Gelfand–Zetlin polytopes and Demazure characters

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Toric geometry

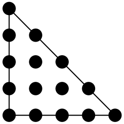
$P \subset \mathbb{R}^n$ — integer convex polytope $\rightsquigarrow X_P^n \subset \mathbb{CP}^N$ — projective toric variety

The Hilbert polynomial H of X_P = the Ehrhart polynomial of P:

$$H(k) = |kP \cap \mathbb{Z}^n|$$

Example: $n = 2$; $P =$  $\rightsquigarrow X_P = \mathbb{P}^2$

$H(k)$ =the number of monomials of degree$\leq k$ in x and y = the number of integer points inside and at the boundary of kP



Generalizations

- Spherical varieties (flag varieties, symmetric varieties, wonderful compactifications,...) $\leftrightsquigarrow$ moment polytopes, string polytopes (generalizations of Gelfand–Zetlin polytopes),...;
  Brion, Kazarnovskii, Littelmann–Berenstein–Zelevinskii,...
- Arbitrary algebraic varieties $\leftrightsquigarrow$ Newton-Okounkov bodies; Kaveh–Khovanskii, Lazarsfeld-Mustata (2009)
- Algebraic varieties with a reductive group action $\leftrightsquigarrow$ moment, multiplicity and string bodies;
  Kaveh–Khovanskii (2010)

Flag varieties

X — the variety of complete flags in $\mathbb{C}^n$:

$$X = \{\{0\} = V^0 \subset V^1 \subset \ldots \subset V^{n-1} \subset V^n = \mathbb{C}^n | \dim V^i = i\}$$

Alternatively, $X = GL_n(\mathbb{C})/B$, where B — upper-triangular matrices.

projective embeddings of $X \leftrightarrow$ irreducible representations of $GL_n(\mathbb{C})$ with strictly dominant weights

Flag varieties and Gelfand–Zetlin polytopes

$\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ — a strictly dominant weight of the group $GL_n(\mathbb{C})$, i.e. $\lambda_i < \lambda_{i+1}$ for all $i = 1, \ldots, n-1$.

V_λ — the highest weight irreducible GL_n-module with the highest weight λ

$X \hookrightarrow \mathbb{P}(V_\lambda)$ - projective embedding

$P_\lambda \subset \mathbb{R}^d$ (where $d = n(n-1)/2$) — the Gelfand–Zetlin polytope (a convex integer polytope).

The integer points inside and at the boundary of P_λ parameterize a natural basis (*Gelfand–Zetlin basis*) in V_λ.

Gelfand–Zetlin polytope

The Gelfand–Zetlin polytope P_λ is defined by inequalities:

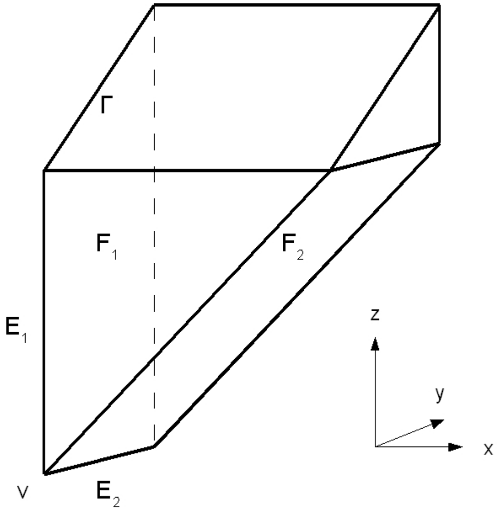
$$
\begin{array}{ccccccccc}
\lambda_1 & & \lambda_2 & & \lambda_3 & & \dots & & \lambda_n \\
& \lambda_{1,1} & & \lambda_{1,2} & & \dots & & \lambda_{1,n-1} & \\
& & \lambda_{2,1} & & \dots & & \lambda_{2,n-2} & & \\
& & & \ddots & & \dots & & & \\
& & & \lambda_{n-2,1} & & \lambda_{n-2,2} & & & \\
& & & & \lambda_{n-1,1} & & & &
\end{array}
$$

where $(\lambda_{1,1}, \dots, \lambda_{1,n-1}; \lambda_{2,1}, \dots, \lambda_{2,n-2}; \dots; \lambda_{n-2,1}, \lambda_{n-2,2}; \lambda_{n-1,1})$ are coordinates in $\mathbb{R}^d$, and the notation

$$
\begin{array}{ccc}
a & & b \\
& c &
\end{array}
$$

means $a \leq c \leq b$.

Gelfand–Zetlin polytopes



The Gelfand–Zetlin polytope for GL_3:

$$\begin{matrix} \lambda_1 & & \lambda_2 & & \lambda_3 \\ & x & & y & \\ & & z & & \end{matrix}$$

Toric geometry

$P \subset \mathbb{R}^n$ — integer convex polytope $\rightsquigarrow X_P^n \subset \mathbb{CP}^N$ — projective toric variety

Faces $\Gamma \subset P \leftrightarrow$ irreducible toric subvarieties $X_\Gamma \subset X_P$

The Hilbert polynomial H of $X_\Gamma =$ the Ehrhart polynomial of Γ.

This is important for intersection theory on X_P since

$$[X_F] \cdot [X_G] = [X_{F \cap G}]$$

if F and G are transverse.

Flag varieties

$w \in S_n$ —permutation $\rightsquigarrow$ Schubert variety $X^w \subset X$

$X^w = \overline{B^- w}$, where B^- — lower-triangular matrices and w acts on the standard basis vectors e_i by the formula $e_i \mapsto e_{w(i)}$.

The codimension of X^w is equal to the length $l(w)$ of w.

Goal: find faces of the Gelfand–Zetlin polytope responsible for the Hilbert polynomial of the Schubert variety $X^w \subset X \hookrightarrow \mathbb{P}(V_\lambda)$.

Gelfand–Zetlin polytopes

Kogan faces — faces of the Gelfand–Zetlin polytope given by the equations of the type $\lambda_{i,j} = \lambda_{i+1,j}$.

$$\begin{matrix} \lambda_{i,j} & & \lambda_{i,j+1} \\ & \lambda_{i+1,j} & \end{matrix}$$

Kogan face $F \rightsquigarrow$ permutation $w(F)$

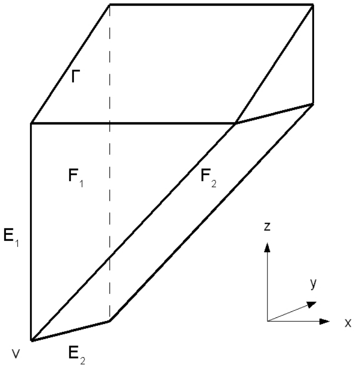
1. $\lambda_{i,j} = \lambda_{i+1,j} \rightsquigarrow$ transposition $s_{i+j} = (i+j, i+j+1)$.
2. compose s_{i+j} by going from left to right in each row and by going from the bottom row to the top one.

We say that a Kogan face F is *reduced* if the decomposition for $w(F)$ obtained this way is reduced.

Kogan faces $\leftrightarrow$ *pipe-dreams* (Kogan 2000)

Gelfand–Zetlin polytopes

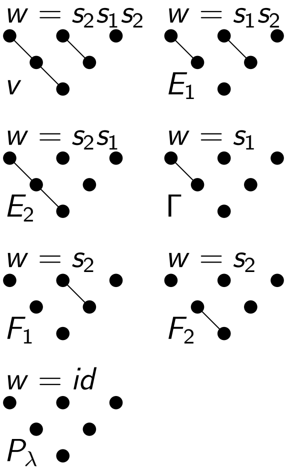
Kogan faces for $n = 3$



$\lambda_1 \qquad \lambda_2 \qquad \lambda_3$

$x \qquad y$

z



Note that the faces F_1 and F_2 have the same permutation.

Demazure characters

$X^w \subset X \hookrightarrow \mathbb{P}(V_\lambda)$

$\mathcal{L}_\lambda$ — restriction to $X^w \subset \mathbb{P}(V_\lambda)$ of the tautological line bundle

$V^-_{\lambda,w} := H^0(X^w, \mathcal{L}_\lambda)^*$ is a B^--module called *Demazure module*

$\chi^w(\lambda)$ — the character of $V^-_{\lambda,w}$ called *Demazure character*

Choose a basis of weight vectors in $V^-_{\lambda,w}$. The *character* $\chi^w(\lambda)$ of $V^-_{\lambda,w}$ is

$$\chi^w(\lambda) := \sum_{\mu\in\Lambda} m_{\lambda,w}(\mu)e^\mu,$$

where Λ is the weight lattice of GL_n and $m_{\lambda,w}(\mu)$ is the multiplicity of the weight μ in $V^-_{\lambda,w}$.

Demazure characters

an integer point $z \in P_\lambda \rightsquigarrow$ the weight $p(z) \in \Lambda$

This extends to the linear map $p : \mathbb{R}^d \to \mathbb{R}^{n-1}$ from the space $\mathbb{R}^d$ with coordinates $\lambda_{i,j}$ to the space $\mathbb{R}^n$ with coordinates u_i:

$$u_i = \sum_{j=1}^{n-i} \lambda_{i,j}, \quad u_n = \lambda_1 + \ldots + \lambda_n$$

$S \subset P_\lambda$ — subset $\rightsquigarrow$ *character* $\chi(S)$

$$\chi(S) := \sum_{z \in S \cap \mathbb{Z}^d} e^{p(z)}.$$

Example: $S = P_\lambda \rightsquigarrow \chi(S) = \chi^{id}(\lambda)$

Demazure characters

Theorem (VK, Evgeny Smirnov, Vladlen Timorin, 2010)

For each permutation $w \in S_n$, the Demazure character $\chi^w(\lambda)$ is equal to the character of the union of the reduced Kogan faces in the Gelfand–Zetlin polytope P_λ, whose permutation is w:

$$\chi^w(\lambda) = \chi\left(\bigcup_{w(F_\lambda)=w} F_\lambda\right).$$

This theorem generalizes a formula of Postnikov–Stanley (2009) for 132–*avoiding* (=*Kempf*=*dominant*) permutations.

A permutation w is Kempf if and only if there is a unique reduced Kogan face F such that $w(F) = w$.

Hilbert functions

Corollary

The dimension of the space $H^0(X^w, \mathcal{L}_\lambda|_{X^w})$ *is equal to the number of integer points in the union of all reduced Kogan faces with permutation* w*:*

$$\dim H^0(X^w, \mathcal{L}_\lambda) = \left| \bigcup_{w(F)=w} F_\lambda \cap \mathbb{Z}^d \right| .$$

In particular, the Hilbert function $H_{w,\lambda}(k) := \dim H^0(X^w, \mathcal{L}_\lambda^{\otimes k})$ *is equal to the Ehrhart polynomial of* $\bigcup_{w(F_\lambda)=w} F_\lambda$*, that is,*

$$H_{w,\lambda}(k) = \left| \bigcup_{w(F_\lambda)=w} kF_\lambda \cap \mathbb{Z}^d \right|$$

for all positive integers k*.*

Degrees of Schubert varieties

Denote by $\mathbb{R}F \subset \mathbb{R}^d$ the affine span of a face F. In the formulas displayed below, the volume form on $\mathbb{R}F$ is normalized so that the covolume of the lattice $\mathbb{Z}^d \cap \mathbb{R}F$ in $\mathbb{R}F$ is equal to 1.

Corollary

The degree $\deg_\lambda(X^w)$ of the Schubert variety X^w in the embedding $X_w \hookrightarrow \mathbb{P}(V_\lambda)$ can be computed as follows:

$$\deg_\lambda(X^w) = (d - l(w))! \sum_{w(F_\lambda)=w} \mathrm{Volume}(F_\lambda)$$

Applications and open problems

Multiplying Schubert cycles in the flag variety = intersecting faces of the Gelfand–Zetlin polytope

$\Downarrow$

- Schubert calculus on the variety of complete flags:

$$[X^v]\cdot[X^w]=\sum_{u\in S_n} c_{vw}^u [X^u]$$

  Gelfand–Zetlin polytopes $\dashrightarrow$ a positive combinatorial formula for c_{vw}^u?

- Schubert varieties in G/B for other reductive groups G (e.g. for $G=Sp_{2n}$) $\dashrightarrow$ faces of string polytopes?
- Polytopes $\dashrightarrow$ Description of the cohomology rings of spherical varieties by generators and relations?