What we learned so far:

$\lambda_1 \geq \dots \geq \lambda_N \qquad \ell_i = \lambda_i - i\theta$

$$\prod_{i<j} (\ell_i - \ell_j) \frac{\Gamma(\ell_i - \ell_j + \theta)}{\Gamma(\ell_i - \ell_j + 1 - \theta)} \qquad \prod_i w(\ell_i)$$

$[a, b] \qquad \cancel{\alpha = \frac{a}{N}, \ \beta = \frac{b}{N}} \quad \hat{a} = \frac{a}{N}, \ \hat{b} = \frac{b}{N}$

$w(a) = w(b) = 0 \qquad w(x) = \exp\left(-N\, V\left(\frac{x}{N}\right) + \varepsilon_N\right). \quad (*)$

($V(x)$ - continuous, with $x \ln x$ behavior at $\hat{a}, \hat{b}$).

Theorem (intermediate step!) $\quad G_N(z) = \frac{1}{N} \sum \frac{1}{z - \ell_i/N} \qquad G(z) = \int \frac{\mu(x)\,dx}{z - x}$

$$N^{\frac{1}{2} - \varepsilon} \left| G_N(z) - G(z) \right| \xrightarrow{N \to \infty} 0, \text{ in prob. / moments}$$

$\boxed{\text{skip it!}}$ "limit shape / equilibrium measure"

$\mu(x)\,dx$ is found as a solution to a variational problem

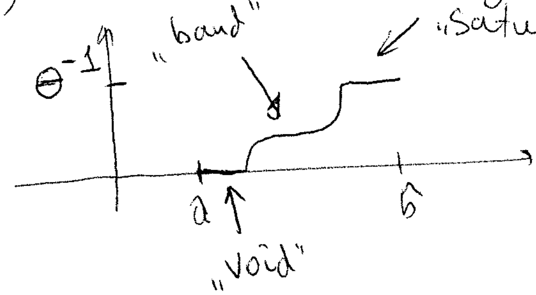
It also satisfies an equation:

$$\frac{w(x)}{w(x-1)} = \frac{\varphi_N^+(x)}{\varphi_N^-(x)} \qquad \begin{aligned} &\varphi_N^+(Nz) \to \varphi^+(z) \\ &\varphi_N^-(Nz) \to \varphi^-(z) \\ &\text{analytic!} \end{aligned} \qquad \left(\frac{\varphi^+(x)}{\varphi^-(x)} \approx \exp\left(-\frac{\partial}{\partial x} V(x)\right) \text{ if we ignore } \varepsilon_N \right)$$

$$R(z) := \varphi^-(z) \exp(-\theta G(z)) + \varphi^+(z) \exp(\theta G(z)) \text{ is}$$

analytic where $\varphi^\pm$ are.

$0 \leq \mu(x) \leq 1 \quad$ E.g. it might look as



bands $\cancel{[\alpha_i, \beta_i]} \quad [\alpha_i^+, \alpha_i^-]$

Define $Q(z) = \cancel{R} \, \varphi^-(z) \exp(-\theta G(z)) - \varphi^+(z) \exp(\theta G(z))$.

?

$$R^2(z) - Q^2(z) = 4 \varphi^-(z) \varphi^+(z).$$

$$Q(z) = \sqrt{\;.....\;}$$

$Q(z)$ is analytic outside ~~branch~~, bands. On bands it ~~ha~~ should have ~~singularities~~ discontinuities, and therefore branchpoints at the endpoints of bands.

$$Q(z) = H(z) \sqrt{\prod_{i=1}^{K} (z - d_i^-)(z - d_i^+)}.$$

[BGG]
Theorem: Assume $K = 1$ (one band) and $H(z) \neq 0$ on $[\hat{a}, \hat{b}]$, then for any analytic in $[\hat{a}, \hat{b}]$ $f(x)$

$$\sum_{i=1}^{N} \left[f\left(\frac{l_i}{N}\right) - \mathbb{E} f\left(\frac{l_i}{N}\right) \right]$$ is asymptotically Gaussian.

For $f = \frac{1}{z-x}$ the covariance is

$$\lim_{N \to \infty} N^2 \left(\cancel{\mathbb{E} G_N^2(z) - \mathbb{E} G_N(z)} \; \mathbb{E} G_N(z) G_N(w) - \mathbb{E} G_N(z) \mathbb{E} G_N(w) \right) =$$

$$= - \frac{\theta^{-1}}{2(z-w)^2} \left(1 - \frac{zw - \frac{1}{2}(d^+ + d^-) zw - d^- d^+}{\sqrt{(z-d^-)(z-d^+)(w-d^-)(w-d^+)}} \right)$$

[For general analytic $\to$ integration of this formula]

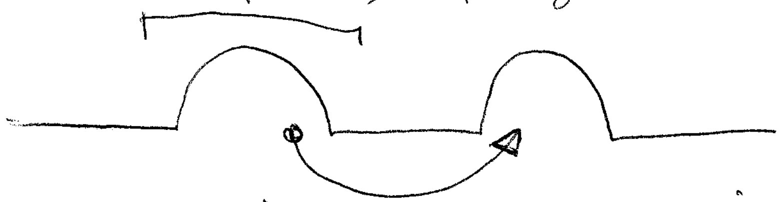
1] Depends only on d^-, d^+. $\longrightarrow$ universal !

Covariance of a log-correlated Gaussian field (section of 2d GFF).

The same as in Random Matrices (E.g. GO/U/S E).

2) If there are several bands, then the result (↑ gaussianity) is <u>not</u> true.

# of particles → filling fraction



leads to a macroscopic <u>discrete</u> change in in observable. E.g. (dropping analyticity) one can count the filling fractions, which → to <u>discrete</u> gaussians.

A) <u>Fixed</u> filling fractions, general K → same article of [Borodin – G. – Guionnet] (different covariance → GFF in multiply-connected regions)

B) Fluctuations of filling fractions → $\boxed{\text{discrete gaussians}}$
→ [Borot – G. – Guionnet, in progress].

$\boxed{3}$ $H(z) \neq 0$ – How serious is that?

Our running example of binomial weight $w(x)$
$H(z) =$ <u>constant</u>, so it is trivial.

Many more examples → easy check

This is not <u>generic</u>. However:

[Borot - G. - Guionnet] : in „off critical" case it is enough to check condition only near bands, and then it becomes „generic"

How is it proven?

Define $\Delta G_N(z) = N\left(G_N(z) - \cancel{G(z)\,\mathbb{E}\,G_N(z)} - G_N(z)\right)$

We aim to prove that its moments are Gaussian (finite) as $N \to \infty$.

<u>The proof combines 2 ideas:</u>

1. Find the leading order of $\mathbb{E}\,\Delta G_N(z)$ by expanding the Nekrasov equation.
2. Access higher moments by differentiation of 1

We start from 2.

Recall
$$P(\lambda) = \frac{1}{Z} \prod_{i<j} (\ell_i - \ell_j) \frac{\Gamma(\ell_i - \ell_j + \theta)}{\Gamma(\ell_i - \ell_j + 1 - \theta)} \prod w(\ell_i)$$

Replace w by $w_t(\ell_i) = w(\ell_i) \cdot \left(1 + \frac{t}{w - \ell_i/N}\right)$

<u>Claim:</u>
$$\frac{\partial}{\partial t}\Big|_{t=0} \mathbb{E}_{P_t}\, \Delta G_N(z) =$$

$$= \mathbb{E}\, \Delta G_N(z)\, \Delta G_N(w) - \mathbb{E}\, \Delta G_N(z)\, \mathbb{E}\, \Delta G_N(w) \quad \text{, i.e. covariance.}$$

<u>Proof.</u>
$$\frac{\partial}{\partial t} \mathbb{E}_{P_t}\, \Delta G_N(z) = \frac{\partial}{\partial t} \sum P_t\, \Delta G_N(z) =$$

$$= \cancel{\frac{t}{\frac{\partial}{\partial t} Z_t}} + \frac{\sum \frac{\partial}{\partial t} \prod_{i<j} (\ )\prod_i w_t(\ell_i)\, \Delta G_N(z) \cdot Z_t - }{Z_t^2}$$

$$- \frac{\frac{\partial}{\partial t}(Z_t) \sum \prod_{i<j} (\ ) \prod_i w_t(\ell_i)\, \Delta G_N(z)}{}$$

$$\frac{\partial}{\partial t}\Big|_{t=0} \prod_{i<j}(\ \ ) \prod_{i=1}^{N} W_t(\ell_i) =$$

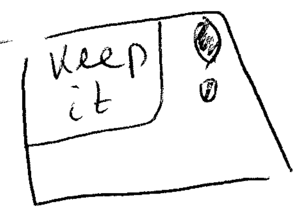
$$= \prod_{i<j}(\ \ ) \prod_{i=1}^{N} w(\ell_i) \cdot \underbrace{\sum_{i=1}^{N} \frac{1}{w-\ell_i/N}}_{N G_N(w)}$$

Therefore, we get
$$\frac{\sum \left(N G_N(w)\right) \Delta G_N(z) \prod_{i<j}(\ldots) \prod_{i=1}^{N} w(\ell_i)}{Z_0} -$$

$$- \frac{\sum \left(N G_N(w)\right) \prod_{i<j}(\ldots) \prod_{i=1}^{N}(\ldots) \ \sum \Delta G_N(z) \prod_{i<j}() \prod_{i=1}^{N}(\ldots)}{Z_0^2} =$$

$$= \mathbb{E}\, N G_N(w)\, \Delta G_N(z) - \mathbb{E}\, N G_N(w)\ \mathbb{E}\, \Delta G_N(z) =$$
$$= \mathbb{E}\, \Delta G_N(w)\, \Delta G_N(z) - \mathbb{E}\, \Delta G_N(w)\ \mathbb{E}\, \Delta G_N(z).$$

Conclusion: If we know the leading $N \to \infty$ behavior of $\mathbb{E}_{P_t} \Delta G_N(z)$ and can differentiate it, then we know the covariance.



Similarly: $$W_{t_1,\ldots t_k} = W \cdot \prod_{\alpha=1}^{k} \left(1 + \frac{t_\alpha}{w_\alpha - \ell_i/N}\right)$$

$$\frac{\partial^k}{\partial t_1 \ldots \partial t_k} \mathbb{E}_{P_{t_1,\ldots t_k}} \Delta G_N(z) = \text{joint cumulant of } \Delta G_N(z), \Delta G_N(w_1) \ldots \Delta G_N(w_k).$$

Their asymptotic $\to 0$ $\Rightarrow$ asymptotic gaussianity

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Now return to ①. How to control $\mathbb{E}\, \Delta G_N(z)$?

[Break — 3 minutes]

$$\frac{w(x)}{w(x-1)} = \frac{\varphi^+(Nz)}{\varphi^-(Nz)} \qquad \text{(Recall)}$$

$$R_N(z) = \varphi^+(Nz)\, \cancel{\exp(}\, \mathbb{E} \prod_{i=1}^{N} \left(1 + \frac{1}{N} \frac{\theta}{z - \frac{\ell_i}{N} - \frac{1}{N}}\right) +$$

$$+ \varphi^-(Nz)\, \mathbb{E} \prod_{i=1}^{N} \left(1 - \frac{1}{N} \frac{\theta}{z - \ell_i/N}\right) \qquad \text{is analytic.}$$

[Note $\dfrac{w_t(x)}{w_t(x-1)} \longrightarrow$ φ^+ and φ^- are simply multiplied by two linear factors ?!]

Expanding $\mathbb{E}\prod$ $\longrightarrow$ in the first order we get

$$\varphi^+(Nz) \exp(\theta G(z)) + \varphi^-(Nz) \exp(-\theta G(z)) = R(z)$$

(also analytic).

In the second order:

$$\prod_{i=1}^{N} \left(1 - \frac{1}{N} \frac{\theta}{z - \ell_i/N}\right) = \exp \sum_{i=1}^{N} \ln\left(1 - \frac{1}{N} \frac{\theta}{z - \ell_i/N}\right) =$$

$$= \exp \sum_{i=N}^{N} \left(-\frac{1}{N} \frac{\theta}{z - \ell_i/N}\right) - \frac{1}{2} \frac{1}{N^2} \left(\frac{\theta}{z - \ell_i/N}\right)^2 + \dots$$

$$-\theta G(z) + \theta\big(G(z) - G_N(z)\big) \qquad \frac{1}{2} \frac{\theta^2}{N} G'(z) + \dots$$

$$R_N(z) = R(z) + \frac{\theta}{N} \mathbb{E}\, G_N(z) \begin{pmatrix} \varphi^-(Nz) \exp(-\theta G(z)) + \\ \varphi^+(Nz) \exp(\theta G(z)) \end{pmatrix} +$$

$+$ explicit combination of $G'(z)$, $\exp(\pm\theta G(z))$ $+ \; \bar{\bar{o}}\left(\frac{1}{N}\right) +$

$A(z)$. analytic outside $[a,b]$

$$+ \underline{\underline{O}}\left(\left|\frac{1}{N} \Delta G_N(z)\right|^2 \exp\left(\left|\frac{1}{N} \Delta G_N(z)\right|\right) \right)$$

Next term in the expansion.

What do we know from intermediate step theorem?

$|\Delta G_N(z)| \approx N^{1/2 - \varepsilon} \Rightarrow$ This term is $\underline{\underline{O}}(N^{-1+2\varepsilon})$

But suppose for now that we proved $\bar{\bar{o}}(N^{-1})$ already.

Then we can ignore it

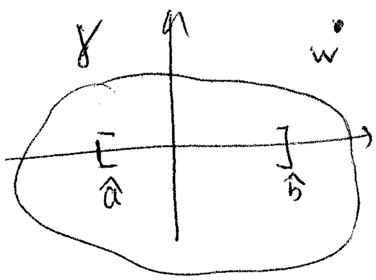
$$\Theta\left(\mathbb{E}\Delta G_N(z)\right) Q(z) = N\left(R(z)-R_N(z)\right) + \bar{\bar{o}}(1) + A(z)$$

No idea what it is, but it is analytic!

we know it! (in terms of $G(z)$)

what to do? We need to use analytic information on $Q(z)$ (its $\sqrt{\ }$ behavior)

$$\Theta\, \mathbb{E}\Delta G_N(z) \sqrt{(z-d^-)(z-d^+)}\; H(z) = \dots$$



Divide by $(2\pi i)(z-w)H(z)$, integrate around γ.

$$\frac{1}{2\pi i}\oint \frac{\Theta\, \mathbb{E}\Delta G_N(z)\sqrt{(z-d^-)(z-d^+)}}{z-w}\,dz = \oint \frac{N(R(z)-R_N(z))}{2\pi i\,(z-w)\,H(z)}\,dz + \bar{\bar{o}}(1) + \frac{1}{2\pi i}\oint \frac{A(z)}{(z-w)\,H(z)}\,dz$$

Integrates to 0! Has no poles inside

explicit! ✓

At ∞ $G_N(z) \sim \frac{1}{z^2}$ $\left(G_N(z),\ G(z) \sim \frac{1}{z}\right)$

$\Downarrow$

integrand $\sim \frac{1}{z^2}$, no pole!

& Integral = − residue at **w**

$$-\Theta\, \mathbb{E}\Delta G_N(w)\sqrt{(w-d^-)(w-d^+)} = \bar{\bar{o}}(1) + \frac{1}{2\pi i}\oint \frac{A(z)}{(z-w)\,H(z)}\,dz$$

$\boxed{\text{Done!}}$

what remains? Differentiate in t_a and prove the bound on $\mathbb{E}\,|\Delta G_N(z)|^K$ (need to be finite, but we have only $\left(N^{(\frac{1}{2}+\varepsilon)}\right)^K$ so far).

This is done simultaneously.

$$w_t(x) = w(x)\cdot \prod \left(1 + \frac{t_a}{w_a - x/N}\right) \qquad \varphi_t^+$$

$$\frac{w_t(Nz)}{w_t(Nz-1)} = \frac{\varphi^+(Nz)}{\varphi^-(Nz)}\cdot \prod_{a=1}^{K} \frac{(w_a - z + t_a)\,(w_a - z + \frac{1}{N})}{(w_a - z)\,(w_a - z + t_a + \frac{1}{N})}$$

$$\varphi_t^-$$

So new φ_t^+, φ_t^- are still analytic $\to$ multiplied by polynomials.
~~So~~ $\Rightarrow$ We have Nekrasov equation (for signed measure).

Expand it again in N. This time do not stop on $\Delta G_N(z)^2$, but continue. Again divide by $H(z)(z-w)$ and integrate. Then differentiate at $t_1 \dots t_K = 0$

Result: $\mathcal{M}^c\left(\Delta G_N(w), \Delta G_N(w_1), \dots \Delta G_N(w_K)\right) =$ (*)

cumulant

~~??~~ $[\xi_1 \dots \xi_K]\ \ln \mathbb{E}\exp(\sum \xi_k z_k)$

$$= \oint \sum_{m=K+2}^{\infty} \frac{\text{Moments of order } m \text{ of } \Delta G_N}{N^{m-1}} + \oint \frac{1}{N}\cdot \text{Moments of order } (K+1) \text{ of } \Delta G_N = O(1)$$

Claim: ~~This implies~~ These identities for all K, and z imply $\mathbb{E}\,|\Delta G_N(z)|^K < \infty$.

Indeed:
1) Sum is finite, since $\mathbb{E}\,|\Delta G_N|^K \le N^{(\frac{1}{2}+\varepsilon)K}$
2) Iterate and improve!

After we did this, $O(1)$ in (*) will give the result.