Regluing of rational functions

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A standard commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \Phi\downarrow & & \downarrow\Phi \\ Y & \xrightarrow[g]{} & Y \end{array}$$

To do a surgery, one needs to make Φ discontinuous.

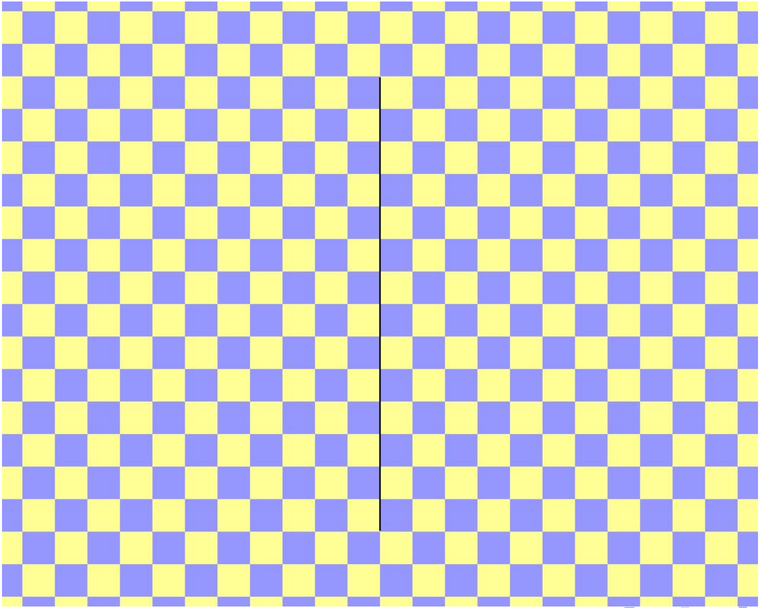
An example of regluing

A branch of the analytic function

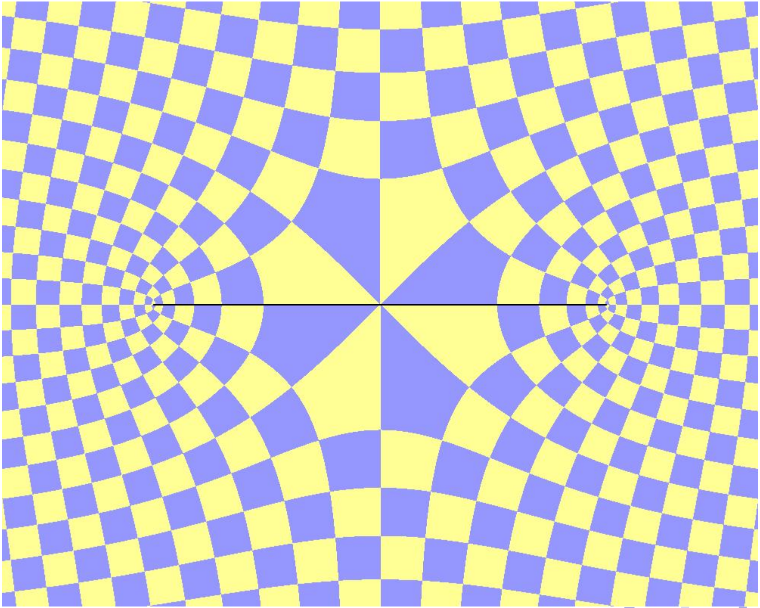
$$j(z) = \sqrt{z^2 + 1}$$

is defined on the complement to $[-i, i]$. It reglues the segment $[-i, i]$ into $[-1, 1]$.

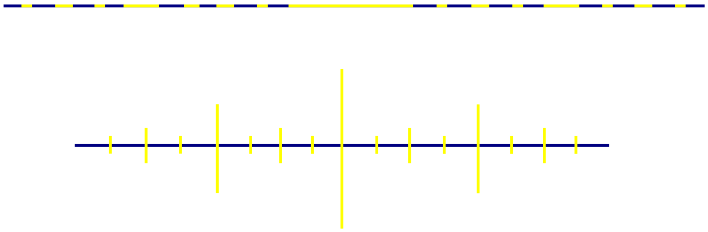
A regluing: before



A regluing: after

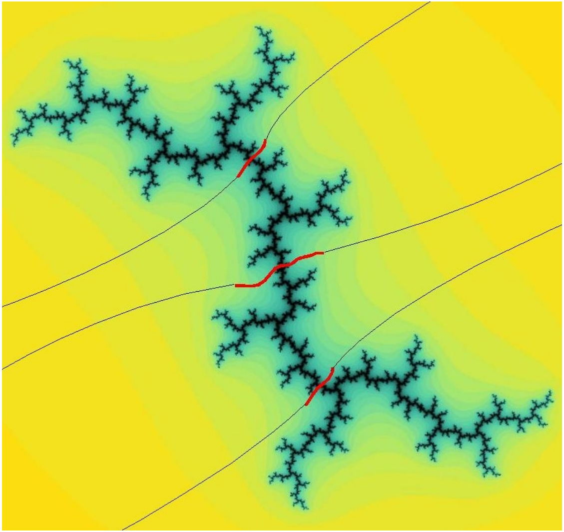


The Julia set of $z \mapsto z^2 - 3$ is a Cantor subset of $\mathbb{R}$. Reglue all complementary segments. We obtain the map $z \mapsto z^2 - 2$, whose Julia set is a segment!

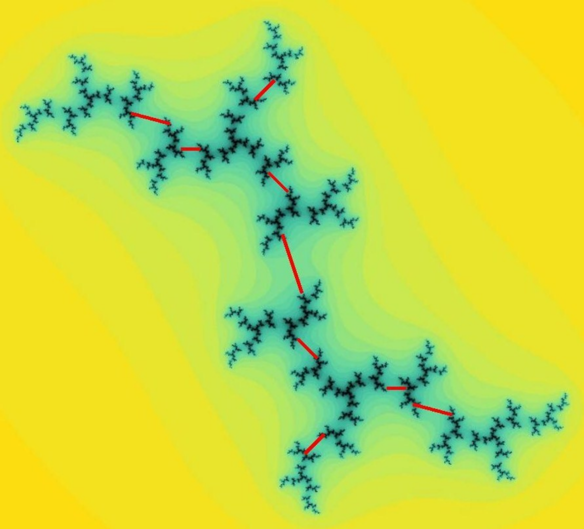


More generally, let f be a quadratic polynomial $z \mapsto z^2 + c$, where c is the landing point of an external parameter ray R. Suppose that the Julia set of f is locally connected, and all periodic points are repelling. Also, consider a quadratic polynomial g, for which the corresponding parameter value belongs to R. Thus the Julia set of g is disconnected. Then $\Phi \circ f = g \circ \Phi$ for a regluing Φ.

Regluing: before



Regluing: after



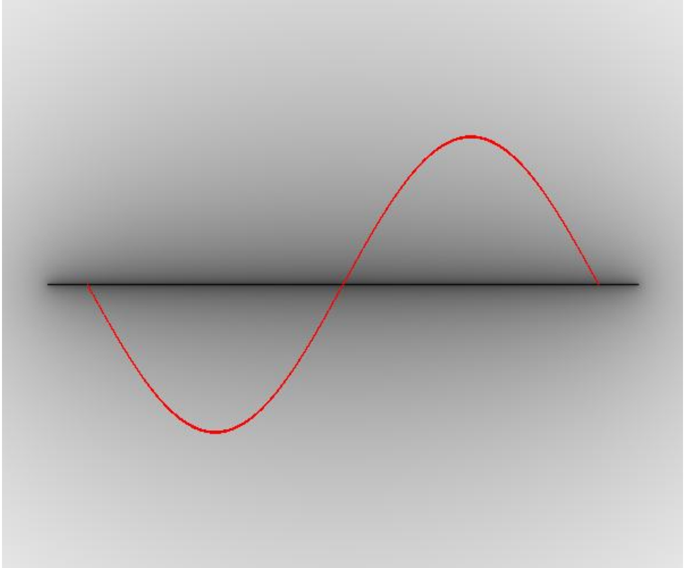
Existence of topological regluing

Let X be a compact metric space, and $\mathcal{A}$ a set of compact subsets of X. We say that $\mathcal{A}$ is *contracted* if for every $\varepsilon > 0$, there are only finitely many elements of $\mathcal{A}$, whose diameter exceeds ε. It is not hard to see that this property is topological, i.e. does not depend on the choice of metric.

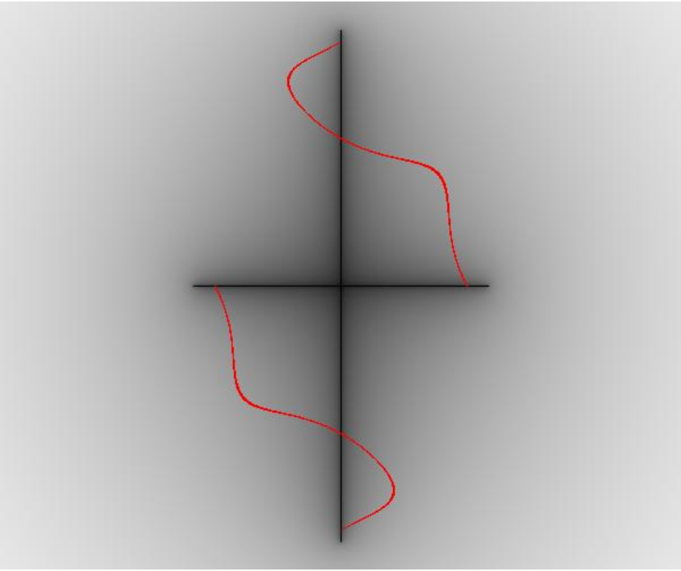
Theorem

Let $\mathcal{A}$ be a contracted set of disjoint simple curves in S^2. There exists a homeomorphism $\Phi : S^2 - \bigcup \mathcal{A} \to S^2 - \bigcup \mathcal{B}$ regluing $\mathcal{A}$ into another set $\mathcal{B}$ of disjoint simple curves.

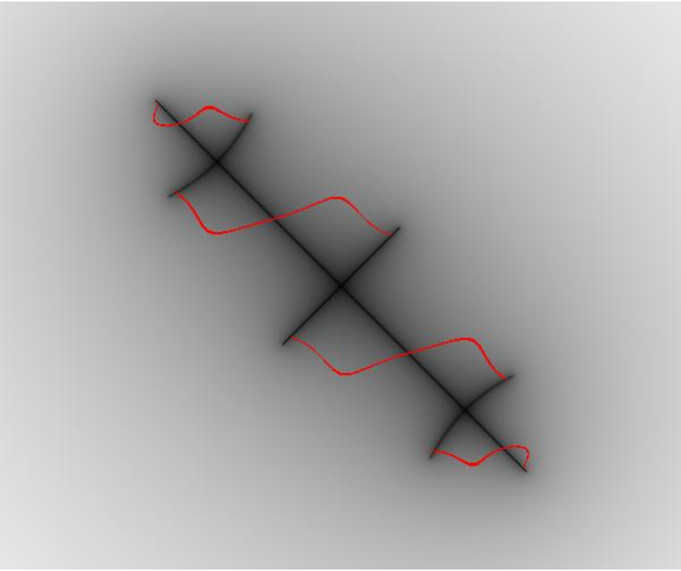
Regluing of $z^2 - 2$ into $z^2 - 2$: step 0



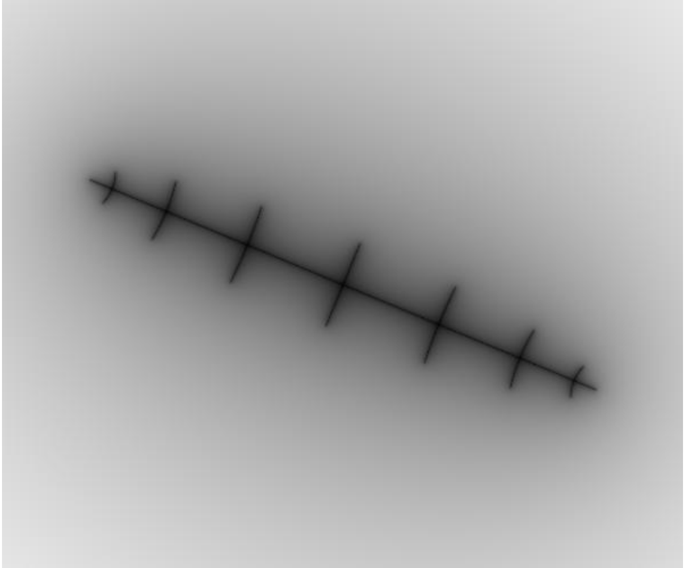
Regluing of $z^2 - 2$ into $z^2 - 2$: step 1



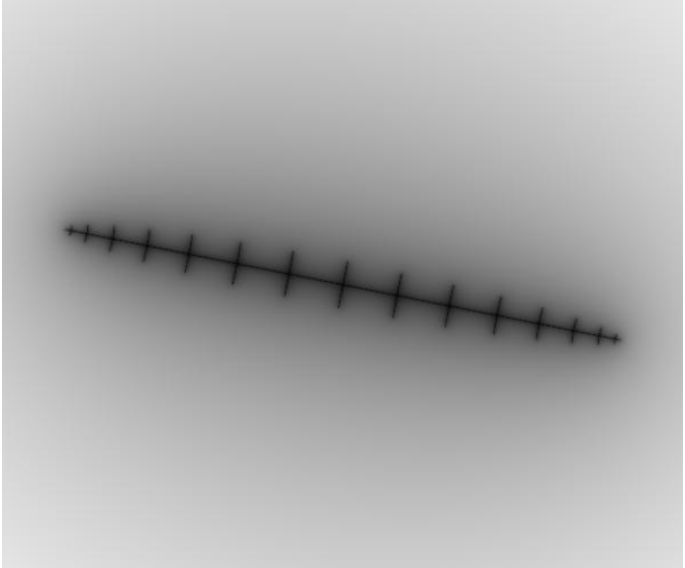
Regluing of $z^2 - 2$ into $z^2 - 2$: step 2



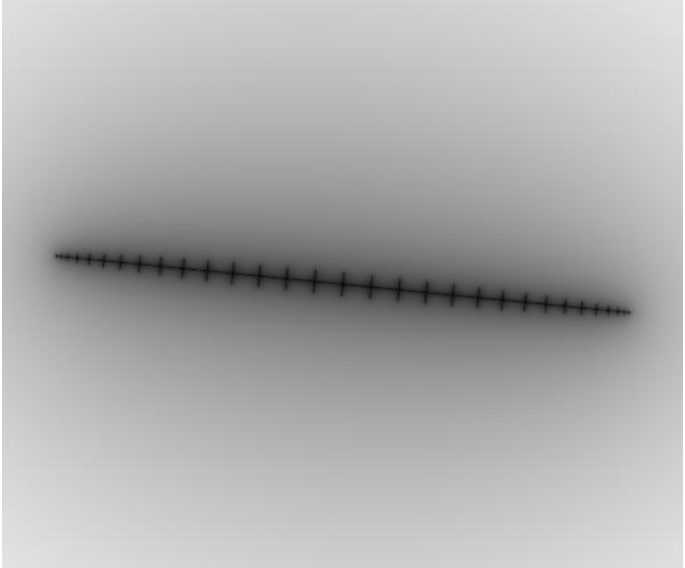
Regluing of $z^2 - 2$ into $z^2 - 2$: step 3



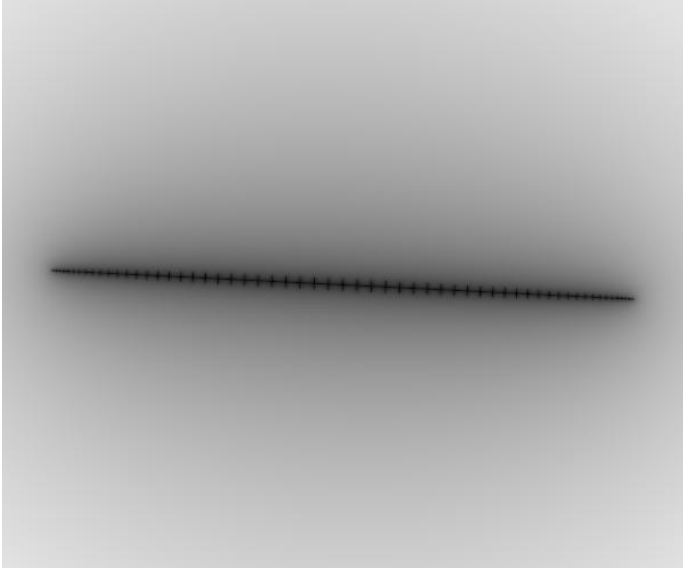
Regluing of $z^2 - 2$ into $z^2 - 2$: step 4



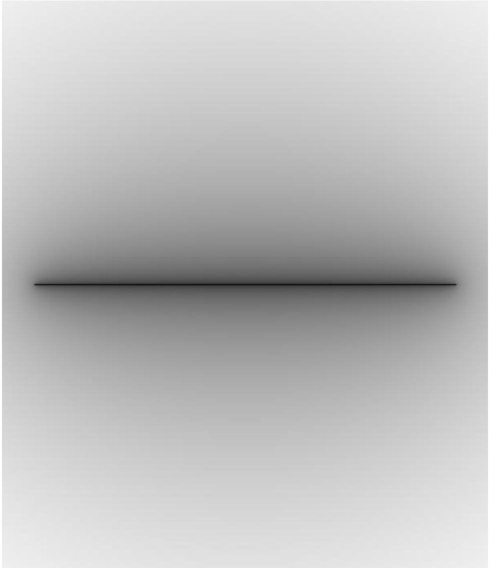
Regluing of $z^2 - 2$ into $z^2 - 2$: step 5



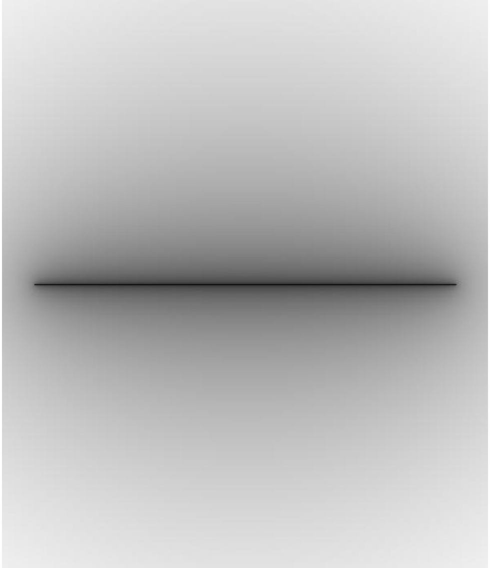
Regluing of $z^2 - 2$ into $z^2 - 2$: step 6



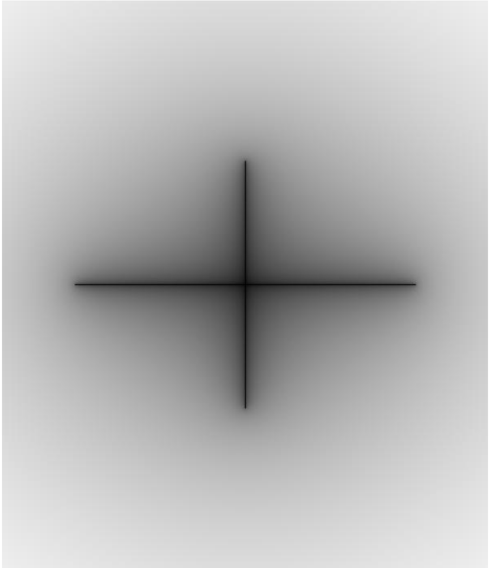
Regluing of $z^2 - 2$ into $z^2 - 2$: the limit



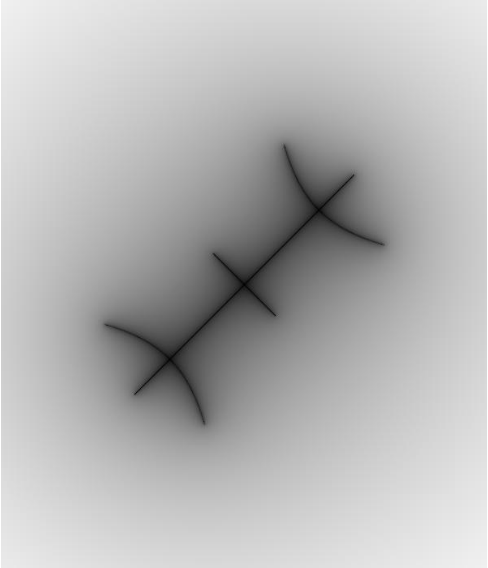
Regluing of $z^2 - 2$ into $z^2 + i$: step 0



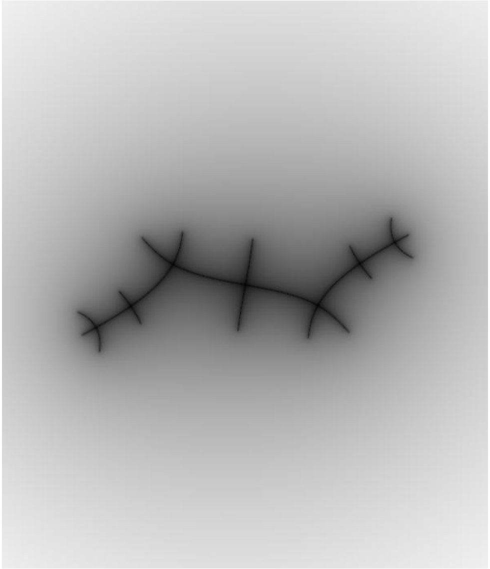
Regluing of $z^2 - 2$ into $z^2 + i$: step 1



Regluing of $z^2 - 2$ into $z^2 + i$: step 2



Regluing of $z^2 - 2$ into $z^2 + i$: step 3



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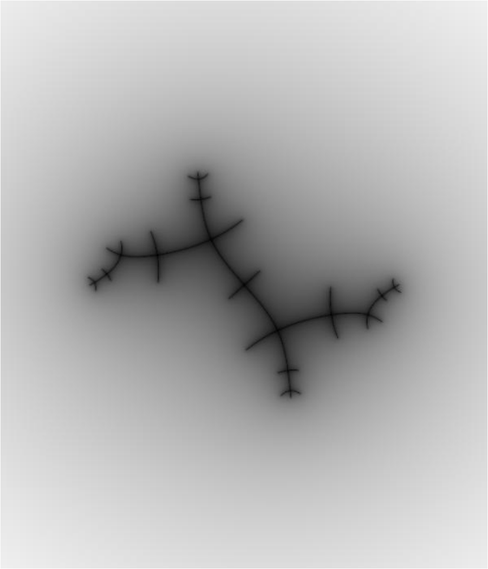

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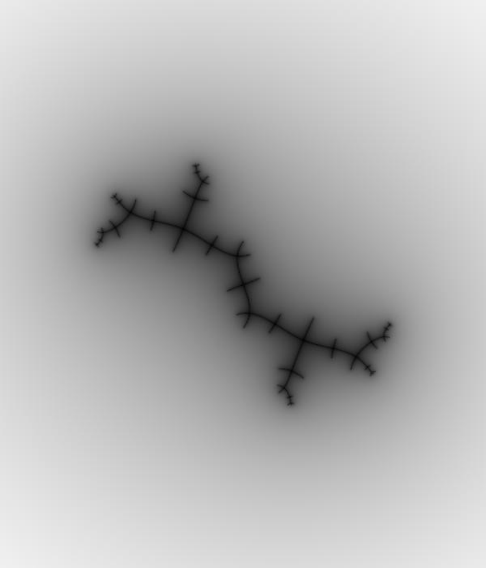

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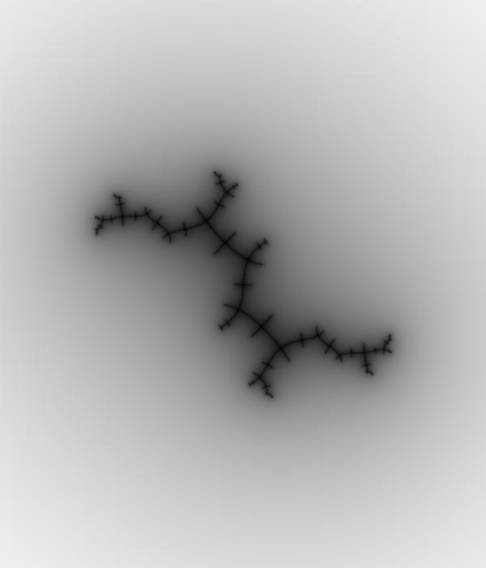
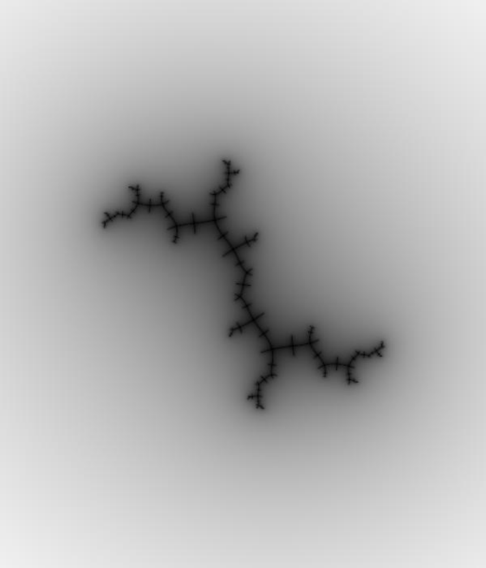
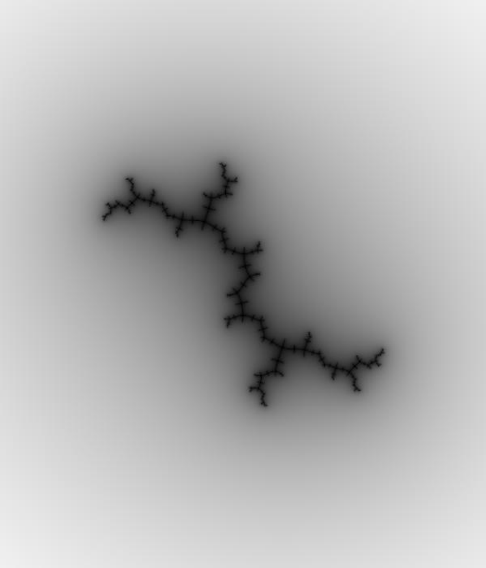


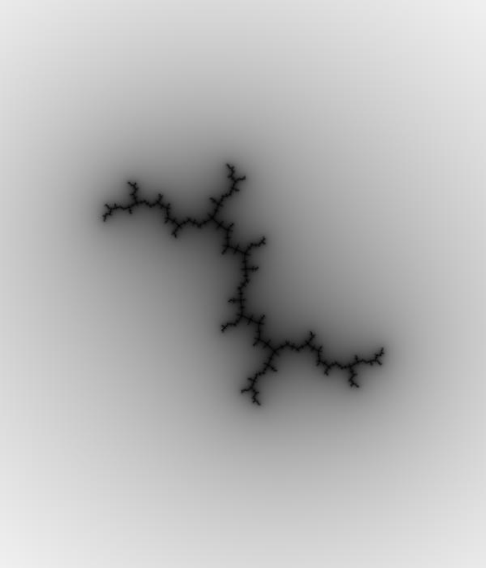
Regluing of $z^2 - 2$ into $z^2 + i$: step 7



Regluing of $z^2 - 2$ into $z^2 + i$: step 8

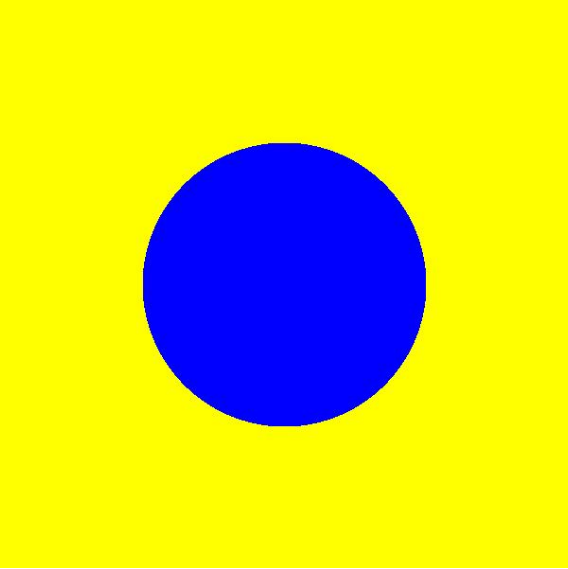


Regluing of $z^2 - 2$ into $z^2 + i$: step 9

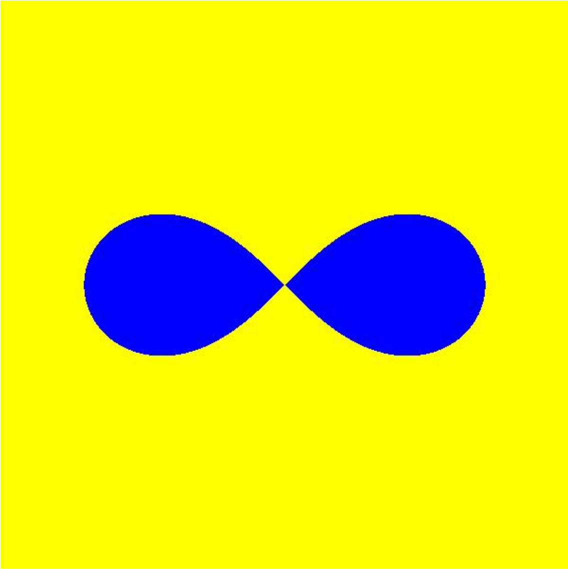


Regluing of $z^2 - 2$ into $z^2 + i$: the limit

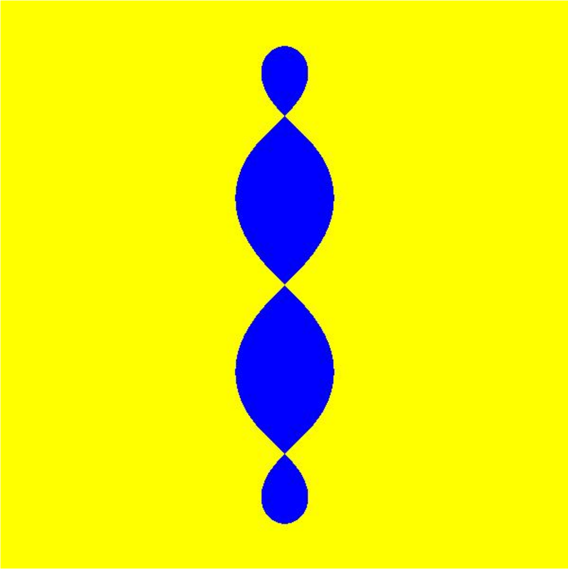
Regluing of z^2 into $z^2 - 2$: step 0



Regluing of z^2 into $z^2 - 2$: step 1



Regluing of z^2 into $z^2 - 2$: step 2



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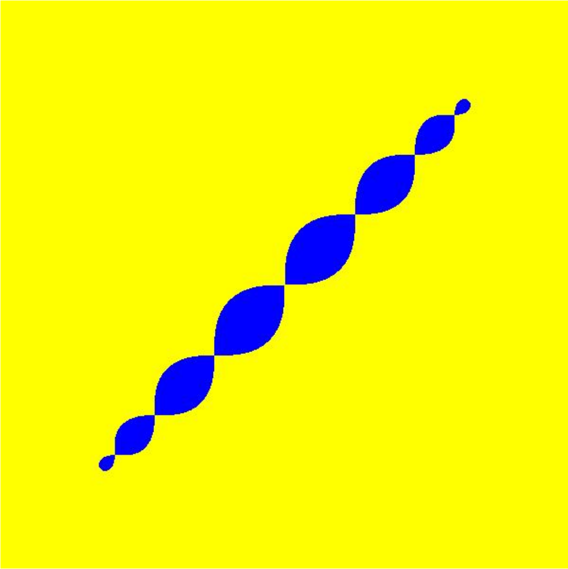

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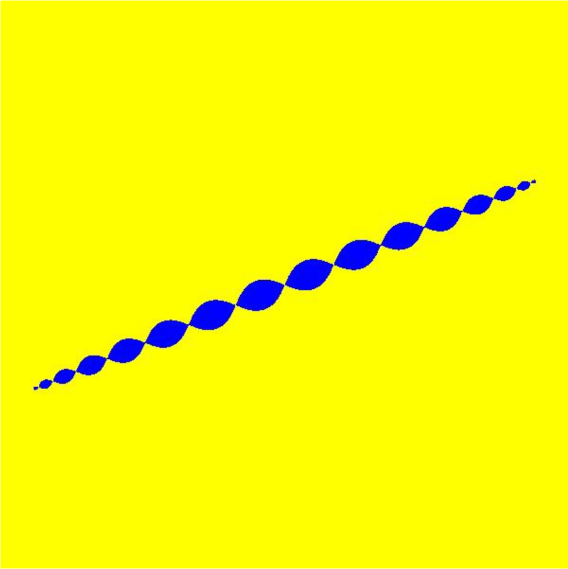

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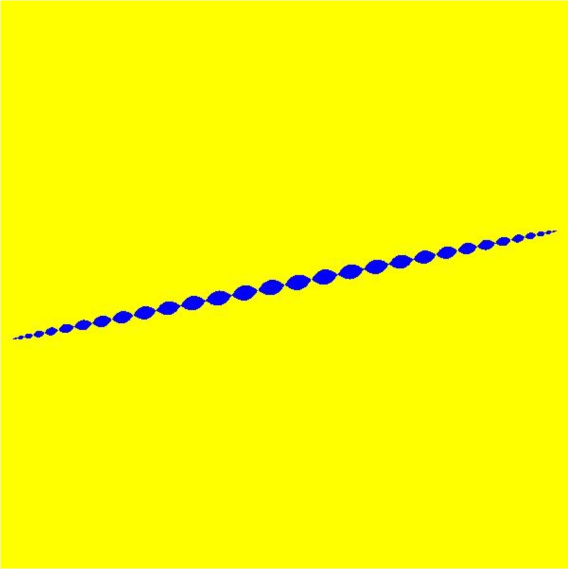

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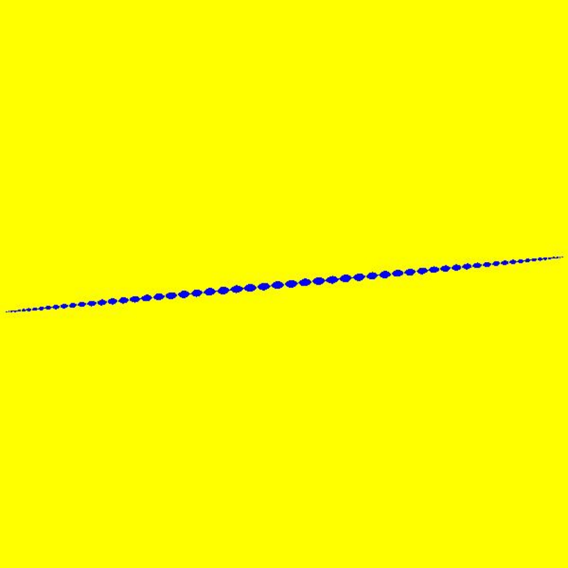

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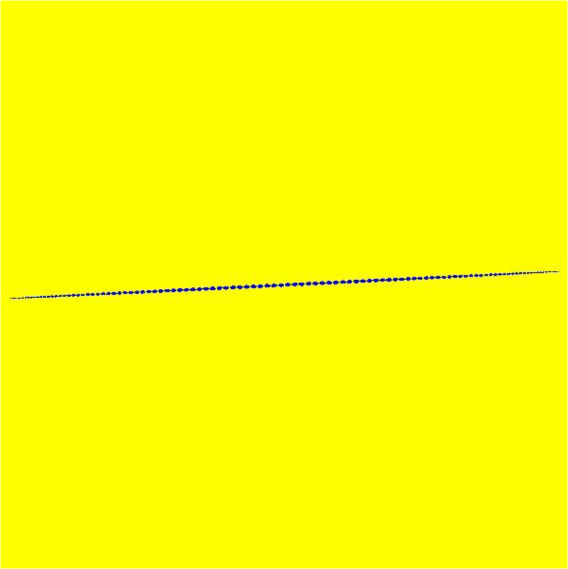
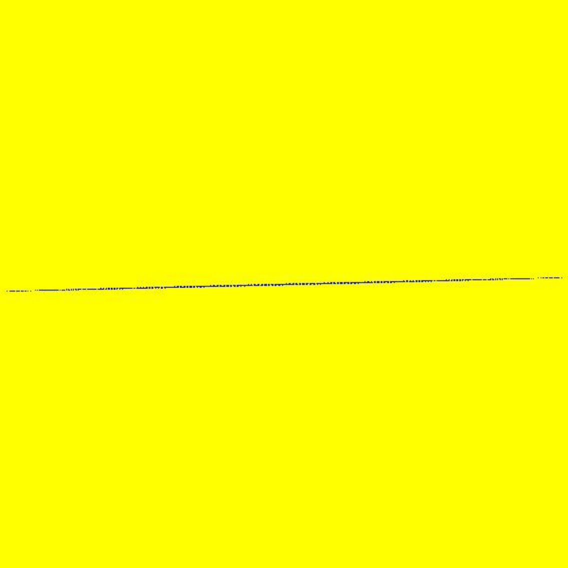
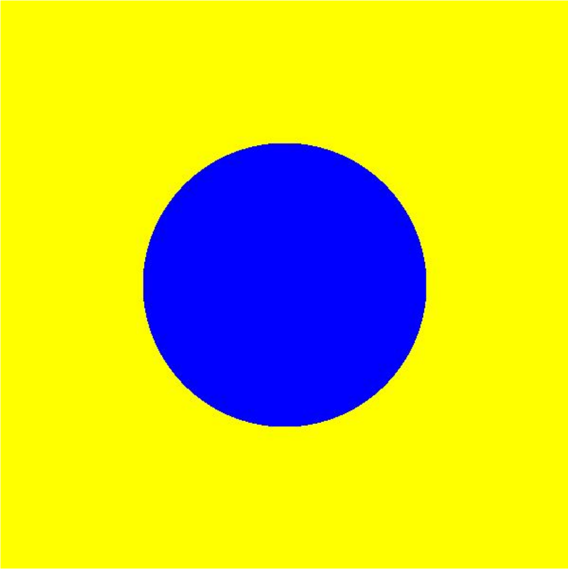


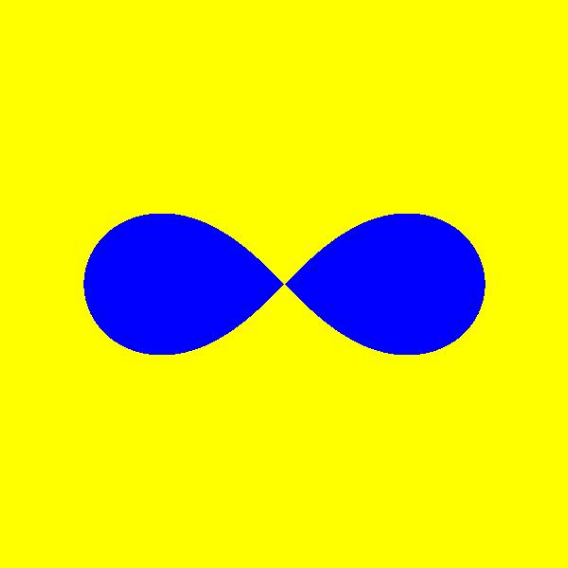
Regluing of z^2 into $z^2 - 2$: step 8



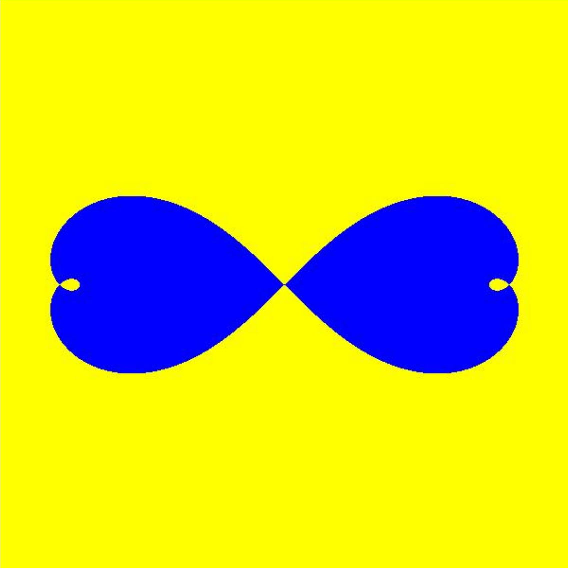
Regluing of z^2 into $(z^2+2)/(z^2-1)$: step 0



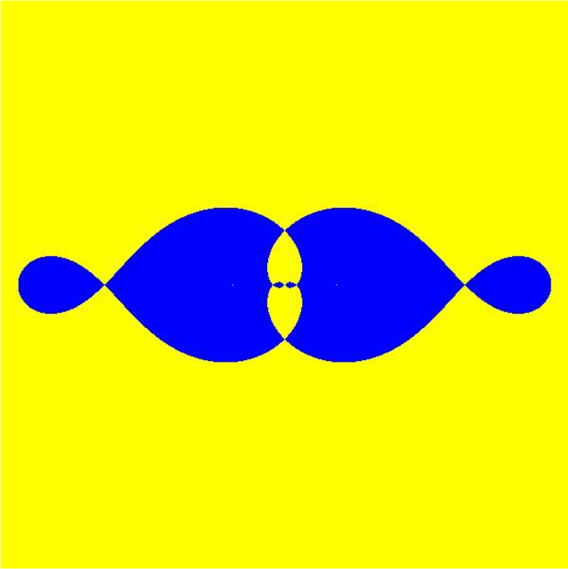
Regluing of z^2 into $(z^2+2)/(z^2-1)$: step 1



Regluing of z^2 into $(z^2+2)/(z^2-1)$: step 2



Regluing of z^2 into $(z^2+2)/(z^2-1)$: step 3



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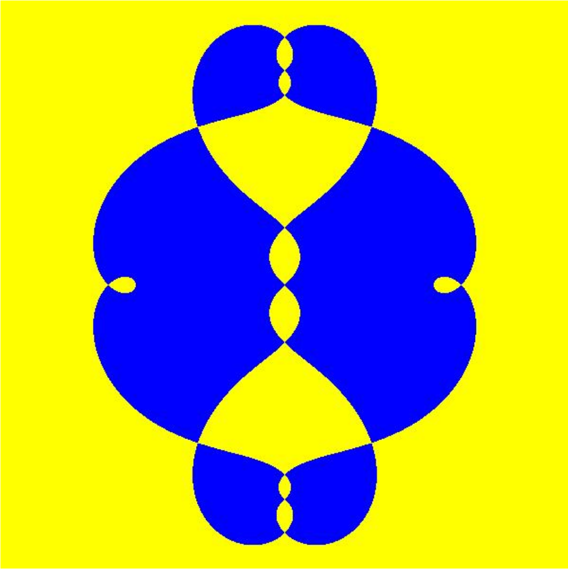

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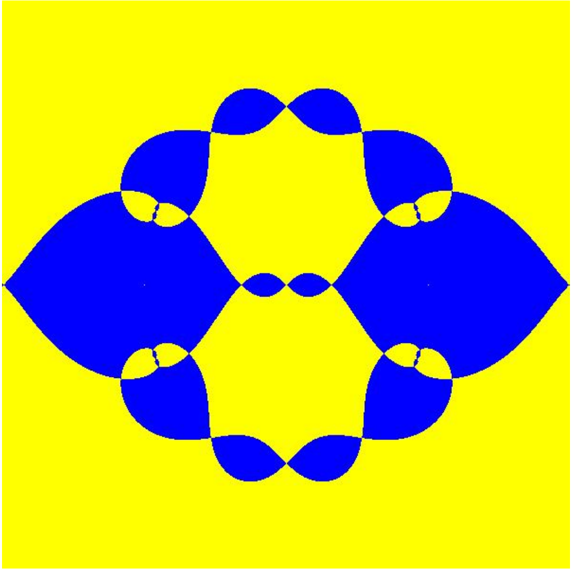

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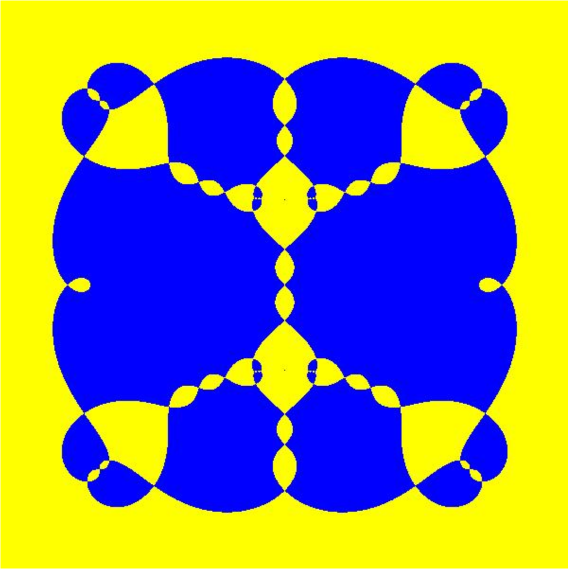

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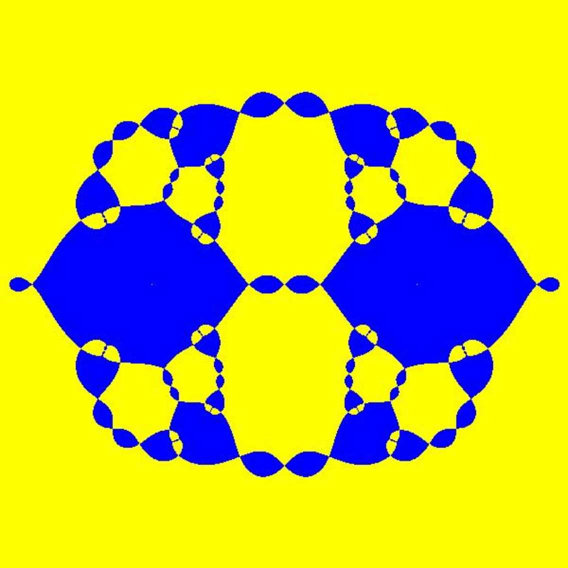
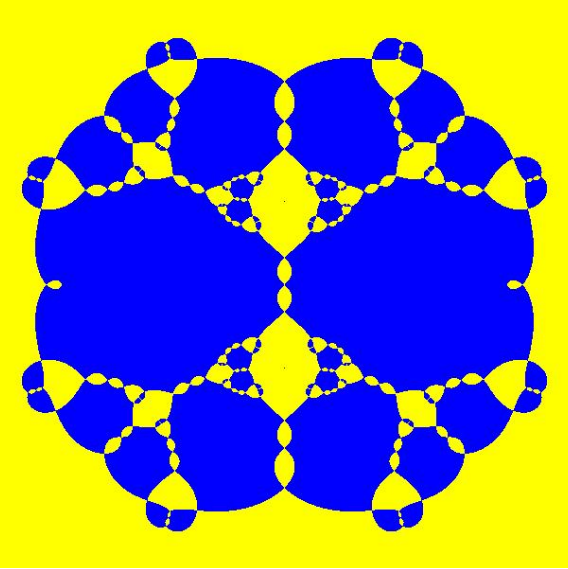
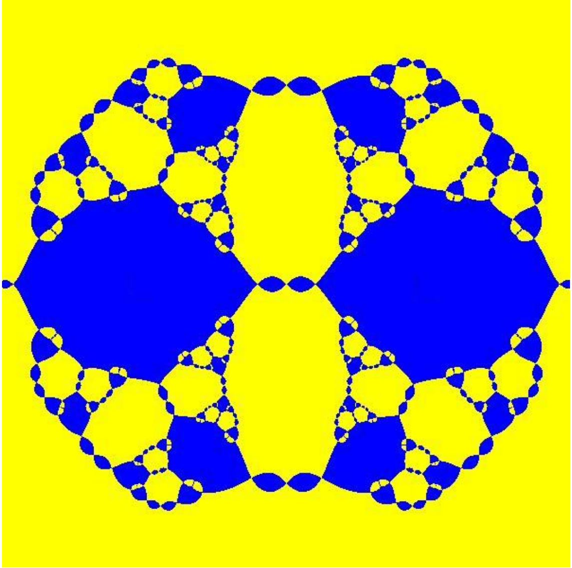


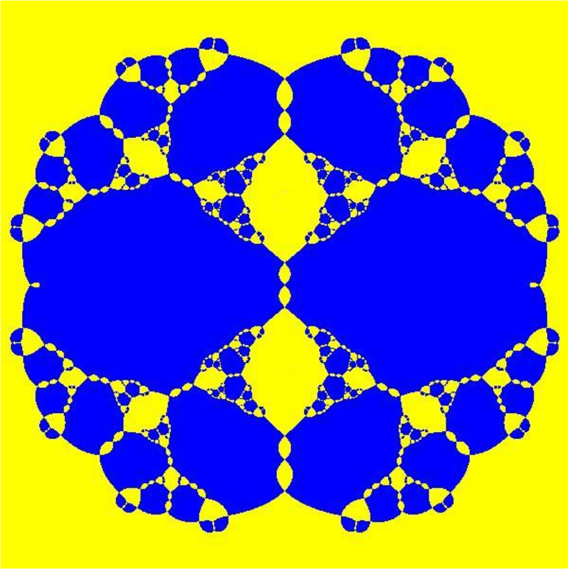
Regluing of z^2 into $(z^2 + 2)/(z^2 - 1)$: step 8



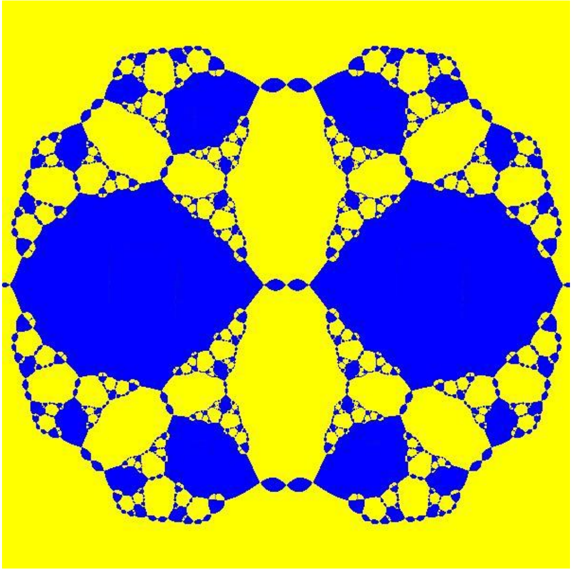
Regluing of z^2 into $(z^2+2)/(z^2-1)$: step 9



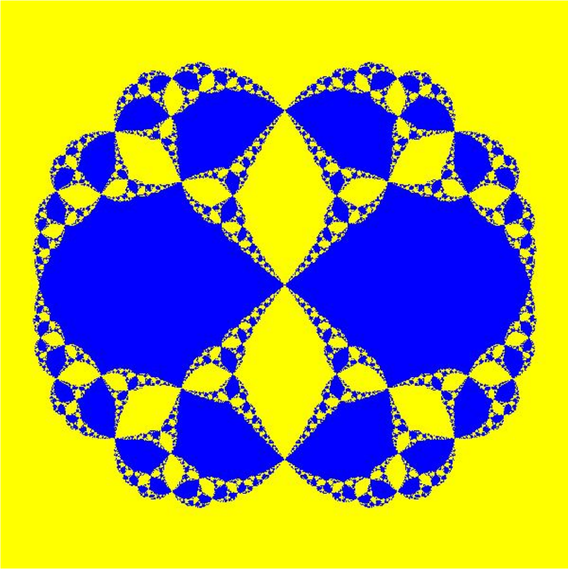
Regluing of z^2 into $(z^2+2)/(z^2-1)$: step 10



Regluing of z^2 into $(z^2+2)/(z^2-1)$: step 11



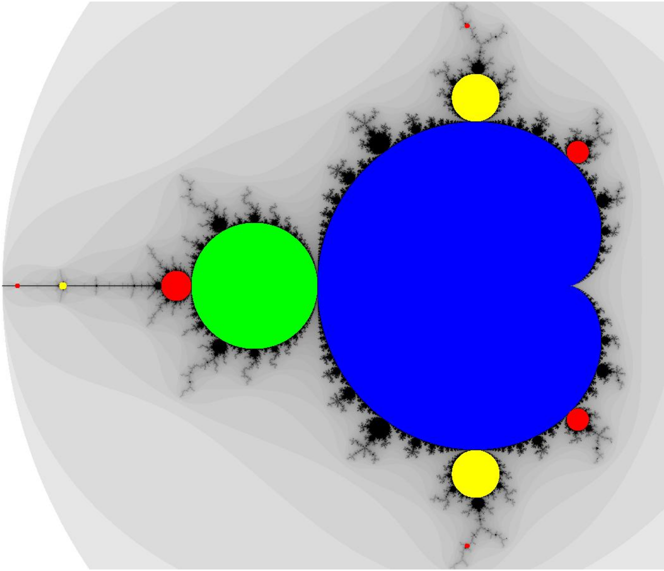
Regluing of z^2 into $(z^2+2)/(z^2-1)$: the limit



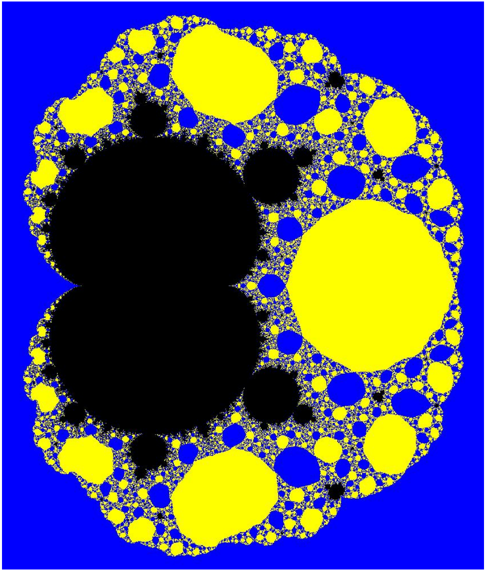
Parameter slices

$Per_k(0) = \{$classes of quadratic rational functions f with marked critical points c_1, c_2 such that $f^{\circ k}(c_1) = c_1\}$.

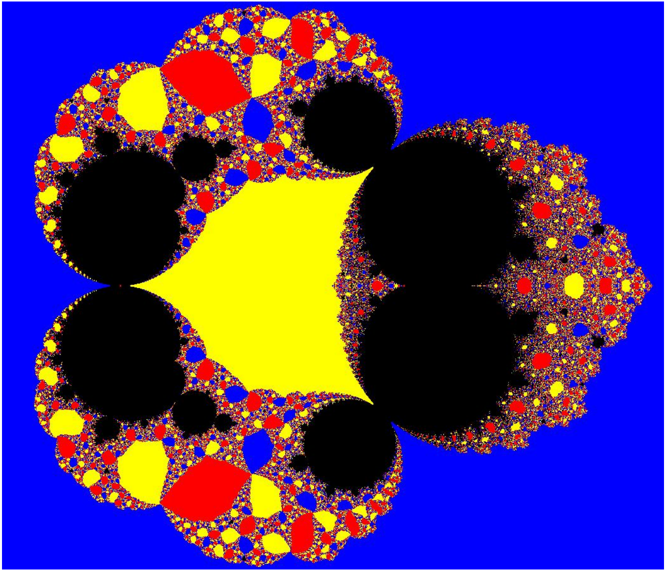
$Per_1(0)$



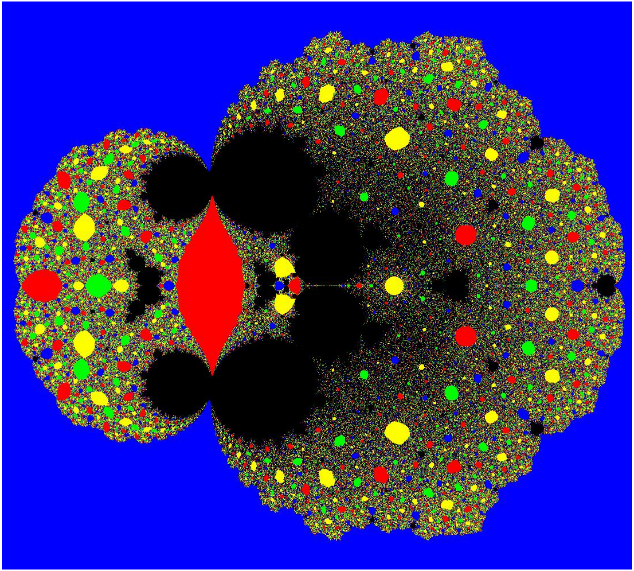
$Per_2(0)$



$Per_3(0)$



$Per_4(0)$



- Suppose that $k > 1$, and f is a quadratic rational function with a k-periodic critical point c_1 and a free critical point c_2.
- f is *hyperbolic rational function of type B* if c_2 lies in the immediate basin of c_1 (but necessarily not in the same component).
- f is *a hyperbolic rational function of type C* if c_2 lies in the full basin of c_1, but not in the immediate basin.
- The set of hyperbolic rational functions with a k-periodic critical point splits into *hyperbolic components*.
- We say that a hyperbolic component is of type B or C if it consists of hyperbolic rational functions of this type.

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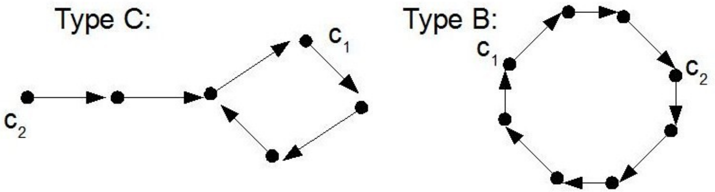
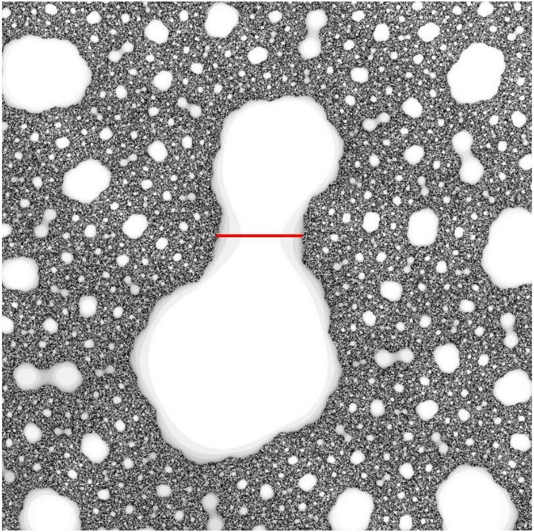
Types of hyperbolic components



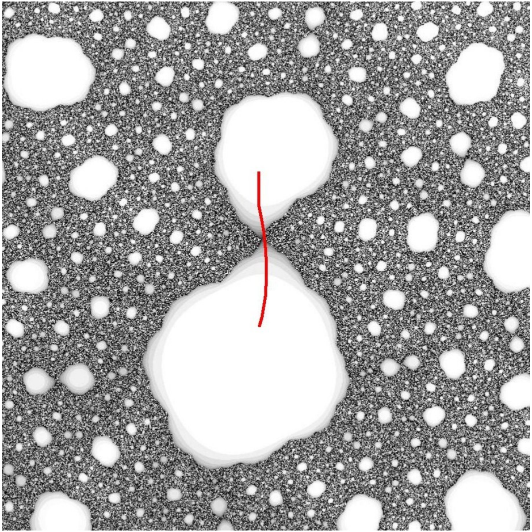
Theorem

If f is on the boundary of a type C hyperbolic component, but not on the boundary of a type B hyperbolic component, then $\Phi \circ f = h \circ \Phi$, where h is the center of a type C hyperbolic component, whose boundary contains f, and Φ is a regluing.

Regluing: before



Regluing: after



Generalized holomorphy

Let Z be a countable union of disjoint simple curves. Assume that Z has zero Lebesgue measure. We say that a map $\Phi : \mathbb{C} - Z \to \mathbb{C}$ is *holomorphic modulo* Z if there is a function $\Psi : Z \to \mathbb{C}$ such that

$$\int_{\mathbb{C}-Z} \Phi \, \overline{\partial} \omega = \int_Z \Psi \, \omega$$

for every smooth (1,0)-form ω on $\mathbb{C}$ with compact support.

Theorem

Consider a quadratic polynomial $f : z \mapsto z^2 + c$ *with connected Julia set such that the critical value* c *is accessible from the basin of infinity. There exists a countable union* Z *of disjoint simple curves of zero area, and a quadratic polynomial* g *with totally disconnected Julia set such that* $\Phi \circ f = g \circ \Phi$ *on* $\mathbb{C} - Z$*, where* $\Phi : \mathbb{C} - Z \to \mathbb{C}$ *is a holomorphic map modulo* Z*.*