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# Topological regluing of rational functions

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**Abstract.** Regluing is a surgery that helps to build topological models for rational functions. It also has a holomorphic interpretation, with the flavor of infinite dimensional Thurston–Teichmüller theory. We will discuss a topological theory of regluing, and just trace a direction, in which a holomorphic theory can develop.

## 1. Introduction

### *1.1. Overview and main results*

Consider a continuous map $f : S^2 \to S^2$. We are mostly interested in the case, where $f$ is a rational function of one complex variable considered as a self-map of the Riemann sphere. The objective is to study the topological dynamics of $f$. In particular, how to modify the topological dynamics in a controllable way? There is an operation that does not change the dynamics at all: a conjugation by a homeomorphism. Let $\Phi : S^2 \to S^2$ be a homeomorphism, and consider the map $g = \Phi \circ f \circ \Phi^{-1}$. Then one can think of $g$ as being "the map $f$ but in a different coordinate system". In particular, all dynamical properties of the two maps are the same. Another example is a semi-conjugacy. Let $\Phi : S^2 \to S^2$ now be a continuous surjective map, but not a homeomorphism. Sometimes, the "conjugation" $g = \Phi \circ f \circ \Phi^{-1}$ still makes sense, although $\Phi^{-1}$ does not make sense as a map. Namely, this happens when $f$ maps fibers of $\Phi$ to fibers of $\Phi$. In this case, $g$ is well-defined as a continuous map. Such map is said to be *semi-conjugate* to $f$.

If we want to perform a surgery on $f$, then we need to consider discontinuous maps $\Phi$ (*if you don't cut, that is not a surgery*). Of course, if we allow badly discontinuous maps $\Phi$, then the "conjugation" $g = \Phi \circ f \circ \Phi^{-1}$, even if it makes sense as a map, would have almost nothing in common with $f$, in particular, it would be hard to say anything about the dynamics of $g$. Thus we must confine ourselves with only nice types of discontinuities. An example is the following: given a simple curve, one can cut along this curve, and then reglue in a different way.

E.g. consider the following map:

$$j(z) = \sqrt{z^2 - 1}.$$

Note that there are two branches of this map that are well-defined and holomorphic on the complement to the interval $[-1, 1]$. We choose the branch that is asymptotic to the identity near infinity and call it $j$. The map $j$ has a continuous extension to each "side" of the interval $[-1, 1]$ but the limit values at different sides do not match. By considering the limit values of $j$ at both sides of $[-1, 1]$, we can say that $j$ reglues this interval into the interval $[-i, i]$ (i.e. the straight line segment connecting $i$ and $-i$). A precise definition of a regluing will be given in Section 2.1. For now, a regluing of a set of disjoint simple curves is a one-to-one map defined and continuous on the complement to these curves and behaving near each curve as the map $j$ considered above.

We are forced to consider regluings of countably many curves. In fact, if we want to reglue some curve in the dynamical picture of a function $f$, then, in order to have a global continuous extension of $\Phi \circ f \circ \Phi^{-1}$, our regluing $\Phi$ must also reglue all pullbacks of this curve under $f$.

Below, we briefly explain the main result. Since the precise definitions are rather lengthy, we will only give a sketch, postponing the detailed statements until the main body of the paper. Let $\Phi$ be a regluing of countably many disjoint simple curves in the sphere (note that the complement to countably many simple curves is not necessarily open but is always dense). Also, consider a branched covering $f : S^2 \to S^2$. Under certain simple topological conditions on $f$ and the curves, we can guarantee that the map $\Phi \circ f \circ \Phi^{-1}$ extends to the whole sphere as a branched covering. We say that this covering is obtained from $f$ by *topological regluing of disjoint simple curves.* With the help of topological regluing, new topological models of rational functions can be obtained from the existent models.

The simplest example is provided by quadratic polynomials. E.g. consider the quadratic polynomial $f(z) = z^2 - 6$ (the particular choice of number 6 does not have any importance; we could as well take any real number bigger than 2). Most points escape to infinity under the iterations of $f$. The set of points that do not escape is a Cantor set $J_f$ called the *Julia set of $f$*. This set lies on the real line. The right-most point of $J_f$ is 3. Note that 3 is fixed under $f$. The left-most point of $J_f$ is $-3$, which is mapped to 3. The biggest component of the complement to $J_f$ in $[-3,3]$ is $(-\sqrt{3},\sqrt{3})$, and all other components are pullbacks of $(-\sqrt{3},\sqrt{3})$ under the iterates of $f$. The points $\pm\sqrt{3}$ are mapped to $-3$. Suppose that we reglue the interval $[-\sqrt{3},\sqrt{3}]$ and all its pullbacks under $f$. Then the Julia set of $f$ collapses into a connected set homeomorphic to the interval. Moreover, we can choose the corresponding regluing map $\Phi$ in such a way that $\Phi\circ f\circ\Phi^{-1}$ extends to the quadratic polynomial $g(z)=z^2-2$, the so called *Tchebyshev polynomial*. The Julia set of $g$ is the interval $[-2,2]=\overline{\Phi(J_f)}$. We will work out this example in detail in Section 5.1.

More generally, let $f$ be a quadratic polynomial $z\mapsto z^2+c$, where $c$ is the landing point of an external parameter ray $\mathcal{R}$. Suppose that the Julia set of $f$ is locally connected, and all periodic points of $f$ in $\mathbb{C}$ are repelling. Also, consider a quadratic polynomial $g$, for which the corresponding parameter value belongs to $\mathcal{R}$. Thus the Julia set of $g$ is disconnected. Then $f$ and $g$ can be obtained one from the other by a regluing (of a set of disjoint simple curves). This is explained in Section 2.3.

However, the main motivation for the notion of regluing was the problem of finding topological models for quadratic rational functions. According to a well-known general observation, the dynamical behavior of a rational function is determined by the behavior of its critical orbits. A quadratic rational function has two critical points. Thus, to simplify the problem, one puts restrictions on the dynamics of one critical point, and leaves the other critical point "free". For example, it makes sense to consider quadratic rational functions with one critical point periodic of period $k$. For $k=1$, we obtain quadratic polynomials. Indeed, the fixed critical point can be mapped to infinity, and a quadratic rational function having infinity as a fixed critical point is necessarily a quadratic polynomial.

Suppose now that $k>1$, and $f$ is a quadratic rational function with a $k$-periodic critical point and the other critical point non-periodic. Recall that $f$ is called a *hyperbolic rational function of type B* if the non-periodic critical point of $f$ lies in the immediate basin of the periodic critical cycle (but necessarily not in the same component, see e.g. [Mi93,R90]). The function $f$ is said to be *a hyperbolic rational function of type C* if the non-periodic critical point of $f$ lies in the full basin of the periodic critical cycle, but not in the immediate basin. The classification of hyperbolic rational functions into types was introduced by M. Rees [R90]. However, a different terminology was used (types II and III instead of types B and C). We use the terminology of Milnor [Mi93], which is more popular and perhaps more suggestive (B stands for "Bi-transitive", and C for "Capture").

Fix $k > 1$. The set of hyperbolic rational functions with a $k$-periodic critical point splits into *hyperbolic components*. We say that a hyperbolic component is of type B or C if it consists of hyperbolic rational functions of this type. There are also type D components, which we do not discuss in this paper. There are no type A components for $k > 1$.

**Theorem 1.** *Let $f$ be a quadratic rational function with a $k$-periodic critical point. If $f$ is on the boundary of a type C hyperbolic component but not on the boundary of a type B hyperbolic component, then $f$ is the continuous extension of $\Phi \circ h \circ \Phi^{-1}$ to the sphere, where $h$ is a critically finite hyperbolic rational function, and $\Phi$ is a regluing of a countable set of disjoint simple curves. Moreover, $h$ can be chosen to be the center of any type C hyperbolic component, whose boundary contains $f$.*

This result, combined with the topological models for hyperbolic critically finite functions given in [R92], provides topological models for most functions on the boundaries of type C components. We will prove Theorem 1 in Section 3. The requirement that $f$ be not on the boundary of a type B component is probably inessential. However, to study functions on the boundary of a type B component, one needs to use different techniques. For $k = 2$, there is only one type B component, and a complete description of its boundary is available [T08]: all functions on the boundary are simultaneously matings and anti-matings. See also [Luo, AY] for other interesting results concerning the case $k = 2$. On the other hand, I do not know any example of a type C hyperbolic component and a type B hyperbolic component, whose boundaries intersect at more than one point. In the proof of Theorem 1, many important ideas of [R92] are used. At some point, we employ an analytic continuation argument similar to that in [AY].

In Section 6, we prove that under some natural assumptions on a set of simple disjoint curves on the sphere, there exists a topological regluing of this set. This statement is a major ingredient in the proof of Theorem 1. The existence result is based on a theory of Moore [Mo16], which gives a topological characterization of spaces homeomorphic to the 2-sphere.

In Section 5, we define an explicit sequence of approximations to a regluing. These approximations are defined and holomorphic on the complements to finitely many simple curves (not necessarily disjoint). The holomorphy of approximations may prove to be important. However, we just introduce the basic notions and postpone a deeper theory for future publications.

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## 2. Topological regluing

We define topological regluing and give simplest examples, in particular, we discuss topological regluing of quadratic polynomials.

### *2.1. Definition and examples*

Let $S^1$ denote the unit circle in the plane. In Cartesian coordinates $(t_1, t_2)$, it is given by the equation $t_1^2 + t_2^2 = 1$. We will also consider the sphere $S^2$ obtained as the one-point compactification of the $(t_1, t_2)$-plane. In this sphere, consider the region $\Delta_\infty$ bounded by $S^1$ and containing $\infty$ (i.e. the outside of the unit circle). The closure of this region in the sphere is denoted by $\overline{\Delta}_\infty$. We will write $\Delta$ for the unit disk $\{t_1^2 + t_2^2 < 1\}$. The Klein 4-group $V_4$ acts on the unit circle. Namely, the two generators of $V_4$ are $s_1 : (t_1, t_2) \mapsto (t_1, -t_2)$ and $s_2 : (t_1, t_2) \mapsto (-t_1, t_2)$.

A continuous map $\alpha : S^1 \to S^2$ is called an *$\alpha$-path* if $\alpha(t_1, t_2) = \alpha(t_1', t_2')$ is equivalent to $t_1' = t_1$. Note that $\alpha(t_1, t_2)$ is then an injective continuous function of $t_1$, and that $t_1' = t_1$ implies $t_2' = \pm t_2$. Similarly, a continuous map $\beta : S^1 \to S^2$ is called a *$\beta$-path* if $\beta(t_1, t_2) = \beta(t_1', t_2')$ is equivalent to $t_2' = t_2$. Thus $\beta$ can be represented as an injective continuous function of $t_2$. Note that every simple path $\gamma : [-1, 1] \to S^2$ can be interpreted either as an $\alpha$-path defined as $\alpha(t_1, t_2) = \gamma(t_1)$ or as a $\beta$-path defined as $\beta(t_1, t_2) = \gamma(t_2)$. Note also that any $\alpha$-path can be considered as a quotient map of $S^1$ by the action of $s_1$. Similarly, any $\beta$-path can be considered as a quotient map of $S^1$ by the action of $s_2$.

With every $\alpha$-path $\alpha$, we can associate a *gluing map* $\pi_\alpha : \overline{\Delta}_\infty \to S^2$, a continuous map that restricts to an orientation-preserving homeomorphism between $\Delta_\infty$ and $S^2 - \alpha(S^1)$ and coincides with $\alpha$ on $S^1$. The gluing map $\pi_\alpha$ thus defined is unique up to a homotopy relative to the unit circle. Similarly, we associate a gluing map $\pi_\beta : \overline{\Delta}_\infty \to S^2$ with every $\beta$-path $\beta$. Note that gluing maps are quotient maps of $\overline{\Delta}_\infty$ by the action of $s_1$ or $s_2$ on $S^1$.

Let $\mathcal{A}$ be a set of disjoint $\alpha$-paths in the sphere. Being disjoint means that $\alpha(S^1) \cap \alpha'(S^1) = \emptyset$ for every pair of different $\alpha$-paths

$\alpha, \alpha' \in \mathcal{A}$. Define the set $\mathrm{Im}\mathcal{A} \subset S^2$ as the union of $\alpha(S^1)$ for all $\alpha \in \mathcal{A}$. We define $\mathrm{Im}\mathcal{B}$ for a set $\mathcal{B}$ of disjoint $\beta$-paths in the same way. For every $\alpha \in \mathcal{A}$, fix a gluing map $\pi_\alpha : \overline{\Delta}_\infty \to S^2$. Define the *ungluing space* $\Upsilon_\mathcal{A}$ of $\mathcal{A}$ as the subset of $\overline{\Delta}_\infty^\mathcal{A}$ given by the following condition. A point $\chi : \mathcal{A} \to \overline{\Delta}_\infty$ of $\overline{\Delta}_\infty^\mathcal{A}$ belongs to $\Upsilon_\mathcal{A}$ if the points $\pi_\alpha \circ \chi(\alpha) \in S^2$ are the same for all $\alpha \in \mathcal{A}$. In other terms, $\Upsilon_\mathcal{A}$ is the equalizer of the gluing maps $\pi_\alpha$, $\alpha \in \mathcal{A}$. Define the map $\pi_\mathcal{A} : \Upsilon_\mathcal{A} \to S^2$ as the map $\chi \mapsto \pi_\alpha \circ \chi(\alpha)$ (note that, by definition, $\pi_\alpha \circ \chi(\alpha)$ does not depend on $\alpha$). Clearly, this map is continuous. Intuitively, $\Upsilon_\mathcal{A}$ is obtained from the sphere by ungluing all curves $\alpha(S^1)$, $\alpha \in \mathcal{A}$, i.e. making topological circles out of them. Note that a point $x \in S^2$ has only one preimage under the map $\pi_\mathcal{A}$ unless $x \in \alpha(S^1)$ for some $\alpha \in \mathcal{A}$. Similarly, we can define the ungluing space $\Upsilon_\mathcal{B}$ and the map $\pi_\mathcal{B} : \Upsilon_\mathcal{B} \to S^2$ for a set $\mathcal{B}$ of disjoint $\beta$-paths.

Denote by $\Upsilon_\mathcal{A}^\circ$ the subset $\pi_\mathcal{A}^{-1}(S^2 - \mathrm{Im}\mathcal{A})$ of the ungluing space $\Upsilon_\mathcal{A}$. Also, for every $\alpha \in \mathcal{A}$, we can define $S^1_\alpha \subset \Upsilon_\mathcal{A}$ as the set of points $\chi \in \Upsilon_\mathcal{A}$ such that $\pi_\mathcal{A}(\chi) \in \alpha(S^1)$. There is a natural homeomorphism $h_\alpha : S^1 \to S^1_\alpha$ that takes a point $u \in S^1$ to the point $\chi \in S^1_\alpha$ defined as follows:

$$\chi(\alpha') = \begin{cases} u, & \alpha = \alpha' \\ \pi_{\alpha'}^{-1} \circ \alpha(u), & \alpha \neq \alpha' \end{cases}$$

Thus the ungluing space splits into the union of the set $\Upsilon_\mathcal{A}^\circ$, which identifies canonically with $S^2 - \mathrm{Im}\mathcal{A}$, and disjoint topological circles $S^1_\alpha$, which identify canonically with $S^1$.

The Klein group $V_4$ acts on $\Upsilon_\mathcal{A}$. Namely, the action on $\Upsilon_\mathcal{A}^\circ$ is trivial, and the action on every circle $S^1_\alpha$ identifies (through the homeomorphism $h_\alpha$) with the standard action of $V_4$ on $S^1$. It is not hard to prove that $\pi_\mathcal{A} : \Upsilon_\mathcal{A} \to S^2$ is a quotient map by the action of $s_1$ (warning: the action is not continuous; it still makes sense to talk about the quotient space by the orbit equivalence relation). Similarly, for a set $\mathcal{B}$ of disjoint $\beta$-paths, the map $\pi_\mathcal{B} : \Upsilon_\mathcal{B} \to S^2$ is a quotient map by the action of $s_2$.

We can now give a definition of a regluing. Here, we will only define a topological regluing of a set of disjoint simple curves, thus the word "regluing" will only be used in this sense until Section 5. Consider a set $\mathcal{A}$ of disjoint $\alpha$-paths and a set $\mathcal{B}$ of disjoint $\beta$-paths in the sphere. A bijective continuous orientation-preserving map $\Phi$ : $S^2 - \mathrm{Im}\mathcal{A} \to S^2 - \mathrm{Im}\mathcal{B}$ is called a regluing of $\mathcal{A}$ into $\mathcal{B}$ according to a given one-to-one correspondence between $\mathcal{A}$ and $\mathcal{B}$ if, for every $\alpha \in \mathcal{A}$ and the corresponding $\beta \in \mathcal{B}$, the map

$$\pi_\beta^{-1} \circ \Phi \circ \pi_\alpha : \pi_\alpha^{-1}(S^2 - \mathrm{Im}\mathcal{A}) \to \pi_\beta^{-1}(S^2 - \mathrm{Im}\mathcal{B})$$

extends to $S^1 \cup \pi_\alpha^{-1}(S^2 - \mathrm{Im}\mathcal{A})$ so that the extension is continuous at all points of $S^1$, and its restriction to $S^1$ is the identity.

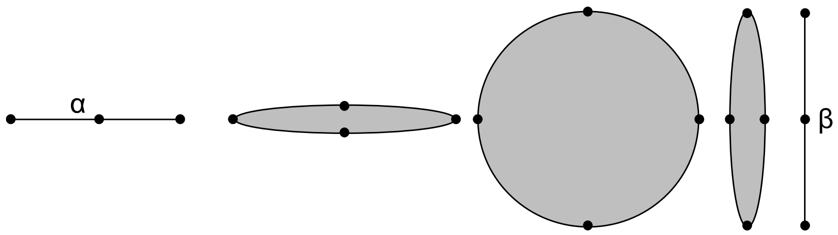


**Fig. 1.** A schematic picture of regluing

*Example 1.* As an example, consider a branch of the function

$$j(z) = \sqrt{z^2 - 1}$$

on the sphere with the interval $[-1, 1]$ removed. Then $j$ maps the set $\overline{\mathbb{C}} - [-1, 1]$ to the set $\overline{\mathbb{C}} - [-i, i]$ homeomorphically. Set $\alpha(t_1, t_2) = t_1$ and $\beta(t_1, t_2) = it_2$. The map $j$ reglues $\alpha$ into $\beta$, for a suitable choice of the branch.

**Proposition 1.** *Consider a regluing $\Phi : S^2 - \mathrm{Im}\mathcal{A} \to S^2 - \mathrm{Im}\mathcal{B}$. There exists a homeomorphism $\hat{\Phi} : \Upsilon_{\mathcal{A}} \to \Upsilon_{\mathcal{B}}$ that makes the following diagram commutative:*

$$\begin{array}{ccccc} \Upsilon_{\mathcal{A}} & \xrightarrow{\pi_{\mathcal{A}}} & S^2 & \xleftarrow{\textit{inclusion}} & S^2 - \mathrm{Im}\mathcal{A} \\ {\scriptstyle\hat{\Phi}}\downarrow & & & & \downarrow{\scriptstyle\Phi} \\ \Upsilon_{\mathcal{B}} & \xrightarrow{\pi_{\mathcal{B}}} & S^2 & \xleftarrow{\textit{inclusion}} & S^2 - \mathrm{Im}\mathcal{B} \end{array}$$

*Moreover, the homeomorphism $\hat{\Phi}$ commutes with the action of $V_4$ on the ungluing spaces.*

*Proof.* On $\Upsilon^\circ_{\mathcal{A}}$, we define $\hat{\Phi}$ as $\pi_{\mathcal{B}}^{-1} \circ \Phi \circ \pi_{\mathcal{A}}$. On $S^1_\alpha$, $\alpha \in \mathcal{A}$, define $\hat{\Phi}$ as $h_\beta \circ h_\alpha^{-1}$, where $\beta$ is the $\beta$-path corresponding to the $\alpha$-path $\alpha$.

It is clear that $\hat{\Phi}$ is one-to-one. This map is continuous on $\Upsilon^\circ_{\mathcal{A}}$ by definition. The continuity on $S^1_\alpha$, $\alpha \in \mathcal{A}$, follows from the formula

$$\hat{\Phi} = pr_\beta^{-1} \circ \pi_\beta^{-1} \circ \Phi \circ \pi_\alpha \circ pr_\alpha$$

on $\Upsilon^\circ_{\mathcal{A}}$, where $pr_\alpha : \Upsilon_{\mathcal{A}} \to S^2$ is the projection $\chi \mapsto \chi(\alpha)$. Since the ungluing spaces are Hausdorff and compact, it follows that $\hat{\Phi}$ is a homeomorphism. Since $\hat{\Phi}$ acts as the identity under standard identifications of $S^1_\alpha$ and $S^1_\beta$ with $S^1$, the action of $V_4$ is preserved. □

For every $\alpha$-path $\alpha : S^1 \to S^2$, define a $\beta$-path $\alpha^\#$ as follows: $\alpha^\#(t_1, t_2) = \alpha(t_2, t_1)$. Similarly, the formula $\beta^\#(t_1, t_2) = \beta(t_2, t_1)$ makes a $\beta$-path $\beta$ into an $\alpha$-path $\beta^\#$. For a set $\mathcal{A}$ of $\alpha$- or $\beta$-paths, we can form the set $\mathcal{A}^\# = \{\alpha^\# \mid \alpha \in \mathcal{A}\}$.

**Proposition 2.** *Consider a regluing $\Phi$ of a set $\mathcal{A}$ of disjoint $\alpha$-paths into a set $\mathcal{B}$ of disjoint $\beta$-paths.*

- *The map $\Phi : S^2 - \mathrm{Im}\mathcal{A} \to S^2 - \mathrm{Im}\mathcal{B}$ is a homeomorphism.*
- *The map $\Phi^{-1}$ is a regluing of $\mathcal{B}^\#$ into $\mathcal{A}^\#$.*

*Proof.* This follows immediately from the existence of the homeomorphism $\hat{\Phi}$ and the fact that it commutes with the action of $V_4$ on the ungluing spaces. □

Let $f : S^2 \to S^2$ be a continuous map. Assume that a countable set $\mathcal{A}$ of disjoint $\alpha$-paths satisfies the following conditions:

- *Forward semi-invariance:* for any path $\alpha \in \mathcal{A}$, we have $f \circ \alpha \in \mathcal{A}$ or $f \circ \alpha(t_1, t_2) = f \circ \alpha(-t_1, t_2)$ for all $(t_1, t_2) \in S^1$. In the latter case, $f \circ \alpha(S^1)$ must be disjoint from $\mathrm{Im}\mathcal{A}$. In the former case, the map $\pi^{-1}_{\alpha \circ f} \circ f \circ \pi_\alpha$ must be defined for all points of $\Delta_\infty$ sufficiently close to $S^1$, and must extend to a neighborhood of $S^1$ in $\overline{\Delta}_\infty$ so that the extension is continuous at all points of $S^1$, and its restriction to $S^1$ is the identity.
- *Backward invariance:* we have $f^{-1}(\mathrm{Im}\mathcal{A}) \subseteq \mathrm{Im}\mathcal{A}$.

We say in this case that $\mathcal{A}$ is *$f$-stable*.

**Proposition 3.** *If the set $\mathcal{A}$ is $f$-stable, then there is a continuous map $\hat{f} : \Upsilon_{\mathcal{A}} \to \Upsilon_{\mathcal{A}}$ that makes the following diagram commutative:*

$$\begin{array}{ccc} \Upsilon_{\mathcal{A}} & \xrightarrow{\pi_{\mathcal{A}}} & S^2 \\ \hat{f}\downarrow & & \downarrow f \\ \Upsilon_{\mathcal{A}} & \xrightarrow{\pi_{\mathcal{A}}} & S^2 \end{array}$$

*Moreover, $\hat{f}$ is equivariant with respect to the action of $V_4$ on $\Upsilon_{\mathcal{A}}$.*

*Proof.* Suppose that $f \circ \pi_{\mathcal{A}}(\chi) \notin \mathrm{Im}\mathcal{A}$, then we set $\hat{f}(\chi) = \pi^{-1}_{\mathcal{A}} \circ f \circ \pi_{\mathcal{A}}(\chi)$. Now suppose that $f \circ \pi_{\mathcal{A}}(\chi) \in \alpha(S^1)$ for some $\alpha \in \mathcal{A}$. By the backward invariance, we necessarily have $\pi_{\mathcal{A}}(\chi) \in \alpha'(S^1)$ for some other $\alpha' \in \mathcal{A}$, hence $\chi(\alpha') \in S^1$. In this case, we can set $\hat{f}(\chi) = h_\alpha \circ \chi(\alpha')$. The continuity of $\hat{f}$ at points of $S^1_\alpha$ follows from the forward semi-invariance and the formula

$$\hat{f} = pr^{-1}_\alpha \circ \pi^{-1}_\alpha \circ f \circ \pi_{\alpha'} \circ pr_{\alpha'}$$

on $\Upsilon^\circ_{\mathcal{A}}$. It is straightforward to check that $\hat{f}$ commutes with the action of $V_4$. □

Our main construction is based on the following simple fact:

**Theorem 2.** *Suppose that $f : S^2 \to S^2$ is a continuous map, and $\mathcal{A}$ is an $f$-stable set of disjoint $\alpha$-paths. Let $\Phi$ be a regluing of $\mathcal{A}$ into a set $\mathcal{B}$ of disjoint $\beta$-paths. Then the map $g = \Phi \circ f \circ \Phi^{-1}$ extends to a continuous map from $S^2$ to $S^2$. Moreover, the set $\mathcal{B}$ is $g$-stable, and $\hat{g} = \hat{\Phi} \circ \hat{f} \circ \hat{\Phi}^{-1}$.*

*Proof.* Consider the map $\hat{g} = \hat{\Phi} \circ \hat{f} \circ \hat{\Phi}^{-1}$, which takes $\Upsilon_{\mathcal{B}}$ to $\Upsilon_{\mathcal{B}}$. Since both $\hat{\Phi}$ and $\hat{f}$ are equivariant under the action of $V_4$, this map is also equivariant. It follows that it descends to a continuous map $g : S^2 \to S^2$. □

We would like to apply this theorem as follows. Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational function. For certain classes of rational functions $f$, there are natural ways to produce $f$-stable sets of paths. Then the corresponding map $g$ is often a model for a new rational function. Note that the topological dynamics of $g$ is very easy to understand in terms of the topological dynamics of $f$, because $\Phi$ is a topological conjugation except on $\mathrm{Im}(\mathcal{A})$. A remarkable fact is that in many cases, the regluing $\Phi$ makes sense in a certain holomorphic category, so that the construction may actually produce a rational function $g$ rather than just a continuous map.

Let $X$ be a compact metric space, and $\mathcal{S}$ a set of compact subsets of $X$. Recall that $\mathcal{S}$ *forms a null-sequence* if for every $\varepsilon > 0$, there are only finitely many elements of $\mathcal{S}$, whose diameter exceeds $\varepsilon$. The following theorem is needed for the construction of topological models:

**Theorem 3.** *Let $\mathcal{A}$ be a countable set of disjoint $\alpha$-paths such that the sets $\alpha(S^1)$, $\alpha \in \mathcal{A}$, form a null-sequence. Then*

1. *the ungluing space $\Upsilon_{\mathcal{A}}$ embeds homeomorphically to $S^2$;*
2. *there exists a regluing of $\mathcal{A}$ into some set $\mathcal{B}$ of disjoint $\beta$-paths; moreover, one can arrange that $\beta(S^1)$, $\beta \in \mathcal{B}$ also form a null-sequence.*

The statement of the theorem may seem intuitively obvious (and it is in fact obvious for the case of finite $\mathcal{A}$). Note, however, that the set $\mathrm{Im}\mathcal{A}$ may be everywhere dense in the sphere, and even have full measure. We will prove Theorem 3 in Section 6.3 using Moore's theory. It is useful to know that the property of being a null-sequence is topological, and does not depend on a particular metric:

**Proposition 4.** *Let $X$ be a compact metric space. A set $\mathcal{S}$ of compact subsets of $X$ forms a null-sequence if and only if for every open covering $\mathcal{E}$ of $X$, there is a finite subset $\mathcal{S}' \subset \mathcal{S}$ such that every element of $\mathcal{S} - \mathcal{S}'$ is contained in an element of $\mathcal{E}$.*

In other terms, the number $\varepsilon > 0$ can be replaced with an open covering $\mathcal{E}$. The following proof is rather standard.

*Proof.* We first prove the only if part. Let $\mathcal{E}$ be an open covering of $X$, and $\varepsilon$ its Lebesgue number. Recall that a Lebesgue number of $\mathcal{E}$ is defined as a real number $\varepsilon > 0$ such that every set of diameter less than $\varepsilon$ belongs to an element of $\mathcal{E}$. Set $\mathcal{S}'$ to be the set of all elements of $\mathcal{S}$, whose diameter is at least $\varepsilon$. Then, by the Lebesgue number lemma, every element of $\mathcal{S} - \mathcal{S}'$ is contained in an element of $\mathcal{E}$.

Let us now prove the if part. Choose any $\varepsilon > 0$, and consider the covering $\mathcal{E}$ of $X$ by all $\varepsilon/2$-balls. Then there is a finite subset $\mathcal{S}' \subset \mathcal{S}$ such that every set in $\mathcal{S} - \mathcal{S}'$ is contained in an element of $\mathcal{E}$. It follows that the diameter of any set in $\mathcal{S} - \mathcal{S}'$ does not exceed $\varepsilon$. □

**Corollary 1.** *Let $X$ and $Y$ be compact metric spaces, and $\phi : X \to Y$ a continuous map. If $\mathcal{S}$ is a set of compacts subsets in $X$ that form a null-sequence, then the sets $\phi(A)$, $A \in \mathcal{S}$, form a null-sequence in $Y$.*

*Proof.* Indeed, let $\mathcal{E}$ be any open covering of $Y$. Consider the corresponding covering $\phi^*(\mathcal{E})$ of $X$. By Proposition 4, all elements of $\mathcal{S}$ but finitely many are subsets of elements of $\phi^*(\mathcal{E})$. It follows that all $\phi(A)$, $A \in \mathcal{S}$, but finitely many are subsets of elements of $\mathcal{E}$. □

### *2.2. Regluing of branched coverings*

The setting of branched coverings is most commonly used for topological discussions of rational functions. On one hand, branched coverings are objects of topological nature, and are much more flexible than holomorphic functions. On the other hand, they are nice objects and do not have pathologies of general continuous maps. This is why we want the regluing construction to fit into the contest of topological branched coverings.

Let $f : S^2 \to S^2$ be a branched covering. Consider a set $\mathcal{A}$ of simple disjoint $\alpha$-paths in the sphere such that $\alpha(S^1)$, $\alpha \in \mathcal{A}$, form a null-sequence. We say that $\mathcal{A}$ is *strongly $f$-stable* if it is $f$-stable, and satisfies the following additional assumption: all critical points in $\mathrm{Im}\mathcal{A}$ have the form $\alpha(0,1)$, where $\alpha \in \mathcal{A}$ is a path such that $f \circ \alpha(t_1, t_2) = f \circ \alpha(-t_1, t_2)$ for all $(t_1, t_2) \in S^1$; moreover, these critical points are simple. Note that the null-sequence property is included into the notion of a strongly $f$-stable set of paths.

**Theorem 4.** *Let $f : S^2 \to S^2$ be a branched covering, and $\mathcal{A}$ a strongly $f$-stable set of disjoint $\alpha$-paths. Consider a regluing $\Phi : S^2 - \mathrm{Im}\mathcal{A} \to S^2 - \mathrm{Im}\mathcal{B}$ of $\mathcal{A}$ into some set $\mathcal{B}$ of disjoint $\beta$-paths. Then the map $g = \Phi \circ f \circ \Phi^{-1}$ extends to a branched self-covering of $S^2$.*

We will prove this theorem in Section 6.4. In the statement of the theorem, the curves $\beta(S^1)$, $\beta \in \mathcal{B}$, will automatically form a null-sequence. Indeed, by Theorem 3, there is a regluing $\Phi'$ of $\mathcal{A}$ into some set $\mathcal{B}'$ of disjoint $\beta$-paths such that $\beta'(S^1)$, $\beta' \in \mathcal{B}'$, form a null-sequence. Then $\Phi \circ (\Phi')^{-1}$ extends to the sphere as a continuous map. Moreover, it maps $\mathcal{B}'$ to $\mathcal{B}$. By Corollary 1, the curves $\beta(S^1)$, $\beta \in \mathcal{B}$, must also form a null-sequence.

### *2.3. Topological regluing of quadratic polynomials*

In this section, we will not say much new about the dynamics of quadratic polynomials. However, we can illustrate the idea of regluing using quadratic polynomials as an example.

Let $\mathcal{R}$ be an external ray in the parameter plane of quadratic polynomials (we write quadratic polynomials in the form $p_c(z) = z^2 + c$, thus the parameter plane is the $c$-plane). Suppose that $\mathcal{R}$ lands at a point $c$ on the boundary of the Mandelbrot set. Suppose that the Julia set of $f = p_c$ is locally connected, and that all periodic points of $f$ (except $\infty$) are repelling. The ray $\mathcal{R}$ determines a pair of rays $R^+_f$ and $R^-_f$ in the dynamical plane of $f$ that land at the critical point 0 (for parameter values in the ray $\mathcal{R}$, these two rays crash into 0). Note that there may be more rays landing at 0, but the pair of rays $R^+_f$, $R^-_f$ is distinguished (i.e. determined by the choice of $\mathcal{R}$).

Fix any real number $\rho > 0$. Consider the $\alpha$-path $\alpha_0 : S^1 \to \mathbb{C}$ in the dynamical plane of $f$ defined as follows:

$$\alpha_0(t_1, t_2) = \begin{cases} R^+_f(t_1\rho), & t_1 > 0, \\ 0, & t_1 = 0, \\ R^-_f(-t_1\rho), & t_1 < 0. \end{cases}$$

Here the dynamical rays are parameterized by the values of the Green function, thus $R(t)$ stands for the point in the ray $R$, at which the Green function is equal to $t$. Then, for each $n \geq 0$, the multivalued function $f^{-n} \circ \alpha_0$ has $2^n$ branches, each being an $\alpha$-path. All these paths are called *pullbacks* of $\alpha_0$ under the iterates of $f$. Let $\mathcal{A}$ denote the set of such pullbacks, including $\alpha_0$. Clearly, $\mathcal{A}$ is strongly $f$-stable.

Now consider the quadratic polynomial $g = p_{c_0}$, where $c_0 = \mathcal{R}(2\rho)$ is the point on the external parameter ray $\mathcal{R}$ with parameter $2\rho$ (the external parameter rays are parameterized by the value of the Green function at the critical value). This means that, in the dynamical plane of $g$, the value of the Green function at the critical value $c_0$ is equal to $2\rho$. Therefore, the value of the Green function at the critical point 0 is equal to $\rho$. There are exactly two rays that are bounded and contain 0 in their closures (here by a ray we mean any gradient curve of the Green function). Denote these rays by $R^+_g$ and $R^-_g$. These two rays can also be parameterized by the values of the Green function, thus the parameter runs through the interval $(0, \rho)$. Let $z^+$ and $z^-$ be the landing points of the rays $R^+_g$ and $R^-_g$ (these rays land because the angle of $\mathcal{R}$ cannot be a rational number with an odd denominator, which is the only case where one of the rays $R_g^+$ and $R_g^-$ can crash into a precritical point). Define the following $\beta$-path $\beta_0 : S^1 \to \overline{\mathbb{C}}$ in the dynamical plane of $g$:

$$\beta_0(t_1,t_2) = \begin{cases} z^+, & t_2 = 1, \\ R_g^+(|t_1|\rho), & t_2 > 0, \\ 0, & t_2 = 0, \\ R_g^-(|t_1|\rho), & t_2 < 0, \\ z^-, & t_2 = -1. \end{cases}$$

Let $\mathcal{B}$ denote the set of all pullbacks of $\beta_0$, including $\beta_0$.

There is a natural one-to-one correspondence between the sets of paths $\mathcal{A}$ and $\mathcal{B}$. For any path $\alpha \in \mathcal{A}$, the point $\alpha(1,0)$ belongs to a unique external ray of angle $\theta$. There is a unique path $\beta \in \mathcal{B}$ such that the ray of angle $\theta$ crashes into $\beta(1,0)$. We will make this path $\beta$ correspond to the path $\alpha$.

**Theorem 5.** *There exists a regluing $\Phi$ of $\mathcal{A}$ into $\mathcal{B}$ such that $g(x) = \Phi \circ f \circ \Phi^{-1}(x)$ at all points $x$, where the right-hand side is defined.*

Before we proceed with the proof of this theorem, we need to recall the definition of symbolic itineraries. The union $\Gamma_f$ of the rays $R_f^+$ and $R_f^-$ together with their common landing point 0 divides the complex plane into two connected components. For a point $x$ not in $f^{-n}(\Gamma_f)$, we define $\sigma_f^n(x)$ to be 0 or 1 depending on whether or not $f^{\circ n}(x)$ is separated from the critical value $c$ of $f$ by $\Gamma_f$. The sequence of numbers $\sigma_f^n(x)$, $n = 1, 2, \dots$ (which may be finite or infinite depending on whether or not $x$ is eventually mapped to $\Gamma_f$) is called the *symbolic itinerary* of $x$. Similarly, we define $\Gamma_g$ to be the union of $\{0\}$ and the external rays in the dynamical plane of $g$ that crash into 0 (they have the same external angles as the rays $R_f^+$ and $R_f^-$). The definition of symbolic itineraries carries over to the dynamical plane of $g$, where we use $\Gamma_g$ instead of $\Gamma_f$. In the dynamical plane of $g$, as well as in the dynamical plane of $f$, there cannot be two different points in the Julia set with the same symbolic itinerary. This is a basic Poincaré distance argument, see e.g. [Mi06].

*Proof of Theorem 5.* Consider the complement $U$ to the closure of $\mathrm{Im}\mathcal{A}$. Since the closure of $\mathrm{Im}\mathcal{A}$ contains the Julia set of $f$, the set $U$ is an open subset of the Fatou set. Actually, $U$ is the complement in the Fatou set to $\mathrm{Im}\mathcal{A}$.

We first define the map $\Phi$ just on $U$. Let $\phi_f$ be the Böttcher parameterization for $f$, i.e. the inverse of the Böttcher coordinate function. Since the Fatou set of $f$ contains no precritical points except for $\infty$, the map $\phi_f$ is a holomorphic isomorphism between the unit disk $\Delta$ and the Fatou set of $f$ such that $\phi_f(z^2) = f \circ \phi_f(z)$ for all $z \in \Delta$. Similarly, let $\phi_g$ be the Böttcher parameterization for $g$. Note that $\phi_g$ is not everywhere defined on the unit disk, because the Fatou set of $g$ contains the critical point 0 and its preimages. However, $\phi_g$ is well-defined on the complement to $\phi_f^{-1}(\mathrm{Im}\mathcal{A})$ in the unit disk, and gives a holomorphic isomorphism between this complement and the complement to $\overline{\mathrm{Im}\mathcal{B}}$ in the sphere. The set $\phi_f^{-1}(\mathrm{Im}\mathcal{A})$ consists of countably many intervals of the form $[z, z/|z|]$. Set $\Phi = \phi_g \circ \phi_f^{-1}$. The map $\Phi$ is defined on $U$. Clearly, we have $g = \Phi \circ f \circ \Phi^{-1}$ on $\Phi(U)$. Note also that $\Phi$ preserves the values of the Green function.

It is easy to see that $\Phi$ extends continuously to each side of each curve $\alpha(S^1)$, $\alpha \in \mathcal{A}$. The extension preserves the values of the Green function. It follows that, for every $\alpha \in \mathcal{A}$ and the corresponding $\beta \in \mathcal{B}$, the function $\pi_\beta^{-1} \circ \Phi \circ \pi_\alpha$ extends to $S^1 \cup \pi_\alpha^{-1}(U)$ so that the extension is continuous at all points of $S^1$, and its restriction to $S^1$ is the identity.

We now need to show that for any point $z \in \mathbb{C} - \mathrm{Im}\mathcal{A}$ and any sequence $z_n \in U$ converging to $z$, the sequence $w_n = \Phi(z_n)$ converges to a well-defined point in the dynamical plane of $g$, and that this point does not depend on the choice of the sequence $z_n$. Since $\Phi$ is continuous on $U$, it suffices to assume that $z$ belongs to the Julia set of $f$. Any limit point of the sequence $w_n$ must have the same symbolic itinerary as $z$, therefore, this can only be one point. We denote this point by $\Phi(z)$, which is justified by the fact that $\Phi$ extends continuously to $z$. Note that, for different points $z$ in the Julia set of $f$ but not in $\mathrm{Im}\mathcal{A}$, the points $\Phi(z)$ have different symbolic itineraries, and hence are different. This finishes the proof of the theorem. □

The reason for the proof shown above to be so simple is that we know a lot about topological dynamics of both $f$ and $g$. However, we would like to use regluing to describe new kinds of topological dynamics, and that would be necessarily more complicated.

### *2.4. Topological models via regluing*

First, we need to make the notion of topological model more precise. Define an (abstract) *topological model* as the collection of the following data:

- A branched covering $f : X \to X$, where $X$ is a topological space homeomorphic to the 2-sphere.
- A compact fully invariant subset $J \subset X$, called the *Julia set* of $f$. The complement to the Julia set is called the *Fatou set*.
- A complex structure (i.e. a structure of a one-dimensional complex manifold, not necessarily connected) on the Fatou set such that $f$ is holomorphic with respect to this structure.

The topological space $X$, the map $f$ and the set $J$ are called the *model space*, the *model map* and the *model Julia set*, respectively. Instead of referring to a topological model as $(f, X, J)$, we will sometimes simply say "topological model $f$". We will sometimes call $X$ the *dynamical sphere* of $f$. Of course, any rational function is an abstract topological model. We say that two abstract topological models $(f, X_f, J_f)$ and $(g, X_g, J_g)$ are *equivalent* if there is a homeomorphism $\phi : X_f \to X_g$ conjugating $f$ with $g$ and such that $\phi(J_f) = J_g$ and $\phi|_{X_f - J_f}$ is holomorphic. We say that $f$ *models* a rational function $R$ if $f$ is equivalent to $R$ as an abstract topological model.

Although a rational function can be given by a very explicit formula, it may be very hard to see its dynamics from this formula. Even drawing an accurate picture requires a lot of computational power. However, we need a *dynamically explicit* description of rational functions. In other terms, we want to find explicit topological models for rational functions. The notion of abstract topological model formalizes this goal, except for the notion of being (dynamically) explicit, which is, of course, an informal notion. The situation is similar to finding explicit solutions of algebraic equations — the notion of solution is rigorously defined, but the notion of being explicit is informal, and depends on the taste, philosophy, etc.

There are several important combinatorial constructions that modify or combine topological models into new topological models. Among the most well-known are matings and captures. Let us now define another combinatorial operation on topological models that uses regluing. There are interesting relationships between matings, captures and regluings, which we may discuss elsewhere.

Consider a topological model $(f, X, J)$ with a hyperbolic Fatou set (i.e. the complex 1-manifold $X - J$ is of hyperbolic type), and a strongly $f$-stable set $\mathcal{A}$ of disjoint $\alpha$-paths in $X$. We will now assume that $\mathcal{A}$ is *non-wandering*: for every $\alpha \in \mathcal{A}$, there exists $n > 0$ such that $f^{\circ n} \circ \alpha \notin \mathrm{Im}\mathcal{A}$. In particular, $\alpha(0, 1)$ is eventually mapped to a critical point of $f$. Under this assumption, we will define another topological model using regluing.

Define an *accumulation point of* $\mathcal{A}$ as a point $x \in X$ such that every open neighborhood of $x$ intersects infinitely many curves $\alpha(S^1)$, $\alpha \in \mathcal{A}$. All accumulation points of $\mathcal{A}$ belong to the Julia set of $f$. Indeed, since $f : X - J \to X - J$ is a holomorphic self-map of a hyperbolic complex 1-manifold, backward $f$-orbits of critical points cannot accumulate in $X - J$ (see e.g. [Mi06]). Since the curves $\alpha(S^1)$, $\alpha \in \mathcal{A}$, form a null-sequence, all accumulation points of $\mathcal{A}$ are also accumulation points of the backward orbits of critical points.

There exists a regluing $\Phi : X - \mathrm{Im}\mathcal{A} \to Y - \mathrm{Im}\mathcal{B}$ of $\mathcal{A}$ into some set $\mathcal{B}$ of disjoint $\beta$-paths in a topological sphere $Y$. Moreover, the curves $\beta(S^1)$, $\beta \in \mathcal{B}$, form a null-sequence. We set $g$ to be the continuous extension of $\Phi \circ f \circ \Phi^{-1}$, which exists by Theorem 2. Since $f$ is a branched covering, and $\mathcal{A}$ is strongly $f$-stable, the map $g$ is also a branched covering by Theorem 4. Define the Fatou set of $g$ as the set of all points $y \in Y$ such that, for some nonnegative integer $n$, we have $g^{\circ n}(y) \notin \mathrm{Im}\mathcal{B}$ (thus $\Phi^{-1}$ is defined at this point) and $\Phi^{-1}(g^{\circ n}(y)) \in X - J$. Clearly, if this condition is satisfied for one particular $n$, then it also holds for all bigger $n$. Therefore, the Fatou set of $g$ thus defined is fully invariant.

It remains to define a complex structure on the Fatou set of $g$ invariant under $g$. Take a point $y$ in the Fatou set of $g$ and its iterated image $y' = g^{\circ n}(y)$ such that $y' \notin \mathrm{Im}\mathcal{B}$ and $x' = \Phi^{-1}(y')$ is in the Fatou set of $f$. Since $\mathcal{A}$ does not accumulate in the Fatou set of $f$, there is a neighborhood $U$ of $x'$ disjoint from $\mathrm{Im}\mathcal{A}$, and a holomorphic embedding $\xi : U \to \mathbb{C}$ such that $\xi(x') = 0$. Note that $\Phi : U \to \Phi(U)$ is a homeomorphism. Therefore, $\xi \circ \Phi^{-1}$ is an embedding of the neighborhood $\Phi(U)$ of $y'$ into $\mathbb{C}$. Finally, if $k$ is the local degree of $g^{\circ n}$ at $y$, then we can define a local complex coordinate near $y$ as a branch of $\sqrt[k]{\xi \circ \Phi^{-1} \circ g^{\circ n}}$. We have now defined a complex coordinate near every point of the Fatou set of $g$. It is easy to check that all transition functions are holomorphic, and that $g$ is holomorphic with respect to the obtained complex structure on the Fatou set.

We have defined a topological model $g$. This topological model will be called the model obtained from $(f, X, J)$ by regluing of $\mathcal{A}$. It is not hard to check that $J$ lifts to the ungluing space $\Upsilon_{\mathcal{A}}$ as a compact subset $\hat{J}$ fully invariant under both $\hat{f}$ and the action of $V_4$. Moreover, the set $\hat{\Phi}(\hat{J})$ is precisely the lift of the Julia set of $g$ to the ungluing space $\Upsilon_{\mathcal{B}}$.

## 3. Boundary points of type C hyperbolic components

In this section, we discuss some general properties of quadratic rational functions on the boundaries of type C hyperbolic components, preparing for the proof of Theorem 1.

### *3.1. Equicontinuous families and holomorphic motion*

Below, we recall a standard fact about equicontinuous families widely used in complex dynamics:

**Proposition 5.** *Let $\Lambda$ be a topological space, and $X$ a metric space. Consider an equicontinuous family $\mathcal{F}$ of maps from $\Lambda$ to $X$. Let $S \subset \Lambda$ be a subset, and $\nu : \Lambda \to X$ a continuous map such that $\nu(\lambda) \in \mathcal{F}(\lambda)$ for all $\lambda \in S$. Then for all $\lambda^* \in \overline{S}$, we have*

$$\nu(\lambda^*) \in \overline{\mathcal{F}(\lambda^*)}.$$

*Proof.* Assume the contrary: $d(\nu(\lambda^*), \mathcal{F}(\lambda^*)) = \varepsilon > 0$, where $d$ denotes the distance in $X$. There is a neighborhood $V'$ of $\lambda^*$ in $\Lambda$ such that

$$d(\nu(\lambda), \nu(\lambda^*)) < \frac{\varepsilon}{2}$$

for all $\lambda \in V'$. This follows from the continuity of $\nu$. On the other hand, there is a neighborhood $V''$ of $\lambda^*$ such that

$$d(f(\lambda), f(\lambda^*)) < \frac{\varepsilon}{2}$$

for all $\lambda \in V''$ and all $f \in \mathcal{F}$. This follows from the equicontinuity of $\mathcal{F}$. Therefore, for every $\lambda \in V' \cap V''$ and every $f \in \mathcal{F}$, we have

$$d(\nu(\lambda^*), f(\lambda^*)) \leq d(\nu(\lambda^*), \nu(\lambda)) + d(\nu(\lambda), f(\lambda)) + d(f(\lambda), f(\lambda^*)) <$$

$$< d(\nu(\lambda), f(\lambda)) + \varepsilon.$$

Take $\lambda \in V' \cap V'' \cap S$ and $f \in \mathcal{F}$ such that $\nu(\lambda) = f(\lambda)$. Then $d(\nu(\lambda^*), f(\lambda^*)) < \varepsilon$, a contradiction. □

Let $\Lambda$ be a complex analytic manifold, and $A$ a set. Recall that a *holomorphic motion* over $\Lambda$ is a map $\mu : \Lambda \times A \to \overline{\mathbb{C}}$ such that $\mu(\lambda, a) \neq \mu(\lambda, b)$ for $a \neq b$ and the map $\mu_a : \lambda \mapsto \mu(\lambda, a)$ is holomorphic for every $a \in A$. We do not require that $A \subset \overline{\mathbb{C}}$ and that $a \mapsto \mu(\lambda_0, a)$ is the identity for some $\lambda_0 \in \Lambda$, although these conditions are usually included into a definition. Thus we use the term "holomorphic motion" in a slightly more general sense. The following well-known fact is very simple but important (see e.g. [MSS]):

**Theorem 6.** *Let $\Lambda$ be a Riemann surface and $\mu : \Lambda \times A \to \overline{\mathbb{C}}$ a holomorphic motion. Then the family of functions $\mu_a$, $a \in A$, is equicontinuous.*

*Proof.* If $A$ is finite, then the statement is obvious. Suppose that $A$ is infinite, and take three different points $a_1, a_2, a_3 \in A$. We can use the following generalization of Montel's theorem: if a family of holomorphic functions on $\Lambda$ is such that the graphs of all functions in the family avoid the graphs of three different holomorphic functions, and these three graphs are disjoint, then the family is equicontinuous. In our case, we can take $\mu_{a_i}$, $i = 1, 2, 3$. These three holomorphic functions have disjoint graphs, and the graph of any function $\mu_a$, $a \neq a_1$, $a_2$, $a_3$, is disjoint from the graphs of $\mu_{a_i}$. Thus the family of functions $\mu_a$ is equicontinuous. □

The following well-known theorem is proved in [MSS]:

**Theorem 7.** *Let $\Lambda$ be a Riemann surface, and $\mu : \Lambda \times A \to \overline{\mathbb{C}}$ a holomorphic motion. Suppose that $A \subset \overline{\mathbb{C}}$ and that $\mu(\lambda_0, a) = a$ for some $\lambda_0 \in \Lambda$ and all $a \in A$. Then $\mu$ extends to a holomorphic motion $\overline{\mu} : \Lambda \times \overline{A} \to \overline{\mathbb{C}}$, and, for every $\lambda \in \Lambda$, the map $a \mapsto \overline{\mu}(\lambda, a)$ from $\overline{A}$ to $\overline{\mathbb{C}}$ is quasi-symmetric.*

Using this theorem, we can prove the following (cf. e.g. [AY]):

**Proposition 6.** *Consider a holomorphic motion $\mu$ satisfying the assumptions of Theorem 7. Assume that $A$ is an open subset of $\overline{\mathbb{C}}$. Consider a continuous function $\nu : \Lambda \to \overline{\mathbb{C}}$ and the subset $O$ of $\Lambda$ consisting of all $\lambda \in \Lambda$ such that $\nu(\lambda) \in \mu(\lambda, A)$. Then $O$ is open. Moreover, $\nu(\lambda^*) \in \partial\mu(\lambda, A)$ if $\lambda^* \in \partial O$.*

*Proof.* Consider the point $\lambda_0$ from Theorem 7. We can assume without loss of generality that $\lambda_0 \in O$. Then $\nu(\lambda_0)$ is some point $a_0 \in A$. Suppose that $a_0 \neq \infty$. Let $\alpha : S^1 \to A$ be a small loop around $a_0$. Then we have

$$I(\lambda_0) = \int_{S^1} \frac{d\alpha(t)}{\alpha(t) - a_0} = 2\pi i.$$

Set $\varepsilon$ to be the minimal spherical distance between $a_0$ and $\alpha(t)$. There is an open neighborhood $V$ of $\lambda_0$ such that the distance between $\nu(\lambda)$ and $\mu(\lambda, \alpha(t))$ is bigger than $\varepsilon/2$ for all $\lambda \in V$ and all $t \in S^1$. This follows from the continuity of $\nu$ and equicontinuity of $\mu(\cdot, a)$, $a \in A$. We can also assume that $\mu(\lambda, \alpha(t)) \neq \infty$ for all $\lambda \in V$ and $t \in S^1$. Then the integral

$$I(\lambda) = \int \frac{d\mu(\lambda, \alpha(t))}{\mu(\lambda, \alpha(t)) - \nu(\lambda)}$$

(with respect to $t$) is a well-defined and continuous function on $V$. Since the possible values of this integral are discrete, we must have $I(\lambda) = 2\pi i$ for all $\lambda \in V$. Therefore, $\nu(\lambda) \in \mu(\lambda, A)$ for such $\lambda$, and $V \subset O$. Thus we have proved that $O$ is open.

Suppose now that $\lambda^* \in \partial O$. Then $\nu(\lambda^*) \in \overline{\mu(\lambda^*, A)}$ by Proposition 5. On the other hand, we have $\nu(\lambda^*) \notin \mu(\lambda^*, A)$ because $\lambda^* \notin O$. Therefore, $\nu(\lambda^*) \in \partial\mu(\lambda, A)$. $\square$

### *3.2. Parameter curves*

Quadratic rational functions that are conjugate by a Möbius transformation have the same dynamical properties. Therefore, one wants to parameterize conjugacy classes, choosing one (or finitely many) particular representative(s) from every conjugacy class. There are many different ways to do this parameterization, see e.g. [Mi93, R]. For our purposes, it will be convenient to do the following: send the two critical points of a rational function $f$ to 0 and $\infty$ by a suitable Möbius transformation. If $\infty$ is fixed, then we can reduce $f$ to the form $p_c : z \mapsto z^2 + c$. If $\infty$ is not fixed, then we can send a preimage of $\infty$ to 1 (unless 0 is a critical point mapping to $\infty$, i.e. $f$ is conjugate to $z \mapsto 1/z^2$), thus $f$ will have the form

$$R_{a,b}(z) = \frac{az^2 - b}{z^2 - 1}.$$

In any case, $f$ is Möbius conjugate to $p_c$, or to $R_{a,b}$, or to $1/z^2$.

We will now consider the following algebraic curves in $\mathbb{C}^2$:

$$\mathcal{V}_k = \{(a,b) \mid R_{a,b}^{\circ k-1}(\infty) = 1\}, \quad k = 2, 3, \dots$$

These are complex one-dimensional slices of the parameter space of quadratic rational functions. These slices correspond to simple (periodic) types of behavior of one critical point (note that $R_{a,b}(1) = \infty$ so that for all $(a,b) \in \mathcal{V}_k$, the critical point $\infty$ of the function $R_{a,b}$ is periodic of period $k$).

We will identify pairs $(a,b) \in \mathcal{V}_k$ with the corresponding rational functions $R_{a,b}$. For every $(a,b) \in \mathcal{V}_k$, let $\Omega_{a,b}$ denote the immediate basin of the super-attracting fixed point $\infty$ of the rational function $R_{a,b}^{\circ k}$. We will sometimes write $\Omega_F$ instead of $\Omega_{a,b}$ if $F = R_{a,b}$. Define the set

$$\mathcal{B}_k = \{(a,b) \in \mathcal{V}_k \mid 0 \in R_{a,b}^{\circ m}(\Omega_{a,b}),\ m \geq 0\}.$$

This set consists of all parameter values such that the critical point 0 is in the immediate basin of the cycle of $\infty$. Thus $\mathcal{B}_k$ is the union of all type B hyperbolic components in $\mathcal{V}_k$. Define the set

$$\Lambda_k = \mathcal{V}_k - \overline{\mathcal{B}_k}.$$

This is a one-dimensional complex manifold (for smoothness, see e.g. [S, R03]).

Recall that a function $R_{a,b} \in \Lambda_k$ is hyperbolic of type C if $R_{a,b}^{\circ m}(0) \in \Omega_{a,b}$ for some $m > 0$. The set of type C hyperbolic functions is open by Proposition 6.

### *3.3. Notation needed for the proof of Theorem 1*

We will use the following notation throughout the proof of Theorem 1: Let $H \subset \Lambda_k$ be a hyperbolic component of type C, and $f \in \partial H$. Note that the boundary is taken in $\Lambda_k$, so that the boundaries of type B components are automatically excluded. Set $\Omega = \Omega_f$. Also, let $h$ be the center of the hyperbolic component $H$, i.e. the unique critically finite map in $H$. There is a positive integer $k'$ such that for any parameter value $(a, b) \in H$, we have $R_{a,b}^{\circ k'}(0) \in \Omega_{a,b}$, and $k'$ is the minimal integer with this property.

Let $\Delta$ denote the unit disk $\{|z| < 1\}$. There is a holomorphic motion

$$\mu : \Lambda_k \times \Delta \to \overline{\mathbb{C}}$$

such that $\mu((a, b), z)$ is the point in $\Omega_{a,b}$, whose Böttcher coordinate is equal to $z$. By Theorem 7, this holomorphic motion extends to a holomorphic motion

$$\overline{\mu} : \Lambda_k \times \overline{\Delta} \to \overline{\mathbb{C}}$$

By Proposition 6, we have $f^{\circ k'}(0) \in \partial\overline{\mu}(\lambda_0, \Delta) = \partial\Omega$, where $\lambda_0 \in \Lambda_k$ is the parameter value corresponding to $f$. Therefore, $f(0)$ is on the boundary of some Fatou component $V$ containing a point $v$ such that $f^{\circ k'-1}(v) = \infty$.

Note that there may be several Fatou components containing $f(0)$ on their boundary and eventually mapping to $\Omega$. The choice of the Fatou component $V$ is determined by the choice of the hyperbolic component $H$: as the parameter value enters $H$, the critical value corresponding to $f(0)$ enters a Fatou component corresponding to $V$ (here "corresponding" means "included into the holomorphic motion of").

### *3.4. Accessibility and non-recurrence*

The following Proposition shows that the Fatou components of $f$ do not have topological pathologies (cf. [AY]):

**Proposition 7.** *The boundary of $\Omega$ is locally connected. In particular, every boundary point is accessible from $\Omega$.*

*Proof.* Let $\lambda_0$ be the parameter value corresponding to $f$. Then the function $\overline{\mu}_f : z \mapsto \overline{\mu}(\lambda_0, z)$ is a quasi-symmetric homeomorphism between $\overline{\Delta}$ and $\overline{\Omega}$, by Theorem 7. It follows that the boundary of $\Omega$ is locally connected. □

Since $V$ is a pullback of $\Omega$, the boundary of $V$ is also locally connected. In particular, $f(0) \in \partial V$ is accessible from $V$. The following proposition shows that the dynamical properties of $f$ are rather simple:

**Proposition 8.** *The critical point* 0 *of $f$ is non-recurrent.*

*Proof.* Note that all limit points of the orbit of 0 belong to the forward orbit of $\partial\Omega$ under $f$. Therefore, if 0 is recurrent, then $0 \in f^{\circ l}(\partial\Omega)$ for some $l = 0, \ldots, k-1$. In other terms, $0 = f^{\circ l}(\overline{\mu}(\lambda_0, z))$, where $z$ is a point on the unit circle, and $\lambda_0$ is the parameter value corresponding to $f$.

Consider the holomorphic function $\nu(\lambda) = R_\lambda^{\circ l}(\overline{\mu}(\lambda, z))$ on $\Lambda_k$. (Recall that $R_\lambda$ is the rational function $R_{a,b}$ corresponding to the point $\lambda = (a, b) \in \Lambda_k$). This function vanishes at the point $\lambda_0$. However, $\nu$ is not identically equal to zero, because e.g. for $h \in H$, the critical point is not in the orbit of $\partial\Omega_h$. Therefore, we can choose a small loop $\gamma : S^1 \to \Lambda_k$ around $\lambda_0$ such that $\nu \circ \gamma$ loops around 0 (i.e. the algebraic number of full turns is nonzero). Take $\tilde{z} \in \Delta$ sufficiently close to $z$. Then the function $\tilde{\nu}(\lambda) = R_\lambda^{\circ l}(\overline{\mu}(\lambda, \tilde{z}))$ is uniformly close to $\nu$. In particular, $\tilde{\nu} \circ \gamma$ loops around 0. Therefore, there exists a parameter value $\lambda_1$ inside $\gamma$ such that $\tilde{\nu}(\lambda_1) = 0$. This means that 0 lies in $R_{\lambda_1}^{\circ l}(\Omega_{\lambda_1})$, i.e. $\lambda_1 \in \mathcal{B}_k$, a contradiction. □

### *3.5. Restatement of Theorem 1*

In this section, we will restate Theorem 1, and give some details on the particular set of paths that was mentioned but not defined in the statement of Theorem 1. We also need to introduce some more notation. Consider a simple path $\alpha_{-1} : [0, 1] \to \overline{V}$ such that $\alpha_{-1}(0) = f(0)$, $\alpha_{-1}(1) = v$, and $\alpha_{-1}(0, 1] \subset V$. The existence of such path follows from Proposition 7. There is an $\alpha$-path $\alpha_0 : S^1 \to \overline{\mathbb{C}}$ such that $f \circ \alpha_0(t_1, t_2) = \alpha_{-1}(|t_1|)$. This $\alpha$-path is unique up to the change of variables $t_1 \to -t_1$. Let $\mathcal{A}$ be the set of all pullbacks of $\alpha_0$ under iterates of $f$, including $\alpha_0$.

**Proposition 9.** *The curves $\alpha(S^1)$, $\alpha \in \mathcal{A}$ form a null-sequence.*

*Proof.* Since the critical point 0 is non-recurrent, there exists a neighborhood $U$ of $\alpha_0(S^1)$ that does not intersect the post-critical set. Choose a smaller neighborhood $U'$ of $\alpha_0(S^1)$ that is compactly contained in $U$. By the Koebe distortion theorem, the set of pullbacks of $U'$ forms a null-sequence. It follows that the curves $\alpha(S^1)$, $\alpha \in \mathcal{A}$, also form a null-sequence. □

We can conclude that there exists a regluing $\Phi$ of $\mathcal{A}$ into some set of disjoint $\beta$-paths. Moreover, the function $g = \Phi \circ f \circ \Phi^{-1}$ is well-defined as a topological model, since $\mathcal{A}$ is strongly $f$-stable and non-wandering (both properties follow immediately from the definition of $\mathcal{A}$). We can now give a precise statement, from which Theorem 1 follows:

**Theorem 8.** *The map $g$ is equivalent as a topological model to the critically finite hyperbolic rational function $h \in H$.*

Theorem 8 implies Theorem 1. Indeed, if $g = \Phi \circ f \circ \Phi^{-1}$ is conjugate to $h$, then $h = \Psi \circ f \circ \Psi^{-1}$ for some topological regluing $\Psi$. It follows that $f = \Psi^{-1} \circ h \circ \Psi$. Note that $\Psi^{-1}$ is also a regluing.

Thus it remains to prove Theorem 8. We first prove that $g$ is Thurston equivalent to $h$. By a theorem of Mary Rees [R92], Thurston equivalence to a hyperbolic rational function implies semi-conjugacy. We will recall the proof of this theorem. What remains is to prove that all fibers are trivial. Several ideas for this part were also taken from [R92]. Overall, the argument is rather simple, but we need to know from the very beginning that $g$ is well-defined as a branched covering. Here we use results of Section 6 on the existence of topological regluing.

## 4. The proof of Theorem 1

In this section, we prove Theorem 1. We use notation introduced in Sections 3.5 and 3.3.

### *4.1. Backward stability*

Recall the following theorem of Mañe [Ma, TS]:

**Theorem 9 (Backward stability).** *Let $F$ be a rational function with the Julia set $J$. Suppose that there are no recurrent critical points in $J$. Then, for every $\varepsilon, \varepsilon' > 0$, there exist a positive real number $\delta(\varepsilon)$ (depending only on $\varepsilon$), and a positive integer $n_0(\varepsilon, \varepsilon')$ (depending on $\varepsilon$ and $\varepsilon'$) such that for $x, y \in J$*

1. *if $d(x, y) < \delta(\varepsilon)$, then for every $n \geq 0$ and for every $x' \in F^{-n}(x)$, there is a point $y' \in F^{-n}(y)$ such that $d(x', y') < \varepsilon$,*
2. *if $d(x, y) < \delta(\varepsilon)$ and $n \geq n_0(\varepsilon, \varepsilon')$, then for every $x' \in F^{-n}(x)$, there is a point $y' \in F^{-n}(y)$ such that $d(x', y') < \varepsilon'$.*

*Here $d$ is any metric on $\overline{\mathbb{C}}$ compatible with the topology.*

By Proposition 8, this theorem is applicable to the function $f$. From the backward stability of $f$ on its Julia set $J_f$, we can deduce the backward stability of $g$ on its Julia set $J_g$:

**Proposition 10 (Backward stability of $g$ on $J_g$).** *The map $g$ is backwards stable on $J_g$, in the sense of Theorem 9.*

*Proof.* Let $\hat{f}$ be the lift of $f$ to the ungluing space $\Upsilon_{\mathcal{A}}$, and $\hat{J}_f$ the lift of the Julia set $J_f$. It is easy to see that $\hat{f}$ is backwards stable on $\hat{J}_f$. This follows from the backward stability of $f$ on $J_f$ and the fact that the fibers of the projection $\pi_{\mathcal{A}} : \hat{J}_f \to J_f$ form a null-sequence. Since backward stability is a topological property, $\hat{g}$ is backwards stable on $\hat{J}_g$. It follows that $g$ is backwards stable on $J_g$. □

*4.2. Thurston equivalence*

The map $g$ is critically finite. Indeed, the critical values of $g$ are $\Phi \circ f(\infty)$ and $\Phi(v)$. The forward orbit of $\Phi \circ f(\infty)$ under $g$ is the $\Phi$-image of the forward orbit of $f(\infty)$ under $f$. Similarly, the forward orbit of $\Phi(v)$ under $g$ is the $\Phi$-image of the forward orbit of $v$ under $f$. It remains to note that both $f(\infty)$ and $v$ have finite forward orbits under $f$. In this section, we will prove that the map $g$ is Thurston equivalent to $h$.

We first recall the notion of Thurston equivalence. A branched self-covering of the sphere with finite post-critical set is sometimes called a *Thurston map.* Recall that the post-critical set is the union of forward orbits of all critical values. Two Thurston maps $F$ and $G$ with post-critical sets $P_F$ and $P_G$, respectively, are called *Thurston equivalent* if there exist homeomorphisms $\phi, \psi : S^2 \to S^2$ that make the following diagram commutative

$$\begin{array}{ccc} (S^2, P_F) & \xrightarrow{F} & (S^2, P_F) \\ \psi \downarrow & & \downarrow \phi \\ (S^2, P_G) & \xrightarrow{G} & (S^2, P_G) \end{array}$$

and such that $\phi$ and $\psi$ are homotopic relative to $P_F$ through homeomorphisms, in particular, $\phi|_{P_F} = \psi|_{P_F}$. The following are well-known useful criteria of Thurston equivalence:

**Theorem 10.** *Suppose that $F_t$, $t \in [0,1]$, is a continuous family of Thurston maps of degree 2 such that the number of points in $P_{F_t}$ does not change with $t$. Then $F_0$ is Thurston equivalent to $F_1$.*

Let $C_F$ denote the set of critical points of $f$.

**Theorem 11.** *Let $Z$ be a compact connected locally connected subset of $S^2$ such that $S^2 - Z$ is connected and dense. Suppose that quadratic Thurston maps $F$ and $G$ are such that $F = G$ on $Z$ and $P_F \cup C_F = P_G \cup C_G \subset Z$. Then $F$ and $G$ are Thurston equivalent.*

For completeness, we sketch the proofs of these theorems.

*Proof of Theorem 10.* It suffices to prove that $F_t$ is Thurston equivalent to $F_{t'}$ provided that $t$ is close to $t'$. The ramified coverings $F_t$ and $F_{t'}$ are then uniformly close, therefore, their post-critical sets are also close (this follows from the fact that the (weighted) number of critical points in a disk can be computed as a winding number). In particular, these maps are conjugate on their post-critical sets. Moreover, we can choose a homeomorphism $\phi : S^2 \to S^2$ that is close to the identity, maps $P_{F_t}$ to $P_{F_{t'}}$ and conjugates the dynamics of $F_t$

on $P_{F_t}$ with the dynamics of $F_{t'}$ on $P_{F_{t'}}$. The multivalued function $F_{t'}^{-1} \circ \phi \circ F_t$ has two single valued branches. Indeed, every critical value of $\phi \circ F_t$ is a critical value of $F_{t'}$, and the only $F_{t'}$-preimage of this critical value is the corresponding critical point, since the map is quadratic. Thus, whenever $\phi(F_t(z))$ is a ramification point of $F_{t'}^{-1}$, the function $F_{t'}^{-1} \circ \phi \circ F_t$ can be written as $\sqrt{u^2}$ in some local coordinate. It follows that this function has no ramification points, hence it splits into two single valued branches. One of the branches is close to the identity; denote this branch by $\psi$. The homeomorphisms $\phi$ and $\psi$ thus constructed provide a Thurston equivalence between $F_t$ and $F_{t'}$ (they are homotopic relative to the post-critical set because they are both close to the identity). □

*Proof of Theorem 11.* We can assume without loss of generality that $F = G$ on some neighborhoods of all critical points. This can be arranged e.g. by a small variation of $G$ near its critical points, without changing the post-critical set.

By the same argument as in the proof of Theorem 10, the multivalued function $G^{-1} \circ F$ splits into two single valued branches. Near every critical point, one of the branches of $G^{-1} \circ F$ is the identity. It follows that there is a branch $\phi$ of $G^{-1} \circ F$ that restricts to the identity on $Z$. Indeed, the branches cannot switch outside of the critical points, and $z \in G^{-1}(F(z))$ for all $z \in Z$. There exists a continuous map $\gamma : \overline{\Delta} \to S^2$ that restricts to a holomorphic homeomorphism between $\Delta$ and $S^2 - Z$ (the set $Z$ is infinite because any quadratic map has two different critical values). It is not hard to see (with the help of Carathéodory's theory) that $\gamma^{-1} \circ \phi \circ \gamma$ extends continuously to a self-homeomorphism of $\overline{\Delta}$ that is the identity on the unit circle. Such homeomorphism is isotopic to the identity relative to the unit circle (i.e. through homeomorphisms that are the identity on $S^1$). It follows that $\phi$ is isotopic to the identity through homeomorphisms of $S^2$, whose restrictions to $Z$, in particular, to the post-critical set $P_F = P_G$, are the identity. The theorem follows. □

Consider a simple curve in the parameter space $\Lambda_k$ that connects $f$ to $h$ and lies entirely in $H$ except for the endpoint $f$. Let $f_t$, $t \in [0, 1]$ be the corresponding one-parameter family of rational functions so that $f_0 = f$ and $f_1 = h$. Consider a point $v_t$ depending continuously on $t \in [0, 1]$ and satisfying the following properties:

$$v_0 = v, \quad f_t^{\circ k'-1}(v_t) = \infty.$$

Then $v_1 = h(0)$. Define a quadratic rational function $Q_t$ by the property that the critical values of $Q_t$ coincide with $f_t(\infty)$ and $v_t$. We can even assume that the corresponding critical points of $Q_t$ are $\infty$ and 0, respectively (explicit formulas for such functions are very easy to obtain; they are given in Section 5). There exists a continuous family of $\alpha$-paths $\alpha_t$, $t \in [0,1)$ connecting the two preimages of $v_t$ under $f_t$ and having the property that $\alpha_t(-t_1,t_2) = -\alpha_t(t_1,t_2)$. Additionally, we can assume that $\alpha_t$ converge to the constant path 0 uniformly as $t \to 1$. Moreover, this continuous family can be chosen in such a way that for $t = 0$, we obtain the same $\alpha$-path $\alpha_0$ as that introduced in Section 3.5. (Recall that $\alpha_0$ and all its pullbacks form the set $\mathcal{A}$ of $\alpha$-paths that is used in our regluing construction). The multivalued analytic function $Q_t^{-1} \circ f_t$ has a holomorphic branch on the complement to $\alpha_t(S^1)$. Denote this branch by $j_t$. The choice of particular branch is not important; however, we need that $j_t$ depend continuously on $t$. This can obviously be arranged.

Set $q_t = j_t \circ Q_t$. This function is defined on the complement to $Q_t^{-1}(\alpha_t(S^1))$, i.e. on the complement to a pair of disjoint simple curves. Intuitively, the function $q_t$ is obtained from $f_t$ by regluing the path $\alpha_t$, or, to be more precise, we have $q_t = j_t \circ f_t \circ j_t^{-1}$ wherever the right-hand side is defined (however, $q_t$ is defined on a larger set). Note that $q_t$ is always defined and analytic in a neighborhood of 0, because $Q_t(0) = v_t$ avoids the curve $\alpha_t(S^1)$. In particular, 0 is a critical point of $q_t$. Another critical point is $j_t(\infty) = \infty$. Moreover, the critical orbits of $q_t$ are finite and of constant size. More precisely, we have

$$q_t^{\circ k'}(0) = j_t \circ f_t^{\circ k'-1}(v_t) = \infty, \quad q_t^{\circ k}(\infty) = j_t \circ f_t^{\circ k}(\infty) = \infty.$$

In other words, the orbits of 0 and $\infty$ under $q_t$ have the same dynamics as the orbits of the same points under $h$. The relations given above can be easily proved using that $q_t = j_t \circ Q_t$ and that $Q_t \circ j_t = f_t$ wherever the left-hand side is defined. The construction of the map $q_t$ will reappear in Section 5, where more details can be found.

The maps $q_t$ are critically finite but they are not Thurston maps because of discontinuities. However, one can "approximate" $q_t$ by branched coverings $\hat{q}_t$ that differ from $q_t$ only in a small neighborhood of $Q_t^{-1}(\alpha_t(S^1))$. Of course, in a small neighborhood of $Q_t^{-1}(\alpha_t(S^1))$, we cannot hope to make the maps $\hat{q}_t$ uniformly close to $q_t$, thus the word "approximate" comes in the quotation marks. However, we can do the following. Choose a sufficiently small Jordan neighborhood $D$ of $\alpha_t(S^1)$. Then $Q_t^{-1}(D)$ is a pair of disjoint Jordan disks $D_1$ and $D_2$. The boundaries $\partial D_1$, $\partial D_2$ get mapped to the same Jordan curve by $q_t$, moreover, $q_t$ acts as an orientation-preserving homeomorphism on $\partial D_i$, $i = 1, 2$. Therefore, it is possible to extend $q_t$ homeomorphically inside $D_1 \cup D_2$. The extension thus obtained is our map $\hat{q}_t$. We can choose $\hat{q}_t$ to vary continuously with $t$. Additionally, we can arrange that $\hat{q}_1 = h$; in any case, $\hat{q}_1$ is Thurston equivalent to $h$. Thus $\hat{q}_t$ is a continuous family of Thurston maps, whose post-critical sets are of constant cardinality. By Theorem 10, we obtain

**Lemma 1.** *The map $\hat{q}_0$ is Thurston equivalent to $\hat{q}_1 = h$.*

What remains to prove is the following

**Lemma 2.** *The maps $\hat{q}_0$ and $g$ are Thurston equivalent.*

*Proof.* Recall that $g = \Phi \circ f \circ \Phi^{-1}$ wherever the right-hand side is defined. Here $\Phi$ is a regluing of the set $\mathcal{A}$ of $\alpha$-paths consisting of the path $\alpha_0$ and all its pullbacks under $f_0 = f$. The corresponding set of $\beta$-paths in the dynamical sphere of $g$ is $\mathcal{B}$. Let $\beta_0$ be the $\beta$-path corresponding to the $\alpha$-path $\alpha_0$.

Consider the map $j_0 \circ \Phi^{-1}$. It is defined on the complement to $\mathrm{Im}\mathcal{B}$, it establishes a homeomorphism between $S^2 - \mathrm{Im}\mathcal{B}$ and $j_0(S^2 - \mathrm{Im}\mathcal{A})$, and extends continuously to $\beta_0(S^1)$. Denote the extension by $\Psi^{-1}$. Then $\Psi$ is a homeomorphism between the complement to $j_0(\mathrm{Im}\mathcal{A} - \alpha_0(S^1))$ the complement to $\mathrm{Im}\mathcal{B} - \beta_0(S^1)$.

We claim that $\Psi^{-1} \circ g(y) = q_0 \circ \Psi^{-1}(y)$ for every point $y \in S^2$ in the complement to $\mathrm{Im}\mathcal{B} - \beta_0(S^1)$. Indeed, the point $x = \Psi^{-1}(y)$ is well-defined and lies in $S^2 - j_0(\mathrm{Im}\mathcal{A} - \alpha_0(S^1))$. The point $Q_0(x)$ cannot be in $\mathrm{Im}\mathcal{A}$, otherwise

$$x \in Q_0^{-1}(\mathrm{Im}\mathcal{A}) = j_0(f^{-1}(\mathrm{Im}\mathcal{A})) \in j_0(\mathrm{Im}\mathcal{A} - \alpha_0(S^1)).$$

Therefore, $j_0 \circ Q_0(x) = q_0(x) = q_0 \circ \Psi^{-1}(y)$ is well-defined. Clearly, $\Psi^{-1} \circ g(y)$ is also well-defined. Thus both parts of the equality $\Psi^{-1} \circ g = q_0 \circ \Psi^{-1}$ are defined on the complement to $\mathrm{Im}\mathcal{B} - \beta_0(S^1)$. The equality itself holds wherever both parts are defined, because at least on some dense set, we have

$$\Psi^{-1} \circ g = j_0 \circ \Phi^{-1} \circ \Phi \circ f \circ \Phi^{-1} = j_0 \circ f \circ \Phi^{-1} =$$

$$= j_0 \circ Q_0 \circ j_0 \circ \Phi^{-1} = q_0 \circ \Psi^{-1}.$$

Note that the set $P_g \cup C_g$ is disjoint from $\mathrm{Im}\mathcal{B} - \beta_0(S^1)$. Consider a simple curve $Z$ disjoint from $\mathrm{Im}\mathcal{B} - \beta_0(S^1)$ and containing $P_g \cup C_g$. (The existence of such curve follows from a simple Baire category argument, see Section 6.2 for more detail.) Then we can define a homeomorphism $\hat{\Psi} : S^2 \to S^2$ such that $\hat{\Psi}^{-1} = \Psi^{-1}$ on $Z$ and on $g(Z)$. Indeed, both curves $Z$ and $g(Z)$ are disjoint from $\mathrm{Im}\mathcal{B} - \beta_0(S^1)$, thus $\Psi^{-1}$ is well-defined on the union of these curves, and acts as a homeomorphism. It is not hard to see that $\Psi^{-1}$ can be extended inside every component of the complement to $Z \cup g(Z)$ homeomorphically. Doing this for all components, we obtain a homeomorphism $\hat{\Psi}^{-1}$.

The restriction of $g$ to $Z$ is the same as the restriction of $\hat{\Psi} \circ q_0 \circ \hat{\Psi}^{-1}$ to $Z$. Indeed, for every $x \in Z$, we have

$$\hat{\Psi} \circ q_0 \circ \hat{\Psi}^{-1}(x) = \hat{\Psi} \circ q_0 \circ \Psi^{-1}(x) = \hat{\Psi} \circ \Psi^{-1} \circ g(x) = g(x).$$

We can even find a continuous "approximation" $\hat{q}_0$ of $q_0$ such that the restriction of $\hat{\Psi} \circ \hat{q}_0 \circ \hat{\Psi}^{-1}$ to $Z$ is still the same. Indeed, we must have $q_0 = \hat{q}_0$ on the set $\Psi^{-1}(Z)$, where $q_0$ is well-defined (as we know, $q_0$ is defined on all points $\Psi^{-1}(x)$, where $x \notin \mathrm{Im}\mathcal{B} - \beta_0(S^1)$).

By Theorem 11, $g$ is Thurston equivalent to $\hat{\Psi} \circ \hat{q}_0 \circ \hat{\Psi}^{-1}$, hence to $\hat{q}_0$. □

### *4.3. Semi-conjugacy*

Recall the following theorem of Mary Rees [R92]:

**Theorem 12.** *Suppose that a Thurston map $F$ of degree 2 is Thurston equivalent to a hyperbolic rational function $G$. Moreover, suppose that there is an $F$-invariant complex structure near the critical orbits of $F$. Then there is a continuous map $\phi : S^2 \to S^2$ such that $\phi \circ F = G \circ \phi$.*

*Proof.* We can assume $F$ to be defined on $\overline{\mathbb{C}}$ (i.e. on a sphere with a global complex structure) and holomorphic on some open set $U$ containing the post-critical set and satisfying $F(U) \Subset U$. We have the diagram

$$\begin{array}{ccc} \overline{\mathbb{C}} & \xrightarrow{F} & \overline{\mathbb{C}} \\ \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_0} \\ \overline{\mathbb{C}} & \xrightarrow{G} & \overline{\mathbb{C}} \end{array}$$

where $\phi_0$ and $\phi_1$ are homeomorphisms holomorphic on $U$; moreover, $\phi_1 = \phi_0$ on $U$, and $\phi_0$ is isotopic to $\phi_1$ relative to $U$ (the existence of such homeomorphisms $\phi_0$ and $\phi_1$ follows from Böttcher's theorem). Consider the multivalued function $G^{-1} \circ \phi_1 \circ F$. Since the critical values of $\phi_1 \circ F$ coincide with ramification points of $G^{-1}$, this multivalued function has a single valued branch $\phi_2$ such that $\phi_2 = \phi_1$ on $F^{-1}(U)$. We have the following diagram:

$$\begin{array}{ccccc} \overline{\mathbb{C}} & \xrightarrow{F} & \overline{\mathbb{C}} & \xrightarrow{F} & \overline{\mathbb{C}} \\ \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_0} \\ \overline{\mathbb{C}} & \xrightarrow{G} & \overline{\mathbb{C}} & \xrightarrow{G} & \overline{\mathbb{C}} \end{array}$$

Similarly, we can define a sequence of homeomorphisms $\phi_n$ with the following properties: $G \circ \phi_n = \phi_{n-1} \circ F$ and $\phi_n = \phi_{n-1}$ on $F^{-(n-1)}(U)$. Then $\phi_n$ is a single valued branch of $G^{-n} \circ \phi_0 \circ F^{\circ n}$.

We would like to prove that the sequence of maps $\phi_n$ converges uniformly. This would follow from the estimate

$$d(\phi_{n+1}(x), \phi_n(x)) \leq C q^{-n},$$

where $0 < q < 1$ is a number independent of $x$. We can assume that $x \notin F^{-n}(U)$, otherwise the left-hand side is zero. Consider a curve $\gamma$ connecting $\phi_0 \circ F^{\circ n}(x)$ with $\phi_1 \circ F^{\circ n}(x)$ in $\overline{\mathbb{C}} - \overline{\phi_0(U)}$ (we can arrange that this open set be connected by choosing a smaller $U$ if necessary). Since $\overline{\mathbb{C}} - \overline{U}$ is compactly contained in $\overline{\mathbb{C}} - \overline{G \circ \phi_0(U)}$, the hyperbolic length of $\gamma$ in $\overline{\mathbb{C}} - \overline{G \circ \phi_0(U)}$ can be made bounded by some constant independent of $x$ and $n$. The length of the pull-back of $\gamma$ under $G^{\circ n}$ is bounded by $Cq^{-n}$ with $0 < q < 1$ by the Poincaré distance argument. The desired estimate now follows.

Let $\phi$ denote the limit of $\phi_n$. Passing to the limit in both sides of the equality $G \circ \phi_n = \phi_{n-1} \circ F$, we obtain that $G \circ \phi = \phi \circ F$. □

We also need the following general fact:

**Proposition 11.** *Suppose that a continuous map $\phi : S^2 \to S^2$ is the limit of a uniformly convergent sequence of homeomorphisms $\phi_n$. Then $\phi$ is onto and, for any point $v \in S^2$, the fiber $\phi^{-1}(v)$ is connected.*

*Proof.* Any point $y$ of $S^2 - \phi(S^2)$ has a neighborhood $V$ such that $V \subset S^2 - \phi(S^2)$. We have $y = \phi_n(x_n)$ for all $n$. Since $\phi_n$ is uniformly close to $\phi$ for large $n$, we must have $\phi(x_n) \in V$, a contradiction.

Consider two points $z$ and $w$ in the fiber $\phi^{-1}(v)$. For large $n$, the points $\phi_n(z)$ and $\phi_n(w)$ are very close to each other. Let $D_n$ be a small closed disk containing both of them, and set $A_n = \phi_n^{-1}(D_n)$. We can assume that the diameter of $D_n$ tends to 0 as $n \to \infty$. The sequence of compact sets $A_n$ has a subsequence that converges in the Hausdorff metric. Denote the limit by $A$. As a Hausdorff limit of compact connected sets, the set $A$ is connected. Moreover, it contains both points $z$ and $w$. We claim that $\phi(A) = v$. Indeed, for any point $a \in A$, there is a sequence $a_n \in A_n$ such that $a_n \to a$. The distance between $\phi_n(a_n)$ and $\phi(a)$ tends to zero, and $\phi_n(a_n) \to v$, hence $\phi(a) = v$. Thus any pair of points in $\phi^{-1}(v)$ belongs to a common connected subset of $\phi^{-1}(v)$. This means that $\phi^{-1}(v)$ is connected. □

Thus, in our setting, we obtain the following

**Theorem 13.** *The map $h$ is semi-conjugate to $g$, i.e. there is a continuous surjective map $\phi : S^2 \to \overline{\mathbb{C}}$ such that $h \circ \phi = \phi \circ g$. Moreover, the fibers of $\phi$ are connected (i.e. $\phi$ is monotone).*

### *4.4. Dynamics of g*

In this section, we discuss some simple dynamical properties of $g$ that we need for the proof that $g$ is conjugate to $h$.

**Lemma 3.** *There is a positive real number $\delta_0$ such that for every pair $x, y \in J_g$ with $0 < d(x,y) < \delta_0$, we have $g(x) \neq g(y)$.*

*Proof.* Since there are no critical points in $J_g$, every point $x \in J_g$ has a neighborhood $U_x$ such that $g|_{U_x}$ is injective. Let $\delta_0$ be the Lebesgue number of the covering $\mathcal{U} = \{U_x\}_{x \in J_g}$. Now assume that $0 < d(x,y) < \delta_0$. Then $x, y \in U$, where $U \in \mathcal{U}$. Since $g$ is injective on $U$, the result follows. □

**Lemma 4.** *There is a self-homeomorphism $M \neq id$ of the dynamical sphere of $g$ such that $M^{\circ 2} = id$, and $g \circ M = g$. Moreover, we have $\phi \circ M = -\phi$, where $\phi$ is the map from Theorem 13.*

*Proof.* Indeed, the multivalued map $g^{-1} \circ g$ splits into two single valued branches. One of these branches is the identity transformation. Let $M$ be the other branch. We have $g \circ M = g$ by definition. It follows that $g \circ M^{\circ 2} = g$, thus $M^{\circ 2}$ is either $M$ or the identity. Since $M \neq id$, we have $M^{\circ 2} \neq M$. Therefore, $M^{\circ 2} = id$.

We have

$$h \circ \phi \circ M = \phi \circ g \circ M = \phi \circ g = h \circ \phi,$$

therefore $\phi(M(x)) = \pm\phi(x)$ for all $x$ in the dynamical sphere of $g$. Near the critical points of $g$ (which are the $\phi$-preimages of 0 and $\infty$), the map $\phi$ is a homeomorphism, and we have the minus sign. By continuity, the sign is the same for all points. □

### *4.5. Triviality of fibers*

In this section, we prove that the continuous map $\phi$ from Theorem 13 is actually a homeomorphism. Note first that the restriction of $\phi$ to the Fatou set of $g$ is injective. This follows from the definition of $\phi$ given in the proof of Theorem 12. Indeed, on every compact set contained in the Fatou set of $g$, the map $\phi$ coincides with a homeomorphism $\phi_n$. We know that fibers of $\phi$ are connected. Thus they lie entirely in $J_g$ (a connected set intersecting the Fatou set of $g$ by just one point is itself a singleton).

**Lemma 5.** *The restriction of $g$ to any fiber of $\phi$ is injective.*

*Proof.* Assume the contrary: $g(x) = g(y)$ for a pair of different points $x$, $y$ such that $\phi(x) = \phi(y)$. Let $M$ be the involution introduced in Lemma 4. Then $y = M(x)$, and

$$\phi(y) = \phi(M(x)) = -\phi(x) = -\phi(y).$$

It follows that $\phi(y) = 0$ or $\infty$, a contradiction. □

**Lemma 6.** *The image of any fiber of $\phi$ under the map $g$ is contained in a fiber of $\phi$*

*Proof.* Indeed, let $Z$ be a fiber of $\phi$. We have $\phi(g(Z)) = h(\phi(Z))$, which is a single point. □

The following proposition is a minor modification of an argument given in [R92].

**Proposition 12.** *Suppose that $\phi$ is not a homeomorphism, i.e. it has a non-trivial fiber. Let $\delta_0$ be the number introduced in Lemma 3. For every $\varepsilon > 0$, there exist $x, y \in J_g$ such that*

$$d(x, y) > \delta_0/2, \quad d(g(x), g(y)) < \varepsilon, \quad \phi(x) = \phi(y).$$

*Proof.* Without loss of generality, we can assume that $\varepsilon < \delta_0/2$. Let $N$ be a positive integer such that every subset of $J_g$ with at least $N$ points has a pair of distinct points on distance $< \delta(\varepsilon)$ (where $\delta(\varepsilon)$ is as in Theorem 9). Consider a nontrivial fiber $Z$ of $\phi$, and a subset $Z_0$ of $Z$ containing $N$ points. (As a connected set of cardinality $> 1$, the set $Z$ must actually have uncountably many points). Define $\varepsilon'$ to be the minimal distance between different points in $Z_0$. Set $n_0 = n_0(\varepsilon, \varepsilon')$ (where $n_0(\varepsilon, \varepsilon')$ is as in Theorem 9). The set $g^{\circ n_0}(Z_0)$ has cardinality $N$. This follows from Lemmas 5 and 6. Therefore, there is a pair of points $x_{n_0}$, $y_{n_0}$ in this set such that $d(x_{n_0}, y_{n_0}) < \delta(\varepsilon)$. Note that $\delta(\varepsilon) \le \varepsilon$, in particular, $d(x_{n_0}, y_{n_0}) < \varepsilon$.

For every $k = 0, \dots, n_0$, we can define $x_k$ and $y_k$ inductively by the following relations:

$$x_k, y_k \in g^{\circ k}(Z_0), \quad g(x_k) = x_{k+1}, \quad g(y_k) = y_{k+1}.$$

Then either $y_k$ is the closest to $x_k$ preimage of $y_{k+1}$ and, by Proposition 10, we have $d(x_k, y_k) < \varepsilon$, or $d(x_k, y_k) \ge \delta_0 - \varepsilon > \delta_0/2$. Indeed, let $y'_k$ be the closest to $x_k$ preimage of $y_{k+1}$. If $y'_k \ne y_k$, then

$$d(x_k, y_k) \ge d(y_k, y'_k) - d(y'_k, x_k) \ge \delta_0 - \varepsilon.$$

Suppose first that $d(x_k, y_k) \ge \delta_0/2$ for some $k$. Then $x = x_k$ and $y = y_k$ have the required properties. Suppose now that $y_k$ is the closest to $x_k$ preimage of $y_{k+1}$, for all $k$. Then we have $d(x_0, y_0) < \varepsilon'$ by Proposition 10. But this contradicts our choice of $\varepsilon'$, since $x_0, y_0 \in Z_0$. □

Suppose that $\phi$ is not a homeomorphism. Then, by Proposition 12, there exist points $x_n$ and $y_n$ in $J_g$ such that

$$d(x_n, y_n) \ge \delta_0/2, \quad d(g(x_n), g(y_n)) < 1/n, \quad \phi(x_n) = \phi(y_n).$$

Passing to suitable subsequences if necessary, we can assume that $x_n$ and $y_n$ converge to some points $x$ and $y$ in $J_g$, respectively. For these limit points, we must have

$$d(x, y) \geq \delta_0/2, \quad g(x) = g(y), \quad \phi(x) = \phi(y).$$

In other terms, $x$ and $y$ belong to the same fiber of $\phi$, and the restriction of $g$ to this fiber is not injective. This contradicts Lemma 5. The contradiction shows that all fibers of $\phi$ are trivial, thus $\phi$ is a homeomorphism.

This finishes the proof of Theorem 8 and Theorem 1.

## 5. Holomorphic regluing

In this section, we just sketch main characters of holomorphic theory of regluing, and give the most basic constructions.

### *5.1. An example*

We consider first a simple example, where we define an explicit sequence of approximations to a regluing. This sequence will consist of partially defined but holomorphic functions.

Let $f$ be the quadratic polynomial $z \mapsto z^2 - 6$. The Julia set $J_f$ of $f$ is a Cantor set that lies on the real line. Recall that the biggest component of the complement to $J_f$ in $[-3, 3]$ is $(-\sqrt{3}, \sqrt{3})$. Suppose we want to reglue the interval $[-\sqrt{3}, \sqrt{3}]$, thus connecting two parts of $J_f$. This is done by the following map:

$$j(z) = \sqrt{z^2 - 3}$$

(which is understood as a branch over the complement to $[-\sqrt{3}, \sqrt{3}]$ that is tangent to the identity at infinity). The inverse map is given by the formula $j^{-1}(z) = \sqrt{z^2 + 3}$, and is defined on the complement to $[-i\sqrt{3}, i\sqrt{3}]$.

Consider the composition $f \circ j^{-1}$. It turns out that this function extends to a quadratic polynomial! Indeed, we have

$$f(\sqrt{z^2 + 3}) = (z^2 + 3) - 6 = z^2 - 3.$$

We denote the polynomial in the right-hand side by $p_{-3}$; in general, $p_c$ stands for the quadratic polynomial $z \mapsto z^2 + c$. On a more conceptual level, in order to see that $f \circ j^{-1}$ is a restriction of a polynomial, it suffices to show that it extends continuously to the interval $[-i\sqrt{3}, i\sqrt{3}]$. This follows from the fact that $f$ folds the interval $[-\sqrt{3}, \sqrt{3}]$ at 0.

In fact, what we really want to consider is not the composition but the "conjugation" $j \circ f \circ j^{-1}$. Since $f \circ j^{-1}$ is a restriction of the polynomial $p_{-3}$, we define the new function $f_1$ as $j \circ p_{-3}$, which has a larger domain than $j \circ f \circ j^{-1}$. Unfortunately, the function $f_1$ is not continuous. The discontinuity of this function is due to the discontinuity of $j$. Actually, the function $f_1$ is defined and holomorphic on the complement to two simple curves — the pullbacks of $[-\sqrt{3}, \sqrt{3}]$ under $p_{-3}$, and it "reglues" these curves in a sense. We would like to get rid of this discontinuity by "conjugating" $f_1$ with yet another regluing map. To this end, we need an injective holomorphic function $j_1$ defined on the domain of $f_1$ and having the same type of discontinuity at the two special curves, where $f_1$ is undefined. We cannot take $j_1 = f_1$ because $f_1$ is, in general, two-to-one (it is the square root of a degree four polynomial). However, we can take

$$j_1 = \sqrt{f_1 - f_1(0)}.$$

The square root may look disturbing but it does not actually create any ramification, so that the function $j_1$ is a union of two branches. These branches are still not everywhere defined (they are defined exactly on the domain of $f_1$) but they are single valued! Indeed, the square root has ramification points exactly where $f_1$ is equal to $f_1(0)$ or $\infty$. But at all such places, namely at 0 and at $\infty$, the function $f_1$ has simple critical points. Thus at these places, the function $j_1$ looks like $\sqrt{u^2}$, where $u$ is some local coordinate, and this does not have any ramification. It is also easy to see that $j_1$ has no nontrivial monodromy around the curves, on which it is undefined (it suffices to look at the monodromy of the square root). We choose the branch of $j_1$ that is tangent to the identity at infinity.

Now finding $f_1 \circ j_1^{-1}$ is easy: Set $w = j_1(z)$, then $w = \sqrt{f_1(z) - f_1(0)}$, and

$$f_1 \circ j_1^{-1}(w) = f_1(z) = w^2 + f_1(0)$$

for all $w$ such that the left-hand side is defined. So this is again a quadratic polynomial! By the way, the number $c_1 = f_1(0)$ is easy to compute:

$$f_1(0) = j(-3) = -\sqrt{6} = -2^{1/2} \cdot 3^{1/2}.$$

The only non-trivial part is the sign of the square root. It is determined by our choice of the branch for $j$ but we skip the corresponding computation.

Next, we define the function $f_2 = j_1 \circ p_{c_1}$. Note that $j_1 = p_{c_1}^{-1} \circ f_1$ on the domain of $f_1$. It follows that

$$f_2 \text{``}=\text{''} p_{c_1}^{-1} \circ f_1 \circ p_{c_1}.$$

This formula looks nice but one needs to be very careful, because the expression in the right-hand side is ambiguous (it should be considered as a single-valued branch over some domain, but what is written does not carry any information on the domain of definition). The right understanding of this formula is that $p_{c_1}^{-1} \circ f_1$ should be thought of as $j_1$ but then it coincides with the formula $f_2 = j_1 \circ p_{c_1}$ that we have used to define $f_2$. In our formulas, one can recognize Thurston's iteration but in a slightly unusual context because we deal with discontinuous holomorphic functions rather than with continuous non-holomorphic functions. There is a precise relation between what is happening and Thurston's theory (better seen on other examples, because, in the case under consideration, the Teichmüller space has dimension 0, and Thurston's theory does not have much to say).

Continuing the same process, we obtain a sequence $f_n$ of functions, each defined on the complement to a finite union of simple curves, with the following recurrence property:

$$f_{n+1} = j_n^{-1} \circ p_{c_n}, \quad c_n = f_n(0),$$

where $j_n$ is a branch of $\sqrt{f_n - c_n}$ defined on the domain of $f_n$ and tangent to the identity at infinity.

In our example, we can compute the numbers $c_n$ explicitly. Consider the sequence $f_1^{\circ m}(0)$. This sequence stabilizes at the second term:

$$-\sqrt{6},\ \sqrt{6}, \sqrt{6}, \dots, \sqrt{6}, \dots$$

The sequence $f_2^{\circ m}(0)$ can be obtained from the first sequence as follows:

$$f_2^{\circ m}(0) = \sqrt{f_1^{\circ m+1}(0) - f_1(0)}.$$

Thus all terms are equal to $\pm\sqrt{2\sqrt{6}} = \pm 2^{1-1/4} \cdot 3^{1/4}$. The determination of signs is a bit tricky, but the correct signs are the following: the first sign is minus, and all other signs are plus. Continuing in the same way, we obtain that

$$f_n^{\circ m}(0) = \pm 2^{1-1/2^n} \cdot 3^{1/2^n},$$

where the first sign is minus, and all other signs are plus. In particular, $c_n = f_n(0) \to -2$ as $n \to \infty$, and the convergence is exponentially fast.

We see that the sequence of polynomials $p_{c_n}$ converges to the polynomial $g : z \mapsto z^2 - 2$. This is the so called *Tchebychev polynomial*. The Julia set of this polynomial is equal to the interval $[-2, 2]$. Note that the orbit of the critical point 0 is finite: $0 \mapsto -2 \mapsto 2$, and 2 is fixed.

Define the sequence of maps $\Phi_n = j_n \circ \dots j_1 \circ j$. The map $\Phi_n$ is defined and holomorphic on the complement to finitely many simple curves. The main property of $\Phi_n$ is that $f_{n+1} = \Phi_n \circ f \circ \Phi_n^{-1}$ wherever the right-hand side is defined, which follows from the definition. Note also that $p_{c_n} = \Phi_{n-1} \circ f \circ \Phi_n^{-1}$ wherever the right-hand side is defined. In our example, it can be shown that the sequence $\Phi_n$ converges uniformly to a map $\Phi$ defined on the complement to countably many curves — in our case, to all iterated pullbacks of $[-\sqrt{3}, \sqrt{3}]$ under $f$. The map $\Phi$ reglues all these horizontal intervals into "vertical curves" so that the Julia set of $f$, which is a Cantor set, gets glued into the interval $[-2, 2]$ under $\Phi$. Passing to the limit as $n \to \infty$ in the identity $p_{c_n} = \Phi_{n-1} \circ f \circ \Phi_n^{-1}$, we obtain that

$$g = \Phi \circ f \circ \Phi^{-1}.$$

Note that the right-hand side is only defined on the complement to countably many curves, but it extends to the complex plane as a holomorphic function.

### *5.2. Thurston's algorithm for quadratic branched coverings*

In the example above, we constructed an explicit sequence of approximations to a topological regluing. We will now define these approximations in a more general context. First, we recall a version of the Thurston algorithm for branched coverings of degree two, and then extend this algorithm to certain classes of partially defined maps.

**Lemma 7.** *For any pair of different points $a, b \in \overline{\mathbb{C}}$, there exists a quadratic rational function with critical values $a$ and $b$.*

An analog of this statement for arbitrary degree is true and well known but the quadratic case is more explicit.

*Proof.* Set

$$R_{a,b}(z) = \begin{cases} \frac{az^2-b}{z^2-1}, & a, b \neq \infty \\ z^2 + b, & a = \infty,\ b \in \mathbb{C} \\ z^{-2} + a, & a \in \mathbb{C},\ b = \infty \end{cases}$$

Then the critical points of $R_{a,b}$ are $\infty$ and 0, and we have

$$R_{a,b}(\infty) = a, \quad R_{a,b}(0) = b.$$

□

Suppose we start with a branched covering $f_1 : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$. Let $R_1$ be a rational function, whose critical values coincide with those of $f_1$. Then there is a global branch $j_1$ of the multi-valued function $R_1^{-1} \circ f_1$. Note that $R_1$ is only defined up to pre-composition with a Möbius transformation. Hence $j_1$ is only defined up to post-composition with a Möbius transformation. Consider the map $f_2 = j_1 \circ R_1$. This map is an extension of $j_1 \circ f \circ j_1^{-1}$, hence it is only defined up to a Möbius conjugacy. Continue the same process with function $f_2$: consider a quadratic rational function $R_2$ with the same critical values, define $j_2$ as a branch of $R_2^{-1} \circ f_2$, etc. We obtain the following diagram:

$$\begin{array}{ccccccccc}
\overline{\mathbb{C}} & \xrightarrow{j_1} & \overline{\mathbb{C}} & \xrightarrow{j_2} & \overline{\mathbb{C}} & \xrightarrow{j_3} & \overline{\mathbb{C}} & \longrightarrow & \dots \\
{\scriptstyle f_1}\big\downarrow & & {\scriptstyle f_2}\big\downarrow & & {\scriptstyle f_3}\big\downarrow & & {\scriptstyle f_4}\big\downarrow & & \\
\overline{\mathbb{C}} & \xrightarrow[j_1]{} & \overline{\mathbb{C}} & \xrightarrow[j_2]{} & \overline{\mathbb{C}} & \xrightarrow[j_3]{} & \overline{\mathbb{C}} & \longrightarrow & \dots
\end{array}$$

The maps $j_n$ are only defined up to post-composition with Möbius transformations. Therefore, we need to make some normalization in order to fix these maps. Below, a natural normalization is suggested for quadratic rational functions. We can ask then the following questions:

1. does the sequence of rational functions $R_n$ converge?
2. does the sequence of maps $\Psi_n = j_n \circ \dots \circ j_1$ converge?

Convergence is understood as uniform convergence with respect to the spherical metric.

It is not hard to show that if $\Psi_n$ converge to a homeomorphism $\Psi$, then the rational functions

$$R_n = \Psi_{n-1} \circ f_1 \circ \Psi_n^{-1}$$

converge to a rational function $R_\infty$. Moreover, $\Psi$ conjugates $f_1$ with $R_\infty$. If $\Psi_n$ converge to a function $\Psi$ (not necessarily a homeomorphism), and the sequence $R_n$ is equicontinuous, then it also converges. Indeed, for every $\varepsilon > 0$, the uniform distance between $\Psi_n$ and $\Psi_m$ is less than $\varepsilon$ for sufficiently large $n$ and $m$. Since all $\Psi_n$ are surjective, it follows that the uniform distance between $j_m \circ \dots \circ j_{n+1}$ and the identity is less than $\varepsilon$. Finally, we can conclude that $R_n$ and

$$R_m = (j_{m-1} \circ \dots \circ j_n) \circ R_n \circ (j_m \circ \dots \circ j_{n+1})^{-1}$$

are uniformly close. Thus $R_n$ converge to some rational function $R_\infty$, and $\Psi$ semi-conjugates $f_1$ with $R_\infty$.

Suppose now that $f_1$ is critically finite. Then Thurston's theory provides a technique to answer the first question. The answer to the second question depends on a particular normalization for $j_n$. However, once the first question is answered in the affirmative, it is in general much easier to answer the second question.

*5.3. A modification for partially defined maps*

We need to consider the case, where $f_1$ is only defined on a part of the sphere (e.g. on the complement to finitely many simple curves). Thurston's algorithm (as described above) can be modified in the following way. Assume that $f_1$ is defined on some open subset $U_1 \subset S^2$, and that $f_1 : U_1 \to f_1(U_1)$ is an orientation-preserving branched covering of degree two with two critical points. Moreover, assume that forward $f_1$-orbits of both critical points are well-defined (i.e. lie in $U_1$). Then we can proceed as before: choose a quadratic rational function $R_1$, whose critical values coincide with those of $f_1$. It is not hard to show that the multi-valued function $R_1^{-1} \circ f_1$ has trivial monodromy, hence splits into two single valued branches.

Let $j_1$ be one of the branches. It is defined on the set $U_1$. Form a new function $f_2 = j_1 \circ R_1$. It is defined on the set $R_1^{-1}(U_1)$. We need to prove that the critical orbits of $f_2$ are well-defined. Indeed, let $c_1$ and $c_2$ be the critical points of $f_1$. We know that the points $f_1^{\circ n}(c_i)$ lie in $U_1$ for all $n \geq 0$ and $i = 1, 2$. Therefore, the points $j_1 \circ f_1^{\circ n}(c_i)$ are well-defined. Note also that $j_1(c_i)$ are the critical points of $f_2$, and that

$$j_1 \circ f_1^{\circ n}(c_i) = f_2^{\circ n} \circ j_1(c_i).$$

(We use repeatedly that $f_2 = j_1 \circ R_1$ and $f_1 = R_1 \circ j_1$.) In particular, the right-hand side is defined. We can now repeat the step of Thurston' algorithm. In this way, we obtain a sequence of maps $f_n$, $j_n$ and rational functions $R_n$, as above. The only difference is that $f_n$ and $j_n$ are not everywhere defined.

We can now be more specific regarding the normalization of $j_n$. Suppose that the critical points of $f_1$ are $\infty$ and 0 (this can be always arranged by a suitable Möbius conjugacy). Let $a_1 = f_1(\infty)$ and $b_1 = f_1(0)$ be the corresponding critical values. We choose $R_1$ in the form $R_{a_1,b_1}$, where the rational functions $R_{a,b}$ were introduced in Lemma 7. This fixes $R_1$, however, we still need to decide between the two branches of $R_1^{-1} \circ f_1$. We will do this in two examples. These examples are, however, the most important.

*Example 2.* Suppose that $a_1 = f_1(\infty) = \infty$, and that $f_1(z) = z^2 + o(z^2)$ near infinity. Then both branches of $R_1^{-1} \circ f_1$ fix $\infty$. However, one branch is tangent to the identity at infinity. We choose $j_1$ to be this branch. Note by the way that $j_1(z) = R_1^{-1} \circ f_1(z)$ is given by the following explicit formula

$$\sqrt{f_1(z) - b_1}.$$

We have $f_2(z) = z^2 + o(z^2)$ near $\infty$ so that we can choose the branch for $j_2$ in the same way, etc.

*Example 3.* Suppose that $f_1(1) = \infty$ and that $\infty$ is not a critical value of $f_1$. Then one branch of $R_1^{-1} \circ f_1$ maps 1 to 1, and the other branch to $-1$. Indeed, $R_1(1) = R_1(-1) = \infty = f_1(1)$. We choose $j_1$ to be the branch that fixes 1. Note that $j_1(z) = R_1^{-1} \circ f_1(z)$ is given by the following explicit formula

$$\sqrt{\frac{f_1(z) - b_1}{f_1(z) - a_1}}.$$

By definition, $j_1$ fixes both 0 and $\infty$. Therefore, $f_2 = j_1 \circ f_1$ also maps 1 to $\infty$, and $\infty$ is not a critical value of $f_2$. Thus we can choose the branch for $j_2$ in the same way, etc.

### *5.4. The first regluing*

Let us start with a quadratic rational function $f$. Consider a simple path $\alpha_0 : [-1, 1] \to \overline{\mathbb{C}}$ such that $f \circ \alpha_0(-t) = f \circ \alpha_0(t)$ for all $t \in [-1, 1]$. In particular, $\alpha_0(0)$ must be a critical point of $f$. The path $\alpha_0$ can be interpreted as the $\alpha$-path

$$(t_1, t_2) \mapsto \alpha_0(t_1).$$

We will sometimes use this interpretation when referring to regluing, e.g. regluing of $\alpha_0$ means regluing of this $\alpha$-path. To fix ideas, we assume that the critical points of $f$ are 0 and $\infty$ and that $\alpha_0(0) = 0$. If all pullbacks of $\alpha_0$ are disjoint, then we can consider a topological regluing of these pullbacks, as defined earlier in the text. However, we will now make a much weaker assumption, namely, that the orbits of $f \circ \alpha_0(1)$ and $\infty$ are disjoint from $\alpha_0[-1, 1]$.

Let $R$ be a rational function, whose critical values are $f \circ \alpha_0(1)$ and $f(\infty)$. (E.g. we can choose $R$ to have the form $R_{a,b}$ as in Lemma 7). Consider the multi-valued analytic function $R^{-1} \circ f$. There are two single valued branches of this function over the complement to $\alpha_0[-1, 1]$. We choose one branch, and call it $j$. Now $f_1 = j \circ R$ satisfies the assumptions of Section 5.3, and we can run Thurston's algorithm for $f_1$. In the following examples, we comment on a specific choice of the branch for $j$.

*Example 4.* Suppose that $f$ is a quadratic polynomial. In a suitable coordinate, it can be written in the form $f(z) = z^2 + b$. The rational function $R$ should have critical values $\alpha_0(1)^2 + b$ and $\infty$, thus we can take $R(z) = z^2 + \alpha_0(1)^2 + b$. The multi-valued function $R^{-1} \circ f$ is given by the following explicit formula:

$$z \mapsto \sqrt{z^2 - \alpha_0(1)^2}.$$

This function has two branches over the complement to $\alpha_0[-1,1]$. One of the branches is tangent to the identity near infinity. We choose $j$ to be this branch. For the corresponding function $f_1 = j \circ R$, we have $f_1 = z^2 + o(z^2)$ near infinity. Therefore, the normalization of $j_n$ can be made as in Example 2.

*Example 5.* Consider a quadratic rational function $f$ that is conjugate neither to a polynomial nor to the map $z \mapsto 1/z^2$. Then we can reduce $f$ to the form

$$z \mapsto \frac{az^2 - b}{z^2 - 1}$$

by a suitable Möbius conjugacy. Note that $a = f(\infty)$ and $b = f(0)$. The rational function $R$ should have critical values $f \circ \alpha_0(1)$ and $f(\infty) = a$. Therefore, we can take

$$R(z) = \frac{az^2 - f \circ \alpha_0(1)}{z^2 - 1}.$$

The multi-valued function $R^{-1} \circ f$ is given by the following explicit formula:

$$z \mapsto \sqrt{\frac{z^2 - \alpha_0(1)^2}{1 - \alpha_0(1)^2}}.$$

Note that one branch of this function takes 1 to 1, and the other branch to $-1$. We define $j$ as the branch that fixes 1. The function $f_1 = j \circ R$ takes 1 to $\infty$, and $\infty$ is not a critical value of $f_1$. Therefore, we can use normalization for $j_n$ introduced in Example 3.

### *5.5. Some questions*

Starting with a quadratic rational function $f$ and a simple path $\alpha_0 : [-1,1] \to \overline{\mathbb{C}}$, we have defined sequences of maps $f_n$, $j_n$ and rational functions $R_n$ (provided that $\alpha_0$ satisfies certain properties given in Section 5.4). For every $n$, the functions $f_n$ and $j_n$ are defined on a complement to finitely many simple curves (possibly intersecting). As before, we can define the maps $\Psi_n = j_n \circ \ldots \circ j_1$. The map $\Psi_n$ is defined on a set $U_n$, which is also a complement to finitely many simple curves. Since every $U_n$ is open and dense, the intersection $\bigcap U_n$ is dense. We can ask whether the maps $\Psi_n$ converge uniformly on $\bigcap U_n$.

In the case where $f_1$ is critically finite, this question is closely related to Thurston's theory. This theory gives an answer to the question on convergence of $R_n$.

Suppose that $R_n$ and $\Psi_n$ converge uniformly. Then the limit $\Psi$ of $\Psi_n$ is a partially defined semi-conjugacy between $f_1$ and the rational function $R_\infty = \lim R_n$ (meaning that $\Psi$ semi-conjugates the restriction of $f_1$ to a certain forward invariant dense subset with the restriction of $R_\infty$ to a certain forward invariant dense subset). Moreover, $\Phi = \Psi \circ j$ is a partially defined semi-conjugacy between $f$ and $R_\infty$.

What can be said about the limit map $\Psi$? To what extent is this map holomorphic? Note that it is holomorphic in the interior of $\bigcap U_n$ (which may well be empty) but the notion of being holomorphic is not even defined on the boundary. Is it possible to define an appropriate notion generalizing holomorphy to the boundary points so that $\Psi$ would be holomorphic everywhere on $\bigcap U_n$? (An attempt to define this notion is made in preprint [Tp1]).

## 6. Existence of topological regluing

In this section, we prove Theorems 3 and 4. Our methods are based on a theory of Moore [Mo16] that gives a purely topological characterization of topological spheres.

### *6.1. A variant of Moore's theory*

Moore [Mo16] defined a system of topological conditions that are necessary and sufficient for a topological space to be homeomorphic to the sphere. He used this system to lay axiomatic foundations of plane topology. One of the most remarkable applications of Moore's theory is a description of equivalence relations on the sphere such that the quotient space is homeomorphic to the sphere. Moore's theory has been further developed in [B, vK, Z], see also [K].

In this section, we will give a "rapid introduction" into a version of Moore's theory. Our axiomatics is very far from being optimal (many axioms can be proved as theorems), and it does not have the purpose of setting a foundation to plane topology, but just serves as a fast working tool to prove that something is a topological sphere. A better system of axioms is given in [Tp2].

Let $X$ be a compact connected Hausdorff space. Recall that a *simple closed curve* in $X$ is the image in $X$ of a continuous embedding $\gamma : S^1 \to X$. Here the map $\gamma$ is called a *simple closed path*. We also define a *simple path* as a continuous embedding of $[0, 1]$ into $X$, and a *simple curve* as the image of a simple path. A *segment* of a simple curve is defined as the image of a closed subinterval in $[0, 1]$ under the corresponding simple path. Similarly, we can define a segment (or an arc) of a simple closed curve.

Suppose we fixed some set $\mathcal{E}$ of simple curves in $X$. The curves in $\mathcal{E}$ are called *elementary curves*. We will always assume that segments of elementary curves are also elementary curves, and that if two elementary curves have only an endpoint in common, then their union is also an elementary curve. We will not state these assumptions as axioms, although, technically, they are. Define an *elementary closed curve* as a simple closed curve, all of whose segments are elementary curves.

We are ready to state the first axiom:

**Axiom 1 (Elementary domain axiom).** Any elementary closed curve $J$ divides $X$ into two connected components called *elementary* domains bounded by $J$.

Since a simple closed curve is homeomorphic to a circle, it makes sense to talk about the (circular) order of points in it. Define an *elementary quadrilateral* in $X$ as an elementary domain bounded by an elementary closed curve $J \subset X$ with a distinguished quadruple of different points $a$, $b$, $c$ and $d$ in $J$. We assume that the points $a$, $b$, $c$ and $d$ appear in $J$ in the same circular order as they are listed. Consider an elementary quadrilateral $Q$ bounded by $J$. A simple curve connecting the segment $[a, b]$ with the segment $[c, d]$ of $J$ is called a *vertical curve*, provided that it lies entirely in $Q$, except for the endpoints. A simple curve connecting the segments $[b, c]$ and $[d, a]$ is called *horizontal*, provided that it lies entirely in $Q$, except for the endpoints. We will also regard $[a, b]$ and $[c, d]$ as horizontal curves, $[b, c]$ and $[d, a]$ as vertical curves.

**Axiom 2 (Extension axiom).** For any elementary quadrilateral $Q$ bounded by an elementary closed curve $J$, and any point $x$ in $J$, there exists a vertical or a horizontal elementary curve with an endpoint at $x$. Moreover, this curve divides $Q$ into at most two elementary quadrilaterals.

From the Extension axiom, it follows easily that the topological boundary of any elementary domain bounded by an elementary closed curve $J$ coincides with $J$.

Define a *grid* in an elementary quadrilateral $Q$ as a system of finitely many horizontal elementary curves and finitely many vertical elementary curves such that all horizontal curves are pair-wise disjoint, all vertical curves are pair-wise disjoint, and every horizontal curve intersects every vertical curve at exactly one point. Using the Elementary domain axiom and the Extension axiom, it is not hard to show that every grid with $n-1$ horizontal and $m-1$ vertical curves divides $Q$ into $mn$ pieces. We will refer to these pieces as *cells of the grid*. Cells can be regarded as elementary quadrilaterals.

**Axiom 3 (Covering axiom).** Consider an elementary quadrilateral $Q$ and an open covering $\mathcal{U}$ of $\overline{Q}$. Then there exists a grid in $Q$ such that the closure of every cell lies in some element of $\mathcal{U}$ (such grid is said to be *subordinate* to $\mathcal{U}$).

The following is the main topological fact we need.

**Theorem 14.** *Let $X$ be a compact connected Hausdorff second countable topological space. Suppose that a set of elementary curves $\mathcal{E}$ in $X$ satisfies the Elementary domain axiom, the Extension axiom and the Covering axiom. Suppose also that there exists an elementary closed curve in $X$. Then $X$ is homeomorphic to the sphere.*

*Proof.* Since there exists an elementary closed curve, by the Elementary domain axiom, there exists an elementary quadrilateral $Q$. It suffices to prove that the closure of this quadrilateral is homeomorphic to the closed disk.

Fix a countable basis $\mathcal{B}$ of the topology in $X$. There are countably many finite open coverings of $\overline{Q}$ contained in $\mathcal{B}$. Number all such coverings by natural numbers. We will define a sequence of grids $G_n$ in $Q$ by induction on $n$. For $n = 1$, we just take the trivial grid, the one that does not have any horizontal or vertical curves. Suppose now that $G_n$ is defined. Let $\mathcal{U}_n$ be the $n$-th covering of $\overline{Q}$. Using the Covering axiom, we can find a grid in each cell of $G_n$ that is subordinate to $\mathcal{U}_n$. Using the Extension axiom, we can extend these grids to a single grid $G_{n+1}$ in $Q$. Thus $G_{n+1}$ contains $G_n$ and is subordinate to $\mathcal{U}_n$.

Consider any pair of different points $x, y \in \overline{Q}$. There exists $n$ such that $x$ and $y$ do not belong to the closure of the same cell in $G_{n+1}$. Indeed, let us first define a covering of $\overline{Q}$ as follows. For any $z \in \overline{Q}$, choose $U_z$ to be an element of $\mathcal{B}$ that contains $z$ but does not include the set $\{x, y\}$. The sets $U_z$ form an open covering of $\overline{Q}$. Since $\overline{Q}$ is compact, there is a finite subcovering. This finite subcovering coincides with $\mathcal{U}_n$ for some $n$. Then, by our construction, the closure of every cell in $G_{n+1}$ is contained in an element of $\mathcal{U}_n$. However, the set $\{x, y\}$ is not contained in an element of $\mathcal{U}_n$. Therefore, $\{x, y\}$ cannot belong to the closure of a single cell.

Consider a nested sequence $C_1 \supset C_2 \supset \ldots \supset C_n \supset \ldots$, where $C_n$ is a cell of $G_n$. We claim that the intersection of the closures $\overline{C}_n$ is a single point. Indeed, this intersection is nonempty, since $\overline{C}_n$ form a nested sequence of nonempty compact sets. On the other hand, as we saw, there is no pair $\{x, y\}$ of different points contained in all $\overline{C}_n$.

Consider the standard square $[0, 1] \times [0, 1]$ and a sequence $H_n$ of grids in it with the following properties:

1. all horizontal curves in $H_n$ are horizontal straight line segments, all vertical curves in $H_n$ are vertical straight line segments;
2. the grid $H_n$ has the same number of horizontal curves and the same number of vertical curves as $G_n$, thus there is a natural one-to-one correspondence between cells of $H_n$ and cells of $G_n$ respecting "combinatorics", i.e. the cells in $H_n$ corresponding to adjacent cells in $G_n$ are also adjacent;

3. the grid $H_{n+1}$ contains $H_n$; moreover, if a cell of $H_{n+1}$ is in a cell of $H_n$, then there is a similar inclusion between the corresponding cells of $G_{n+1}$ and $G_n$;
4. between any horizontal curve of $H_n$ and the next horizontal curve, the horizontal curves of $H_{n+1}$ are equally spaced; similarly, between any vertical curve of $H_n$ and the next vertical curve, the vertical curves of $H_{n+1}$ are equally spaced.

It is not hard to see that any nested sequence of cells $D_n$ of $H_n$ converges to a point: $\bigcap \overline{D}_n = \{pt\}$.

We can now define a map $\Phi : \overline{Q} \to [0,1] \times [0,1]$ as follows. For a point $x \in \overline{Q}$, there is a nested sequence of cells $C_n$ of $G_n$ such that $\overline{C}_n$ contains $x$ for all $n$. Define the point $\Phi(x)$ as the intersection of the closures of the corresponding cells $D_n$ in $H_n$. (Note that they also form a nested sequence according to our assumptions). Clearly, the point $\Phi(x)$ does not depend on a particular choice of the nested sequence $C_n$ (there can be at most four different choices). It is also easy to see that $\Phi$ is a homeomorphism between $\overline{Q}$ and the standard square $[0,1] \times [0,1]$. □

One of the main applications of Moore's theory is the following characterization of equivalence relations on $S^2$ such that the corresponding quotient spaces are homeomorphic to the sphere:

**Theorem 15.** *Let $\sim$ be a closed equivalence relation on $S^2$ such that all equivalence classes are connected and do not separate the sphere. Then the quotient $S^2/\sim$ is homeomorphic to the sphere provided that not all points are equivalent.*

Recall that a closed equivalence relation on $S^2$ is an equivalence relation represented by a closed subset of $S^2 \times S^2$.

### *6.2. Blow-up space*

Consider a countable set $\mathcal{Z}$ of disjoint compact connected locally connected nonseparating sets in $S^2$. Suppose that $\mathcal{Z}$ forms a null-sequence. For every $A \in \mathcal{Z}$, fix a continuous map $\Pi_A : S^2 \to S^2$ such that $\Pi_A$ restricts to an orientation-preserving homeomorphism between $\Delta_\infty$ and $S^2 - A$, and $\Pi_A(\overline{\Delta}) = A$ (we use notation from Section 2.1).

We can now define the *blow-up space* $X_{\mathcal{Z}}$ of $\mathcal{Z}$ as the equalizer of all maps $\Pi_A$, i.e. the subset of the product $(S^2)^{\mathcal{Z}}$ consisting of all functions $\chi : \mathcal{Z} \to S^2$ such that the points $\Pi_A(\chi(A)) \in S^2$ coincide for all $A \in \mathcal{Z}$. Intuitively, $X_{\mathcal{Z}}$ is obtained from $S^2$ by blowing up every $A \in \mathcal{Z}$ according to the map $\Pi_A$. As a closed subset in the compact Hausdorff second countable space $(S^2)^{\mathcal{Z}}$, the space $X_{\mathcal{Z}}$ is also compact, Hausdorff and second countable. As a continuous map from a compact space to a Hausdorff space, the map $\Pi_{\mathcal{Z}}$ is closed. For $A \in \mathcal{Z}$, define the subset $\Delta_A \subset X_{\mathcal{Z}}$ as the set of points $\chi \in X_{\mathcal{Z}}$ such that $\chi(A) \in \Delta$. Clearly, $\Delta_A$ is homeomorphic to the open disk $\Delta$, and $\overline{\Delta}_A$ is homeomorphic to the closed disk $\overline{\Delta}$. The topological disks $\overline{\Delta}_A$, $A \in \mathcal{Z}$, are disjoint and form a null-sequence. Intuitively, the sets $A \in \mathcal{Z}$ are blown up to disks $\overline{\Delta}_A$.

The main result of this section is the following

**Theorem 16.** *Under our assumptions on $\mathcal{Z}$, the blow up space $X_{\mathcal{Z}}$ is homeomorphic to the sphere.*

To prove Theorem 16, we will use the technique introduced in Section 6.1. First note that there is a natural projection $\Pi_{\mathcal{Z}} : X_{\mathcal{Z}} \to S^2$ that takes $\chi \in X_{\mathcal{Z}}$ to the point $\Pi_A(\chi(A))$, $A \in \mathcal{Z}$. By definition, this point is independent of $A$. Consider a finite subset $\mathcal{Z}' \subset \mathcal{Z}$. The blow-up space $X_{\mathcal{Z}'}$ is homeomorphic to the sphere (for a finite set $\mathcal{Z}'$, this is both intuitively obvious and technically simple). Let $\sim_{\mathcal{Z}'}$ be the equivalence relation on $X_{\mathcal{Z}'}$, whose non-trivial classes are pullbacks of sets in $\mathcal{Z} - \mathcal{Z}'$ under $\Pi_{\mathcal{Z}'}$. Let $Y_{\mathcal{Z}'}$ be the quotient space by this equivalence relation. By Theorem 15 of Moore, the space $Y_{\mathcal{Z}'}$ is homeomorphic to the sphere. Here, we use our assumption that $\mathcal{Z}$ form a null-sequence (otherwise, the equivalence relation would not be closed). Define the countable subset set $Z_{\mathcal{Z}'} \subset Y_{\mathcal{Z}'}$ as the set of points in $Y_{\mathcal{Z}'}$, whose fibers under the quotient map $X_{\mathcal{Z}'} \to Y_{\mathcal{Z}'}$ are non-trivial.

There is a canonical projection $\Pi_{\mathcal{Z},\mathcal{Z}'} : X_{\mathcal{Z}} \to X_{\mathcal{Z}'}$ mapping an element $\chi : \mathcal{Z} \to S^2$ to its restriction to $\mathcal{Z}'$. Denote by $\phi_{\mathcal{Z}'}$ the composition of this projection and the quotient map $X_{\mathcal{Z}'} \to Y_{\mathcal{Z}'}$.

We can now define elementary curves in $X_{\mathcal{Z}}$. Take a simple curve $C$ in $Y_{\mathcal{Z}'}$ avoiding the set $Z_{\mathcal{Z}'}$. The pullback of $C$ under $\phi_{\mathcal{Z}'}$ is called an elementary curve in $X_{\mathcal{Z}}$. A standard Baire category argument shows that the set of paths in $S^2$ avoiding a given countable subset of $S^2$ is dense in the space of all paths with uniform topology. It follows that there are many elementary curves in $X_{\mathcal{Z}}$. Clearly, the set of elementary curves in stable under taking segments and concatenations.

**Proposition 13.** *The space $X_{\mathcal{Z}}$ satisfies the Elementary domain axiom and the Extension axiom.*

*Proof.* Let $\mathcal{Z}'$ be a finite subset of $\mathcal{Z}$, and $C$ a simple closed curve in $Y_{\mathcal{Z}'}$ avoiding the set $Z_{\mathcal{Z}'}$. By the Jordan curve theorem, $C$ divides $Y_{\mathcal{Z}'}$ into two connected components, say, $U$ and $U'$. It follows that the pullbacks of $U$ and $U'$ under $\phi_{\mathcal{Z}'}$ are components of the complement to the elementary curve $\phi_{\mathcal{Z}'}^{-1}(C)$ in $X_{\mathcal{Z}}$. The pullbacks of $U$ and $U'$ are connected because all fibers of $\phi_{\mathcal{Z}'}$ are connected. The Extension axiom for $X_{\mathcal{Z}}$ follows from the corresponding property of $Y_{\mathcal{Z}'}$ (where the elementary curves in $Y_{\mathcal{Z}'}$ are understood as simple curves avoiding the set $Z_{\mathcal{Z}'}$). □

**Proposition 14.** *Elementary domains form a basis of topology in* $X_{\mathcal{Z}}$.

*Proof.* We need to prove that, for every open set $V$ in $X_{\mathcal{Z}}$ and every point $\chi_0 \in V$, there exists an elementary domain $D$ that contains $\chi_0$ and is contained in $V$. Moreover, we can assume that $V$ has the following form: there are open sets $U_1, \dots, U_n$ in $S^2$ and elements $A_1, \dots, A_n \in \mathcal{Z}$ such that $V$ consists of all $\chi \in X_{\mathcal{Z}}$ with the property $\chi(A_i) \in U_i$ for $i = 1, \dots, n$. Recall that such open sets $V$ form a basis in the product topology.

Set $\mathcal{Z}' = \{A_1, \dots, A_n\}$. Clearly, there is an open set $V'$ in $X_{\mathcal{Z}'}$ such that $\chi \in V$ if and only if $\Pi_{\mathcal{Z},\mathcal{Z}'}(\chi) \in V'$. (The set $V'$ consists of all $\chi' \in X_{\mathcal{Z}'}$ such that $\chi'(A_i) \in U_i$ for $i = 1, \dots, n$). Set $\chi'_0 = \Pi_{\mathcal{Z},\mathcal{Z}'}(\chi_0)$. Clearly, there is a Jordan domain neighborhood of $\chi'_0$ contained in $V'$, whose boundary maps to a simple closed curve in $Y_{\mathcal{Z}'}$ avoiding the set $Z_{\mathcal{Z}'}$. The corresponding elementary domain is a neighborhood of $\chi_0$ contained in $V$. □

**Proposition 15.** *The space* $X_{\mathcal{Z}}$ *satisfies the Covering axiom.*

*Proof.* Let $Q$ be an elementary quadrilateral in $X_{\mathcal{Z}}$, and $\mathcal{U}$ an open covering of $\overline{Q}$. By Proposition 14, it suffices to consider the case, where all elements of $\mathcal{U}$ are elementary domains. By compactness, we can also assume that $\mathcal{U}$ is finite. For every $U \subset \mathcal{U}$, the curve $\Pi_{\mathcal{Z}}(\partial U)$ intersects only finitely many elements of $\mathcal{Z}$. Let $\mathcal{Z}'$ be a finite subset of $\mathcal{Z}$ such that $\Pi_{\mathcal{Z}}(\partial U)$ are disjoint from $\mathcal{Z} - \mathcal{Z}'$ for all $U \in \mathcal{U}$. The Covering axiom for $X_{\mathcal{Z}}$ now follows from the corresponding statement in $Y_{\mathcal{Z}'}$. □

Theorem 16 now follows from Theorem 14.

### *6.3. Existence of regluings*

In this section, we prove Theorem 3. Let $\mathcal{A}$ be a countable set of disjoint $\alpha$-curves in $S^2$ such that $\alpha(S^1)$, $\alpha \in \mathcal{A}$, form a null-sequence. Consider the set $\mathcal{Z} = \mathrm{Im}\mathcal{A}$. This set satisfies all assumptions of Section 6.2. For every $\alpha$, define $\Pi_\alpha$ as a continuous extension of $\pi_\alpha : \overline{\Delta}_\infty \to S^2$ to the whole sphere $S^2$ such that $\overline{\Delta}$ is mapped to $\alpha(S^1)$. Let $X_{\mathcal{A}}$ be the blow-up space corresponding to the set $\mathcal{Z}$ and the family of projections $\Pi_\alpha$. By Theorem 16, the space $X_{\mathcal{A}}$ is homeomorphic to the sphere. Clearly, the ungluing space $\Upsilon_{\mathcal{A}}$ embeds into $X_{\mathcal{A}}$ as a closed subset (we will identify $\Upsilon_{\mathcal{A}}$ with its image under this embedding). This proves the first part of Theorem 3. Any component of the complement to $\Upsilon_{\mathcal{A}}$ in $X_{\mathcal{A}}$ is a Jordan domain consisting of all points $\chi \in X_{\mathcal{A}}$ such that $\chi(\alpha) \in \Delta$ for some fixed $\alpha$. Thus components of $X_{\mathcal{A}} - \Upsilon_{\mathcal{A}}$ are in one-to-one correspondence with the set $\mathcal{A}$. Let $D_\alpha$ denote the component of $X_{\mathcal{A}} - \Upsilon_{\mathcal{A}}$ corresponding to an $\alpha$-path $\alpha \in \mathcal{A}$.

Recall (Section 2.1) that there is a natural identification $h_\alpha$ between the boundary of $D_\alpha$ ($=S^1_\alpha$) and the unit circle $S^1$. The composition of $h_\alpha : S^1_\alpha \to S^1$ with the projection of $S^1$ to the $t_2$-axis extends to a continuous map $P_\alpha : \overline{D}_\alpha \to [-1,1]$ in such a way that every fiber of this map is a simple curve with endpoints in $S^1_\alpha$. We can now define an equivalence relation $\sim_{\mathcal{A}}$ on $X_{\mathcal{A}}$ as follows. Two different points of $X_{\mathcal{A}}$ are equivalent, if they lie in the same fiber of some $P_\alpha$, $\alpha \in \mathcal{A}$. It is easy to see that the equivalence relation $\sim_{\mathcal{A}}$ is closed, and its equivalence classes are connected and non-separating. Therefore, by Theorem 14, the quotient space is homeomorphic to the sphere. In other terms, there is a quotient map $\Psi : X_{\mathcal{A}} \to S^2$, whose fibers are exactly equivalence classes with respect to $\sim_{\mathcal{A}}$.

Note that, for every $\alpha \in \mathcal{A}$, the map $\beta = \Psi \circ h_\alpha^{-1}$ is a $\beta$-path. Let $\mathcal{B}$ be the set of all such $\beta$-paths. (We have a fixed one-to-one correspondence between $\mathcal{A}$ and $\mathcal{B}$). It is not hard to verify that the curves $\beta(S^1)$, $\beta \in \mathcal{B}$, form a null-sequence. Since the restriction of $\Psi$ to $\Upsilon^\circ_{\mathcal{A}}$ is one-to-one, and, for every $\alpha \in \mathcal{A}$, the map $\Psi \circ h_\alpha^{-1} : S^1 \to S^2$ equals to the $\beta$-path $\beta$ corresponding to $\alpha$, we obtain that the map

$$S^2 - \mathrm{Im}\mathcal{A} \xrightarrow{\pi_{\mathcal{A}}^{-1}} \Upsilon^\circ_{\mathcal{A}} \xrightarrow{\Psi} S^2 - \mathrm{Im}\mathcal{B}$$

is a regluing of $\mathcal{A}$ into $\mathcal{B}$. This concludes the proof of Theorem 3.

### *6.4. Branched coverings*

Let $X$ be a topological sphere and $f : X \to X$ a continuous map. Recall that $f$ is called a *branched covering* at $x \in X$, if there exist neighborhoods $U$ of $x$ and $V$ of $f(x)$ and homeomorphisms $\phi : U \to \Delta$ and $\psi : V \to \Delta$ such that $\psi \circ f \circ \phi^{-1}$ coincides with $z \mapsto z^k$ on $\Delta$, where $k$ is some positive integer. The number $k$ is called the *local degree* of $f$ at $x$. The map $f$ is a branched covering if it is a branched covering locally at every point. The following is a topological criterion for $f$ to be a branched covering.

**Theorem 17.** *Suppose that $y = f(x)$, and there exist simply connected domains $U \ni x$ and $V \ni y$ such that $f : U - \{x\} \to V - \{y\}$ is a covering of degree $k$. Then $f$ is a branched covering of degree $k$ at $x$.*

*Proof.* Both $U-\{x\}$ and $V-\{y\}$ can be homeomorphically identified with the quotient of the upper half-plane $\mathbb{H} = \{\mathrm{Im} z > 0\}$ by the translation $z \mapsto z+1$. Moreover, one can choose the homeomorphisms with the following property: as a point in $\mathbb{H}/\mathbb{Z}$ tends to infinity (i.e. the imaginary part tends to infinity), the corresponding point in $U-\{x\}$ (respectively, in $V-\{y\}$) tends to $x$ (respectively, to $y$). The map $f$ restricted to $U-\{x\}$ lifts to a homeomorphism $F : \mathbb{H} \to \mathbb{H}$ such that $F(z+1) = F(z)+k$. Consider the maps $\Phi(z) = \exp(2\pi i F(z)/k)$ and $\Psi(z) = \exp(2\pi i z)$. These maps descend to homeomorphisms $\phi : U-\{x\} \to \Delta-\{0\}$ and $\psi : V-\{y\} \to \Delta-\{0\}$, respectively. Moreover, we have $\psi \circ f = \phi^k$. Clearly, $\phi$ and $\psi$ extend continuously to $x$ and $y$, respectively, and $\phi(x) = \psi(y) = 0$ so that the equality $\psi \circ f = \phi^k$ still holds. □

**Corollary 2.** *Let $f : S^2 \to S^2$ be a continuous map and $C \subset S^2$ a finite subset such that $f$ is locally injective on $S^2 - C$. Then $f$ is a branched covering.*

Recall that Theorem 4 states a simple condition that guarantees that a topological regluing of a branched covering is also a branched covering. We are now ready to prove this theorem.

*Proof of Theorem 4.* Consider a branched covering $f : S^2 \to S^2$ and a strongly $f$-stable set $\mathcal{A}$ of disjoint $\alpha$-paths. Let $\Phi : S^2 - \mathrm{Im}\mathcal{A} \to S^2 - \mathrm{Im}\mathcal{B}$ be a regluing of $\mathcal{A}$ into a set $\mathcal{B}$ of $\beta$-paths. The map $\Phi \circ f \circ \Phi^{-1}$ extends to a continuous map $g : S^2 \to S^2$, by Theorem 2. We want to prove that $g$ is a branched covering.

Define $C_g$ as the set consisting of points $\Phi(c)$, where $c \notin \mathrm{Im}\mathcal{A}$ is a critical point of $f$, and of points $\beta(1,0)$, where $\beta \in \mathcal{B}$ is the $\beta$-path corresponding to some $\alpha$-path $\alpha \in \mathcal{A}$, and $\alpha(0,1)$ is a critical point of $f$. Clearly, $C_g$ is finite (its cardinality is the number of critical points of $f$). By Corollary 2, it suffices to prove that for every $y \notin C_g$ there exists a neighborhood of $y$, on which $g$ is injective.

First, assume that $y \notin \mathrm{Im}\mathcal{B}$. Then $y = \Phi(x)$ for some point $x \notin \mathrm{Im}\mathcal{A}$. Consider a small neighborhood $V$ of $f(x)$ that does not contain critical values of $f$. Let $U$ be a Jordan neighborhood of $x$, whose boundary is disjoint from $\mathrm{Im}\mathcal{A}$ and which is contained in $f^{-1}(V)$. It is not hard to see that $g$ is injective on the Jordan neighborhood of $y$ bounded by $\Phi(\partial U)$.

Next, assume that $y \in \beta(S^1)$, where $\beta \in \mathcal{B}$ is the $\beta$-path corresponding to some $\alpha$-path $\alpha \in \mathcal{A}$, and $\alpha(S^1)$ contains no critical points of $f$. Consider a small neighborhood $V$ of $f(\alpha(S^1))$ containing no critical values of $f$. Let $U$ be a Jordan neighborhood of $\alpha(S^1)$, whose boundary is disjoint from $\mathrm{Im}\mathcal{A}$ and which is contained in $f^{-1}(V)$. It is not hard to see that $g$ is injective on the Jordan neighborhood of $y$ bounded by $\Phi(\partial U)$.

Finally, assume that $y = \beta(t_1, t_2)$, where $\beta \in \mathcal{B}$ is the $\beta$-path corresponding to some $\alpha$-path $\alpha \in \mathcal{A}$, and $\alpha(0,1)$ is a critical point of $f$. Then we necessarily have $t_2 \neq 0$. Choose a Jordan domain $V$ disjoint from the critical values of $f$ and such that $\partial V$ intersects $f \circ \alpha(S^1)$ exactly by endpoints (so that the critical value $f \circ \alpha(0,1)$ is in $\partial V$). Let $U_1$ and $U_2$ be the pullbacks of $V$ intersecting $\alpha(S^1)$. These are Jordan domains, whose boundaries intersect exactly by the critical point $\alpha(0,1)$. By a suitable choice of $V$, we can arrange that $\partial U_1$ and $\partial U_2$ do not intersect $\mathrm{Im}\mathcal{A} - \alpha(S^1)$. The union of $\Phi(\partial U_1 - \{\alpha(0,1)\})$, $\Phi(\partial U_2 - \{\alpha(0,1)\})$ and the endpoints of $\beta$ is a pair of simple closed curves intersecting exactly by $\beta(1,0)$. It is easy to see that $g$ is injective on the Jordan neighborhood of $y$ bounded by one of these curves. □

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