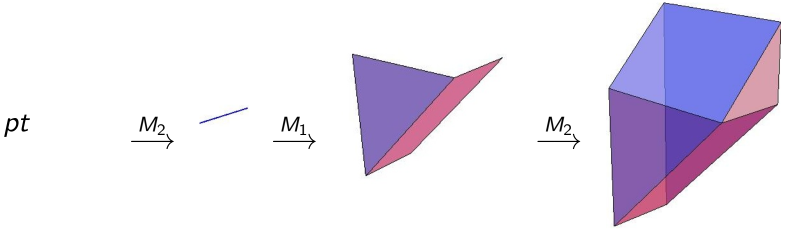

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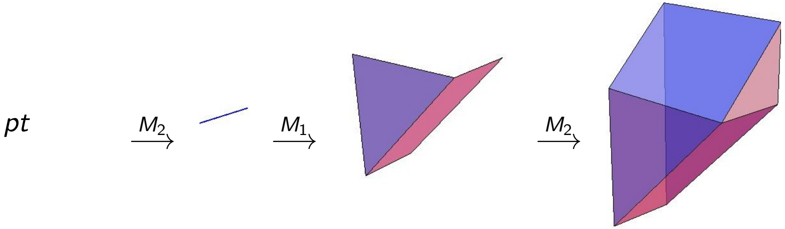

Geometric mitosis: type C

Example Sp_4

Take $\overline{w_0} = s_2 s_1 s_2 s_1$. The corresponding DDO polytope Q_λ is given by inequalities

$$0 \le x \le \lambda_1, \quad z \le x + \lambda_2, \quad y \le 2z,$$

$$y \le z + \lambda_2, \quad 0 \le t \le \lambda_2, \quad t \le \frac{y}{2}.$$

Remark

The polytopes Q_λ coincide with the Newton–Okounkov polytopes of Sp_4/B for the lowest order term valuation v associated with the flag of subvarieties $w_0 X_{id} \subset s_1 s_2 s_1 X_{s_2} \subset s_1 s_2 X_{s_1 s_2} \subset s_1 X_{s_2 s_1 s_2} \subset X$.

Remark

The polytopes Q_λ have 11 vertices so they are not combinatorially equivalent to string polytopes associated with $s_2 s_1 s_2 s_1$ or $s_1 s_2 s_1 s_2$.

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Geometric mitosis: type C

Skew pipe-dreams

Faces that contain the lowest vertex $a_\lambda = (0,0,0,0)$ can be encoded by the diagrams:

	$+ \iff 0 = t$
$+ \iff 0 = x$	$+ \iff t = \frac{y}{2}$
	$+ \iff y = 2z$

Parallelepipeds

The polytope Q_λ admits two different fibrations (by translates of xy- and zt-planes), hence, there are two mitosis operations M_1 and M_2 on faces of Q_λ.

Isotropic flags

$$Sp_4/B = \{(V^1 \subset V^2 \subset V^3 \subset \mathbb{C}^4) \mid \omega|_{V^2} = 0, V^1 = {V^3}^{\perp}\} =$$
$$= \{(a \in l \subset \mathbb{P}^3) \mid l - \text{ isotropic line } \}$$

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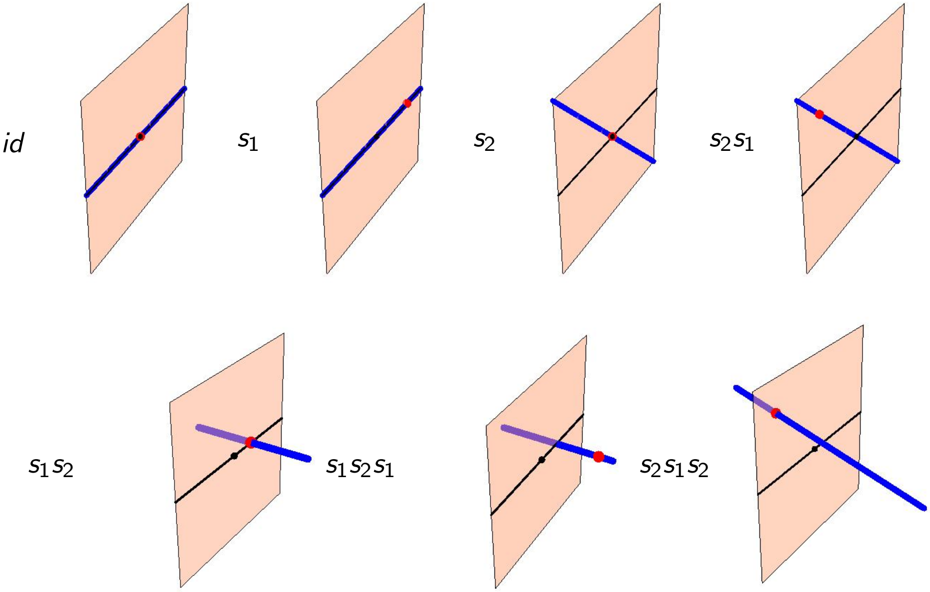
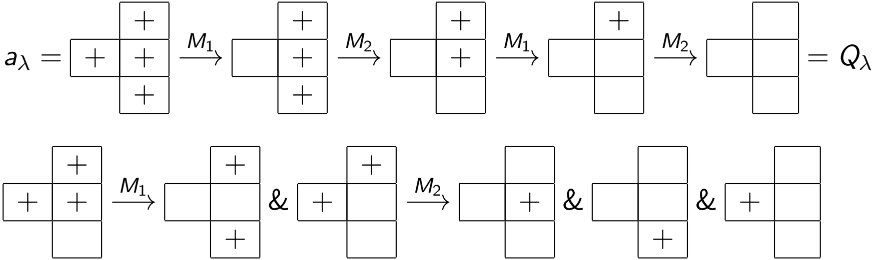
Schubert cycles for Sp_4



Geometric mitosis: type C



Newton–Okounkov polytopes

Valuation

Let $X^n \subset \mathbb{P}^N$ be a projective subvariety with coordinates $(x_1, \dots, x_n)$ in a neighborhood of a smooth point $p \in X$. Define the valuation $v : \mathbb{C}(X) \to \mathbb{Z}^n$ by sending every polynomial $f(x_1, \dots, x_n)$ to $(k_1, \dots, k_n)$ where $x_1^{k_1} \cdots x_n^{k_n}$ is the lowest degree term in f (assuming that $x_1 \succ x_2 \succ \dots \succ x_n$).

Vector space

Let $V \subset \mathbb{C}(X)$ be the vector space spanned by $1, \frac{y_1}{y_0}, \dots, \frac{y_N}{y_0}$, where $(y_0, y_1, \dots, y_N)$ are homogeneous coordinates on $\mathbb{P}^N$.

Example

If $X = \nu_N(\mathbb{P}^1) = \{(u_0^N : u_1 u_0^{N-1} : \dots : u_1^N)\} \subset \mathbb{P}^N$ and $x_1 = \frac{u_1}{u_0}$, then $v(f) =$ the order of zero (or pole) of f at $p = (1:0)$ and $V = \langle 1, x_1, \dots, x_1^N \rangle$.

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Naive definition

The Newton–Okounkov polytope $\Delta_v(X) \subset \mathbb{R}^n$ of X^n is the convex hull of $v(f)$ for all $f \in V$.

Example

$\Delta_v(\nu_N(\mathbb{P}^1)) = [0, N] \subset \mathbb{R}^1$

Example

A toric variety X^n has a natural system of coordinates $(x_1, \ldots, x_n)$ coming from $(\mathbb{C}^*)^n \subset X^n$. For a projective embedding $X^n \subset \mathbb{P}^N$, the space V is spanned by monomials in $x_1, \ldots, x_n$. Hence, the valuation v does not matter, and $\Delta_v(X^n)$ is always the Newton polytope of X.

Observation

If $n!\mathrm{volume}(\Delta_v(X)) = \deg(X)$, then the naive definition coincides with the correct definition.

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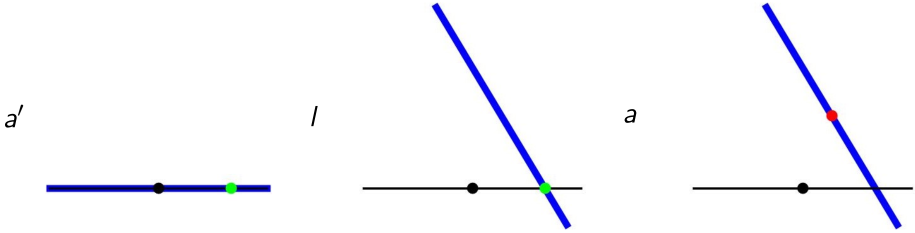
A Newton–Okounkov polytope of GL_3/B

Coordinates on the open Schubert cell

If the flag $(a \in l \subset \mathbb{P}^2)$ is in general position with a fixed flag $(a_0 \in l_0 \subset \mathbb{P}^2)$, then $l \cap l_0 = a' \neq a_0$ and $a \notin l_0$. Hence,

$$a' = (x : 1 : 0); \quad l = \langle a', (y : 0 : 1) \rangle; \quad a = (xz + y : z : 1)$$

are coordinates (assuming that $a_0 = (1 : 0 : 0)$, $l_0 = \{(\star : \star : 0)\}$).



A Newton–Okounkov polytope of GL_3/B

(D.Anderson, 2011)

Consider the embedding $p : GL_3/B \hookrightarrow \mathbb{P}^2 \times (\mathbb{P}^2)^* \hookrightarrow \mathbb{P}^8$;
$p : (a, l) \mapsto a \times l$. Then p takes the flag with coordinates (x, y, z) to

$$\begin{pmatrix} xz+y & z & 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -x \\ -y \end{pmatrix} = \begin{pmatrix} xz+y & -x^2z-xy & -xyz-y^2 \\ z & -xz & -yz \\ 1 & -x & -y \end{pmatrix}$$

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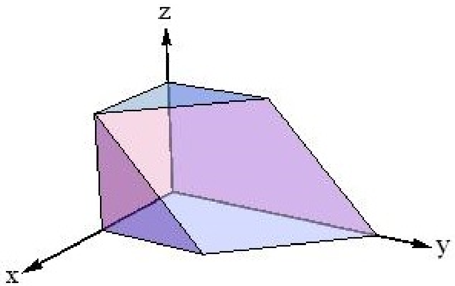

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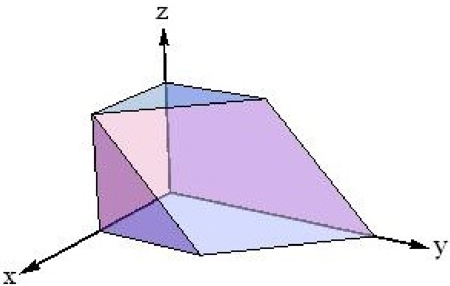

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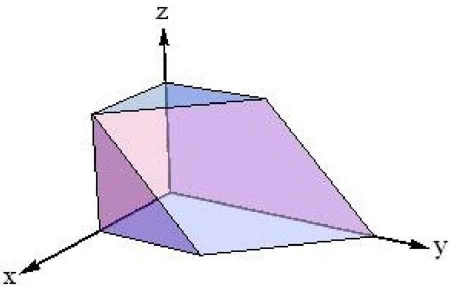
$\Delta_v(p(GL_3/B)) =$ 

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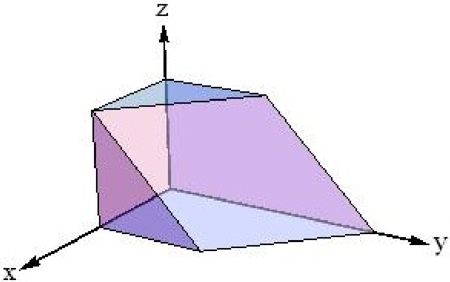
$$\begin{pmatrix} xz+y & z & 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -x \\ -y \end{pmatrix} = \begin{pmatrix} xz+y & -x^2z-xy & -xyz-y^2 \\ z & -xz & -yz \\ 1 & -x & -y \end{pmatrix}$$

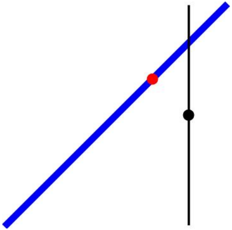
$\Delta_v(p(GL_3/B)) =$



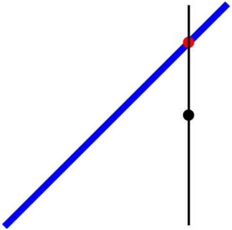
Enumerative geometry

High school geometry problem

How many flags in $\mathbb{P}^2$ are not in general position with respect to three given flags?



Two flags in general position

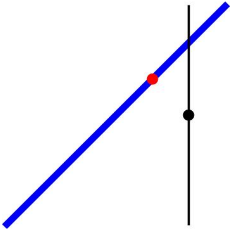


Two flags NOT in general position

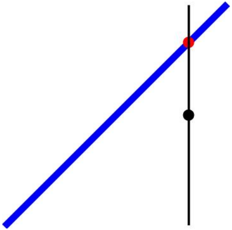
Enumerative geometry

High school geometry problem

How many flags in $\mathbb{P}^2$ are not in general position with respect to three given flags?



Two flags in general position



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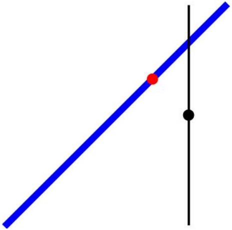

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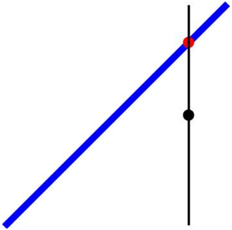

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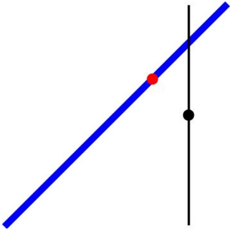

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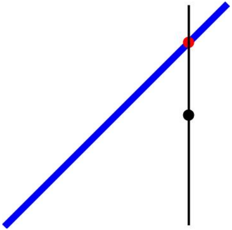
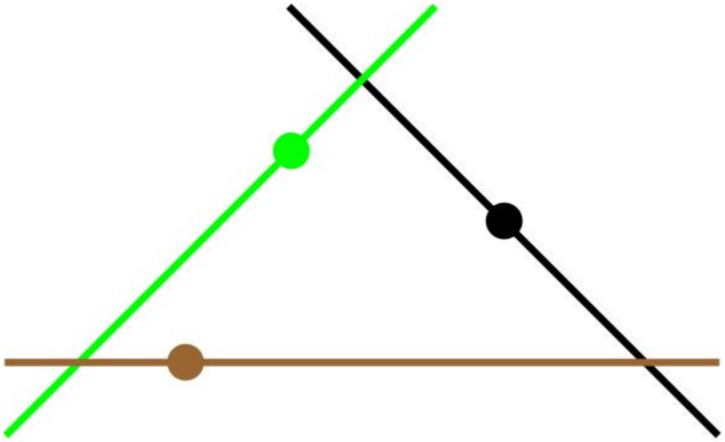


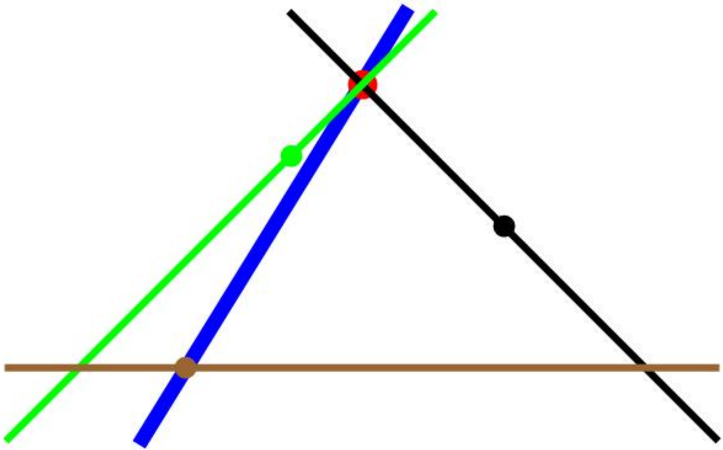
Two flags NOT in general position

Enumerative geometry



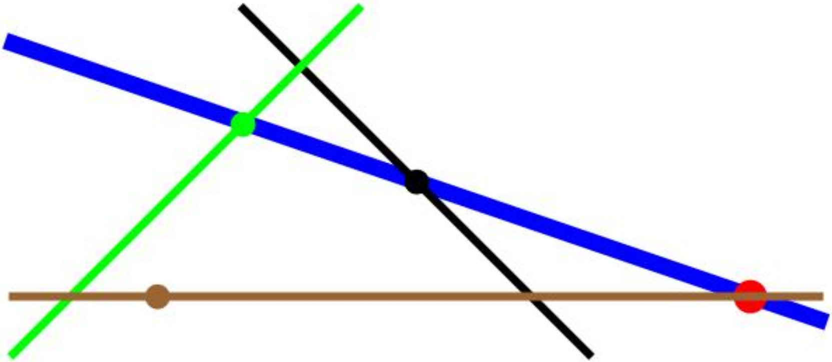
Three flags in the plane

Enumerative geometry



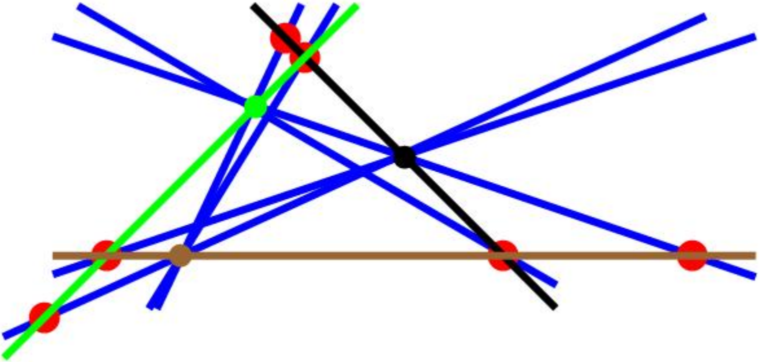
A flag not in general position with respect to three given flags: variant 1

Enumerative geometry



A flag not in general position with respect to three given flags: variant 2

Enumerative geometry



A flag not in general position with respect to three given flags.
Answer: 6.

Valuations on $\mathbb{C}(G/B)$

Decomposition of w_0

Fix a reduced decomposition $\overline{w_0} = s_{i_1} \ldots s_{i_d}$ of the longest element w_0 in the Weyl group of G.

Flag of Schubert varieties

Choose coordinates compatible with the flag $X_{id} \subset X_{s_{i_d}} \subset X_{s_{i_{d-1}} s_{i_d}} \subset \ldots \subset X_{s_{i_2} \cdots s_{i_d}} \subset X$ (coordinates "at infinity").

Flag of translated Schubert varieties

Choose coordinates compatible with the flag $w_0 X_{id} \subset s_{i_1} \ldots s_{i_{d-1}} X_{s_{i_d}} \subset s_{i_1} \ldots s_{i_{d-2}} X_{s_{i_{d-1}} s_{i_d}} \subset \ldots \subset s_{i_1} X_{s_{i_2} \cdots s_{i_d}} \subset X$ (coordinates at the open Schubert cell).

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(coordinates at the open Schubert cell).

Newton–Okounkov polytopes of flag varieties

(Okounkov, 1998)

The *symplectic* Gelfand–Zetlin polytopes coincide with the Newton–Okounkov polytopes of Sp_{2n}/B for the lowest order term valuation v associated with the flag of Schubert varieties for initial subwords of $\overline{w_0} = (s_1)(s_2 s_1 s_2)\dots(s_n s_{n-1}\dots s_2 s_1 s_2 \dots s_{n-1} s_n)$.

(Kaveh, 2013)

The string polytopes associated with $\overline{w_0}$ coincide with the Newton–Okounkov polytopes of X for the highest order term valuation v associated with the flag of Schubert varieties for $\overline{w_0}$.

Example

If $G = GL_n$ and $\overline{w_0} = s_1(s_2 s_1)\cdots(s_{n-1}\cdots s_1)$ then the corresponding string polytopes are exactly Gelfand–Zetlin polytopes.

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Newton–Okounkov polytopes of flag varieties

(Fujita–Naito, Fujita–Oya 2017)

The string polytopes associated with $\overline{w_0}$ coincide with the Newton–Okounkov polytopes of X for the lowest order term-initial subwords valuation v_{in} and for the highest order term-terminal subwords valuation v^{term} associated with the flag of Schubert varieties $\overline{w_0}$. The Nakashima–Zelevinsky polyhedral realizations associated with $\overline{w_0}$ coincide with the Newton–Okounkov polytopes of X for the lowest order term-terminal subwords valuation v_{term} and for the highest order term-initial subwords v^{in} associated with the flag of Schubert varieties $\overline{w_0}$.

Newton–Okounkov polytopes of flag varieties

(E.Feigin–Fourier–Littelmann 2017)

The Feigin–Fourier–Littelmann–Vinberg polytopes coincide with the Newton–Okounkov polytopes of X for a valuation not coming from any longest word decomposition $\overline{w_0}$

(K. 2017)

The Feigin–Fourier–Littelmann–Vinberg polytopes in type A coincide with the Newton–Okounkov polytopes of X for the longest word decomposition $\overline{w_0} = s_1(s_2s_1)\cdots(s_{n-1}\cdots s_1)$ and the lowest order term valuation associated with the flag of translated Schubert subvarieties:

$w_0X_{id} \subset s_{i_1}\dots s_{i_{d-1}}X_{s_{i_d}} \subset s_{i_1}\dots s_{i_{d-2}}X_{s_{i_{d-1}}s_{i_d}} \subset \dots \subset s_{i_1}X_{s_{i_2}\cdots s_{i_d}} \subset X$

Newton–Okounkov polytopes of flag varieties

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The Feigin–Fourier–Littelmann–Vinberg polytopes coincide with the Newton–Okounkov polytopes of X for a valuation not coming from any longest word decomposition $\overline{w_0}$

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The Feigin–Fourier–Littelmann–Vinberg polytopes in type A coincide with the Newton–Okounkov polytopes of X for the longest word decomposition $\overline{w_0} = s_1(s_2s_1)\cdots(s_{n-1}\cdots s_1)$ and the lowest order term valuation associated with the flag of translated Schubert subvarieties:

$w_0X_{id} \subset s_{i_1}\ldots s_{i_{d-1}}X_{s_{i_d}} \subset s_{i_1}\ldots s_{i_{d-2}}X_{s_{i_{d-1}}s_{i_d}} \subset \ldots \subset s_{i_1}X_{s_{i_2}\cdots s_{i_d}} \subset X$

Thank you!