Today we sketch a proof of the theorem | NYU Lecture 4

$$P_N(\lambda_1, \dots \lambda_N) \xrightarrow[N\to\infty]{} S_{P_N} = \sum P_N(\lambda) \frac{S_\lambda(x_1, \dots x_N)}{S_\lambda(1, \dots 1)}$$

I) $\displaystyle \frac{1}{N}\sum_{i=1}^{N} \left(\frac{\lambda_i + N - i}{N}\right)^k \longrightarrow p(k)$

II) $\cancel{\mathbb{E}(\sum \lambda_i + N - i)}$ Covariance $\displaystyle \left(\sum\left(\frac{\lambda_i+N-i}{N}\right)^k, \sum\left(\frac{\lambda_i+N-i}{N}\right)^\ell\right)$

$\longrightarrow \operatorname{cov}(k, \ell)$.

III) Higher moments $\displaystyle \sum\left[\left(\frac{\lambda_i+N-i}{N}\right)^k - \mathbb{E}\left(\frac{\lambda_i+N-i}{N}\right)\right] \longrightarrow$

$\longrightarrow$ Wick's formula

If and only if (Theorem?).

1) $\displaystyle \frac{1}{N}\left(\frac{\partial}{\partial x_1}\right)^k \ln S_{P_N}\Big|_{x_1=\dots=x_N=1} \longrightarrow c_k$ (what if $k=0$?)

2) $\displaystyle \left(\frac{\partial}{\partial x_1}\right)^k \left(\frac{\partial}{\partial x_2}\right)^\ell \ln S_{P_N}\Big|_{x_1=\dots=x_N=1} \longrightarrow d_{k,\ell}$

3) $\displaystyle \prod_{a=1}^{k}\left(\frac{\partial}{\partial x_{i_a}}\right) \ln S_{P_N}\Big|_{x_1=\dots=x_N=1} \longrightarrow 0$, if $|\{i_a\}| > 2$

+ Formulas linking two sets of numbers

How to extract moments from generating function?

$N=1$ $\displaystyle \sum_k x^k P(k) = F(x)$, where $P(k) = P(\xi = k)$

$$\left(x\frac{\partial}{\partial x}\right)^m F(x) = \sum k^m x^k P(k)$$

Plug in $x=1$ to get $\mathbb{E}\,\xi^m$.

What was the key ingredient?

A dif. operator $x\frac{\partial}{\partial x}$, whose eigenfunction is x^k

We mimick the same for Schur gen. func.

$$\mathcal{D}_k = V(x)^{-1} \sum_{i=1}^{N} \left(x_i \frac{\partial}{\partial x_i}\right)^k V(x), \quad \text{where}$$

$$V(x) = \prod_{i<j} (x_i - x_j).$$ I.e. multiply, then differentiate, then divide.

Lemma.

$$\mathcal{D}_k S_\lambda = \sum_{i=1}^{N} (\lambda_i + N - i)^k S_\lambda$$

Proof. $$S_\lambda = \frac{\det\left(x_i^{\lambda_j + N - j}\right)_{i,j=1}^{N}}{V(x)}$$

$$V(x) S_\lambda = \det\left(x_i^{(\lambda_j + N - j)}\right)_{i,j=1}^{N}$$

det = sum of monomials. Each monomial is an eigenfunction. When $\sum\left(x_i \frac{\partial}{\partial x_i}\right)^k$ acts, $\sum(\lambda_i + N - i)^k$ appears. ∎

Corollary.

$$\mathbb{E}\left(\sum_{i=1}^{N} (\lambda_i + N - i)^k\right)^m = \left(\mathcal{D}_k\right)^m S_{P_N}(x_1, \ldots x_N)\Big|_{x_1 = \ldots = x_N = 1}$$

Proof. Same as $N=1$ case.

That's our way to compute moments. In the end, Theorem is based on this lemma. However, there is an important feature, which makes theorems hard:

$V(x)$ vanishes at $x_1 = \dots = x_N = 1$. We need to resolve the singularity.

We will prove only one step: 1), 2), 3) $\Rightarrow$

$$\Rightarrow N^{-k-1} \mathbb{E} \sum (\lambda_i + N - i)^k \longrightarrow \rho(k)$$

Proof.

$$\parallel$$

$$\mathcal{D}_k S_{\rho_N} \big|_{x_1, \dots x_N = 1}.$$

How to apply $\prod (x_i - x_j)^{-1} \sum (x_i \partial_i)^k \prod (x_i - x_j)$ to a function (S_ρ) ?

Each ∂_i acts on

1) $(x_i - x_j)$ turning it into 1.
2) On x_i from other (previous) $x_i \partial_i$
3) On S_ρ
4) On the result of previous differentiation of S_ρ.

Write $\quad S_\rho = \exp(\ln S_\rho)$

Then $\quad \partial_i S_\rho = \partial_i (\ln S_\rho) \cdot S_\rho$

Conclusion $\mathcal{D}_k S_p$ is the sum of the terms of the form

$$S_p \cdot \frac{x_i^{k-q} \prod_{a=1}^{\infty} \partial_i^{\nu_a} (\ln S_p)}{(x_i - x_{j_1}) \cdot \ldots \cdot (x_i - x_{j_m})} \qquad (*)$$

where $q + m + \sum_{a=1}^{\infty} \nu_a = k.$

(Is it clear?)

At this point we need to plug in $x_1 = \ldots = x_N = 1$.
S_p disappears. But the rest explodes!
(why?)

Simplest case : $\frac{x_1^2 \partial_1 (\ln S_p)}{x_1 - x_2}$ $\quad \begin{pmatrix} k=2 \\ q=0, \; m=1 \end{pmatrix}$

Important: S_p and $\mathcal{D}_k$ are both symmetric!

Therefore, we also have the term $\frac{x_2^2 \partial_2 (\ln S_p)}{x_2 - x_1}$

They sum up to $\cancel{x_1^2 \partial_1 \ln S_p}$ $\frac{(x_1^2 \partial_1 - x_2^2 \partial_2) \ln S_p}{x_2 - x_1}.$

This has a well-defined limit $x_1, x_2 \to 1$ $\left(\frac{0}{0}\right)$

~~which is~~ (Expressed through partial derivatives of $\ln S_p$ at 1)

this extends to the general case!

<u>Lemma</u>: $\operatorname*{Sym}_{i, j_1, \dots j_m} (*)$ has a well-defined limit at $x_1 = \dots = x_m = 1$, which is a finite sum of products of partial derivatives of $\ln S_p$ at 1.

<u>Proof.</u> Expand $\ln S_p = \sum_{\vec{r}} A^{\vec{r}} (x_1 - 1)^{r_1} (x_2 - 1)^{r_2} \dots (x_N - 1)^{r_N}$ to reduce to polynomials.

For a polynomial we have:

$$\operatorname*{Sym}_{x_1, \dots x_k} \frac{f(x_1, \dots x_k)}{(x_1 - x_2) \dots (x_1 - x_k)} = \frac{1}{\prod_{i<j} (x_i - x_j)} \sum_{\sigma \in S_k} (-1)^{\sigma} \sigma \left(\frac{f(x_1 \dots x_k)}{(x_1 - x_2) \dots (x_1 - x_k)} \prod_{i<j} (x_i - x_j) \right)$$

This is a polynomial

Skew symmetric polynomial.

Hence, the ratio is a polynomial and therefore has a limit as $x_1, \dots x_N \to 1$.

(Break?)

The terms ~~in~~ $(*)$ make sense. Which of them give leading contribution?

1) For each „type" of term $(*)$ there are $\sim N^{m+1}$ such terms.

Why? # of ways to choose indices out of N

2) Each term gives contribution $\leq N^{\#\text{ non-zero } \nu_a}$. (5)

Why? Because we want to have several factors with derivative w.r.t. 1 variable only.

The factors are created by $\partial_i^{\nu_a} \ln S_p$. Then we only differentiate, which does not create new factors.

So we have a maximization problem

$$q + m + \sum \nu_a = k. \qquad m+1+\#\nu_a \to \max$$

what's the solution?

$q = 0, \quad \nu_1 = \dots = \nu_\ell = 1.$, i.e. the term $[m + \ell = k]$

$$\frac{x_1^k \left(\partial_1 (\ln S_p)\right)^\ell}{(x_1 - x_2) \dots (x_1 - x_{m+1})}$$

Lemma: $\lim_{x_i \to 1} \left(\frac{g(x_1)}{(x_1 - x_2) \dots (x_1 - x_n)} + (n-1 \text{ term } x_1 \leftrightarrow x_n) \right) =$

$$= \frac{\partial^{n-1}}{\partial z^{n-1}} \left(\frac{g(z)}{(n-1)!} \right) \Big|_{z=1}.$$

Proof. Enough to check for $g(x) = (x-1)^k$

1) $k < n-1 \to$ gives 0 by degree considerations (we know, this is a polynomial!)

2) $k > n-1 \longrightarrow$ gives 0 by comparing the multiplicity of 0 at $x_i = 1$

3) $k = n-1$ $\left(\frac{(x_1-1)^{n-1}}{(x_1-x_2) \dots (x_1-x_n)} + (n-1 \text{ term } x_1 \leftrightarrow x_n) \right)$ is a constant! $x_1 \to \infty \Rightarrow$ this constant is 1.

$$p(k) = \sum_{\ell=0}^{k} \frac{k!}{\ell!\,(\ell+1)!\,(k-\ell)!} \left(\frac{\partial}{\partial x}\right)^{\ell} \left((x+1)^k F(x)^{k-\ell}\right)\Big|_{x=0} = \quad 7$$

$$= \frac{1}{2\pi i} \oint_0 \sum_{\ell=0}^{k} \frac{1}{\ell+1} \binom{k}{\ell} \frac{(z+1)^k}{z^{\ell+1}} F(z)^{k-\ell}\, dz =$$

$$= \frac{1}{2\pi i} \oint_0 \frac{(z+1)^k F(z)^{k+1}}{k+1} \sum_{\ell=-1}^{k} \binom{k+1}{\ell+1} \frac{1}{F(z)^{\ell+1} z^{\ell+1}}\, dz =$$

$\ell=-1$, no residue

binomial

$$= \frac{1}{2\pi i} \cdot \frac{1}{k+1} \oint_0 (z+1)^k F(z)^{k+1} \left(1 + \frac{1}{F(z)\, z}\right)^{k+1} dz =$$

$$= [z^{-1}]\ \frac{1}{k+1} \frac{1}{z+1} \left(F(z)(z+1) + \frac{(z+1)}{z}\right)^{k+1}$$

And that's the expression for $p(k)$ we had last time!