Topological models for quadratic rational maps

with a critical 2-cycle and the other critical point on the boundary of its immediate basin

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A family of quadratic rational maps

- Consider the space V_2 of all quadratic rational maps with a super-attracting orbit of period 2.
- the quadratic family $z \mapsto z^2 + c$ is V_1.
- Holomorphic conjugacy classes of maps from V_2 are parameterized by 1 complex number a:

$$f_a = \frac{a}{z^2 + 2z}$$

(here the critical 2-orbit is $\{0, \infty\}$ and the free critical point is -1).

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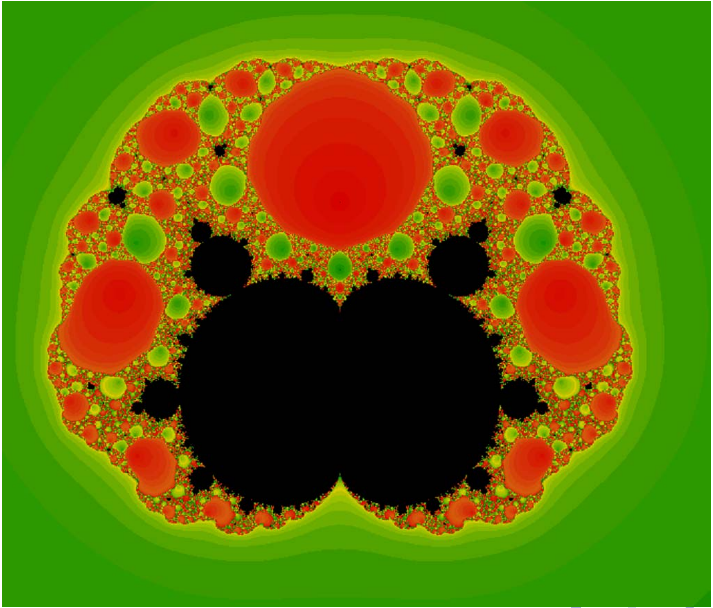
Analog of the Mandelbrot set

Define N as the set of all maps $f \in V_2$ such that the orbit of -1 is bounded. This is an analog of the Mandelbrot set.

The set N



Conjectural description of ∂N (Ben Wittner, 1988)

- ∂N is conjectured to be the "mating" of a part of the Mandelbrot set and a part of the basilica (the Julia set for $z \mapsto z^2 - 1$).
- $\partial M_{1/2}$ = the boundary of the Mandelbrot set with the 1/2-limb removed.
- $J_{1/2}$ = basilica (Julia for $z \mapsto z^2 - 1$) with the 1/2-limb removed.
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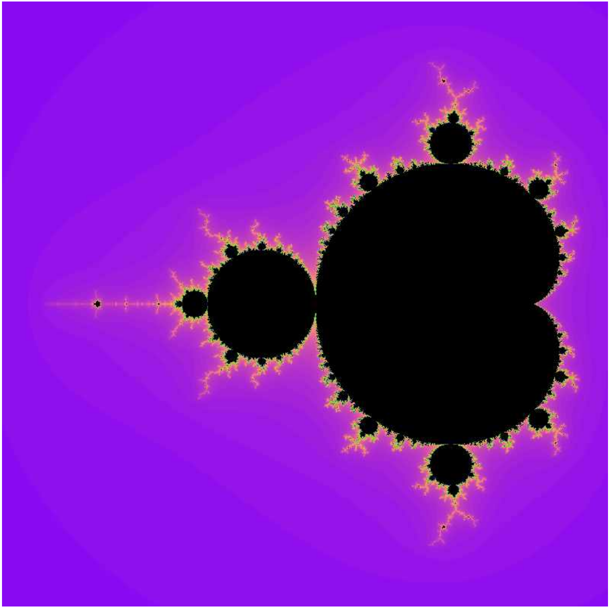
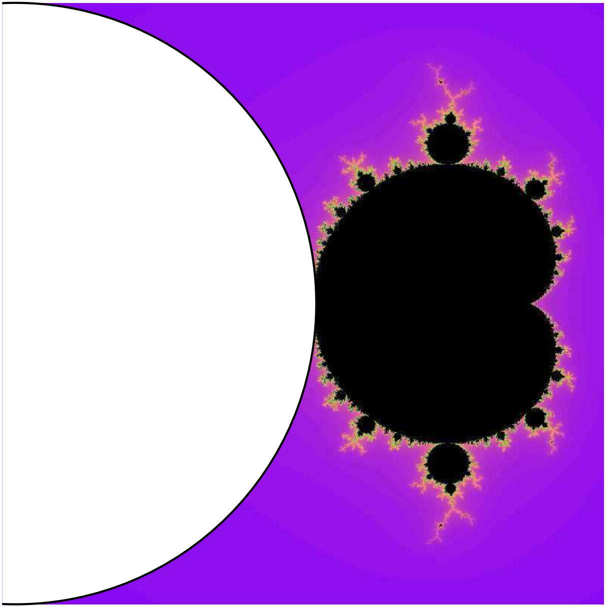
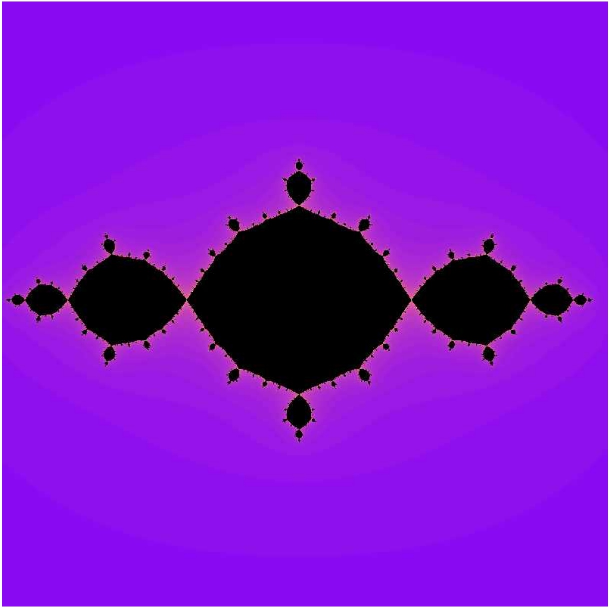
The Mandelbrot set



The Mandelbrot set with 1/2-limb removed



The basilica (the Julia set for $z \mapsto z^2 - 1$)



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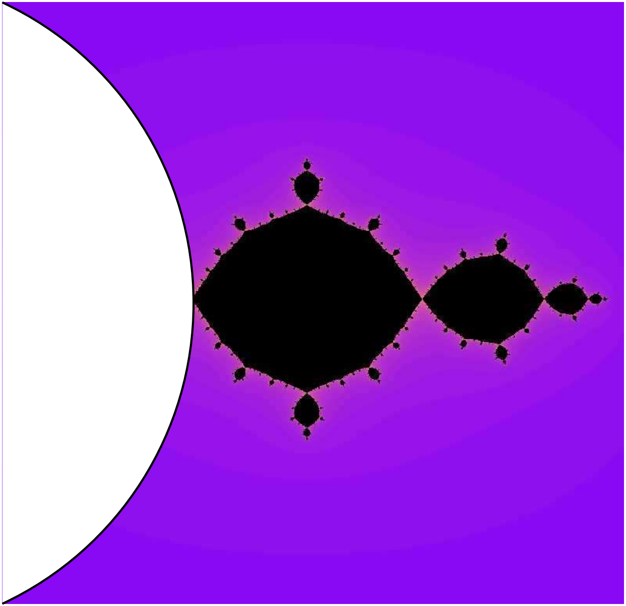

Conjectural description of the Mandelbrot set

- If MLC conjecture is true, then the boundary of the Mandelbrot set is a quotient of the unit circle.
- Connect equivalent points by geodesics in the unit circle.
- The set of geodesics is a *geodesic lamination* in the sense of Thurston (in particular, geodesics do not intersect).

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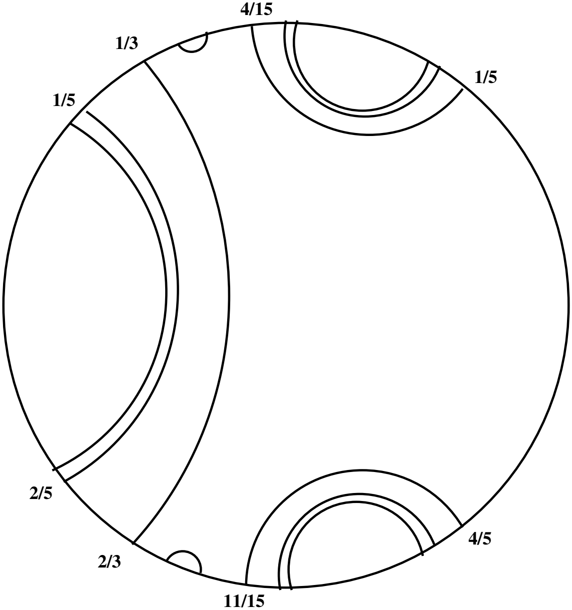
The lamination for the Mandelbrot set



Actual topological description of the basilica

- The basilica is a quotient of the unit circle.
- The corresponding geodesic lamination does not have accumulation points inside the unit disk.

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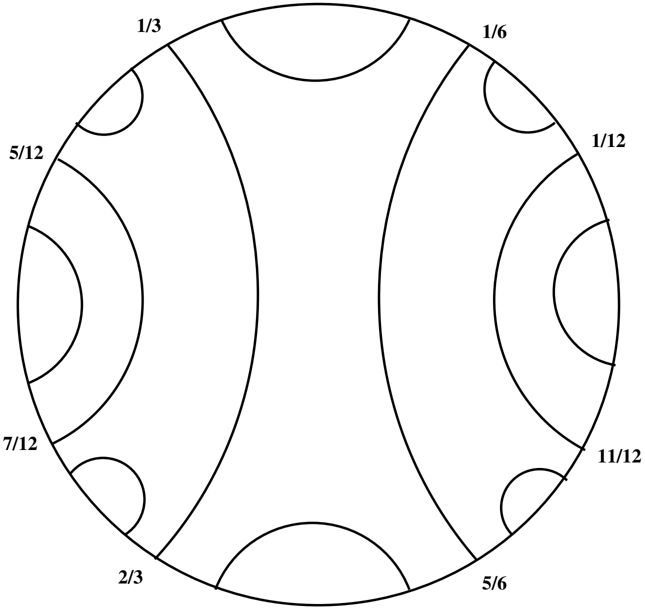

The "mating" of the $\partial M_{1/2}$ and $J_{1/2}$ is done as follows:

- Inside the unit disk, draw the lamination for $\partial M_{1/2}$.
- Outside the unit disk, draw the lamination for $J_{1/2}$.
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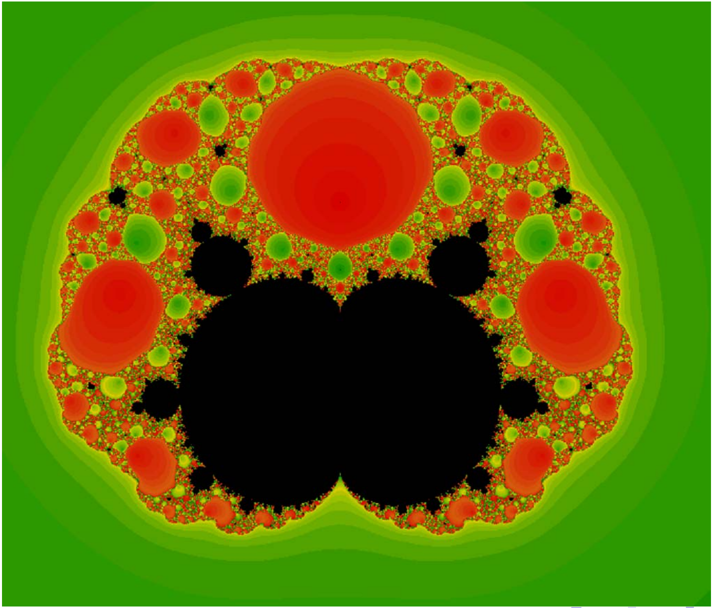

Main results

- *Explicit topological models* are constructed for all maps $f \in V_2$ such that -1 is on the boundary of the immediate basin of $\{0, \infty\}$.
- These maps, together with countably many parabolic maps, form the "exterior boundary" of the set N.
- *All* exterior parameter rays land.
- Periodic rays land at parabolic parameter values (except for 0-ray that lands at point $a = 0$ not corresponding to any quadratic rational map).
- All other rays (including strictly pre-periodic) land at parameter values, for which -1 is on the boundary of the basin of $\{0, \infty\}$

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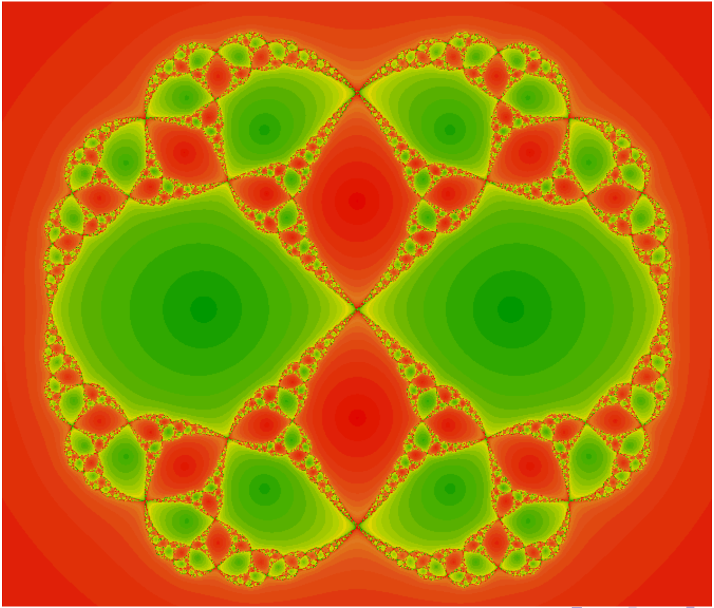
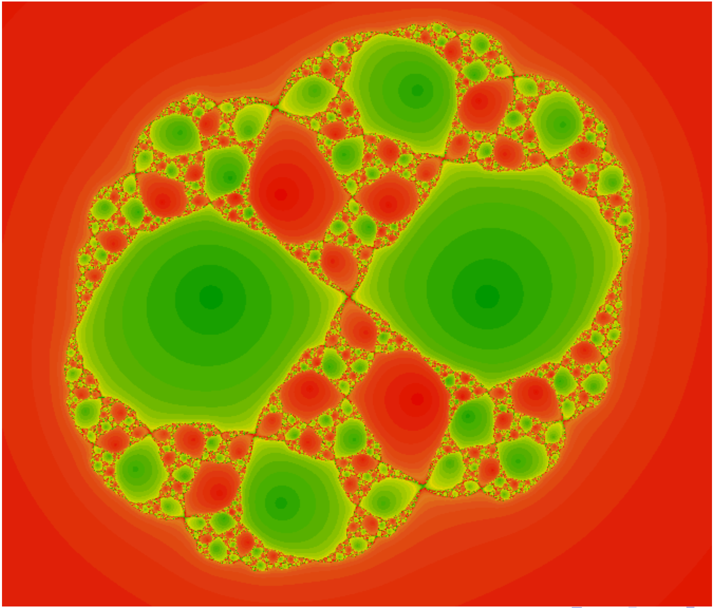
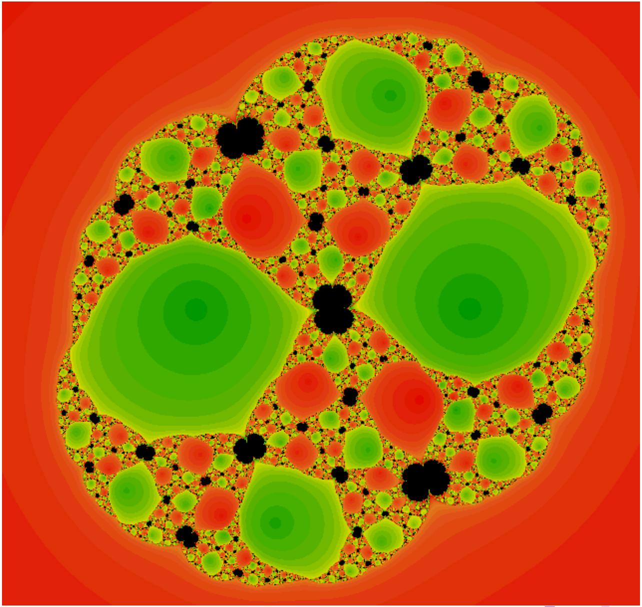
A map from the exterior boundary



Another map from the exterior boundary



A parabolic map from the exterior boundary



Exterior hyperbolic component

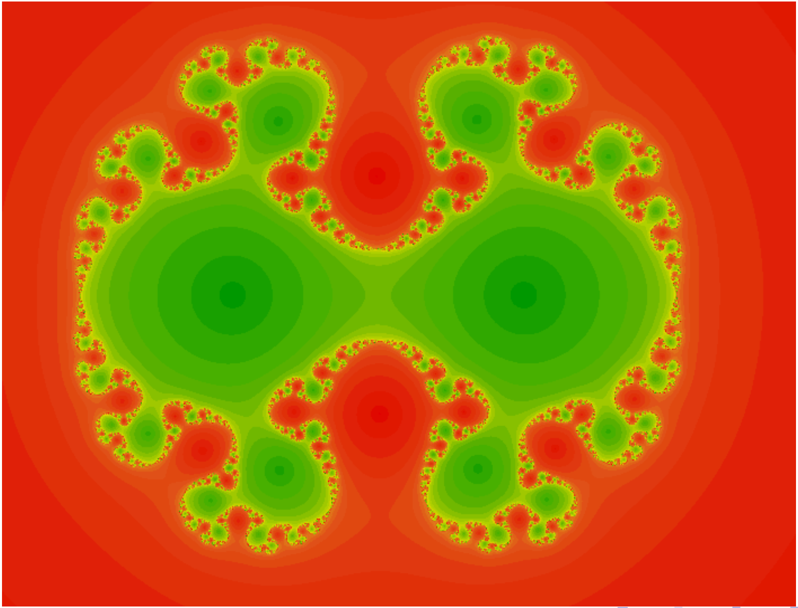
consists of parameter values, for which -1 is in the immediate basin of $\{0, 1\}$.

Theorem (Sullivan) *For such maps, the Julia set is a quasi-circle.*

The complement to the Julia set is the union of open topological disks Ω_0 and Ω_∞:

$$0, -1 \in \Omega_0, \quad \infty \in \Omega_\infty.$$

A map from the exterior hyperbolic component



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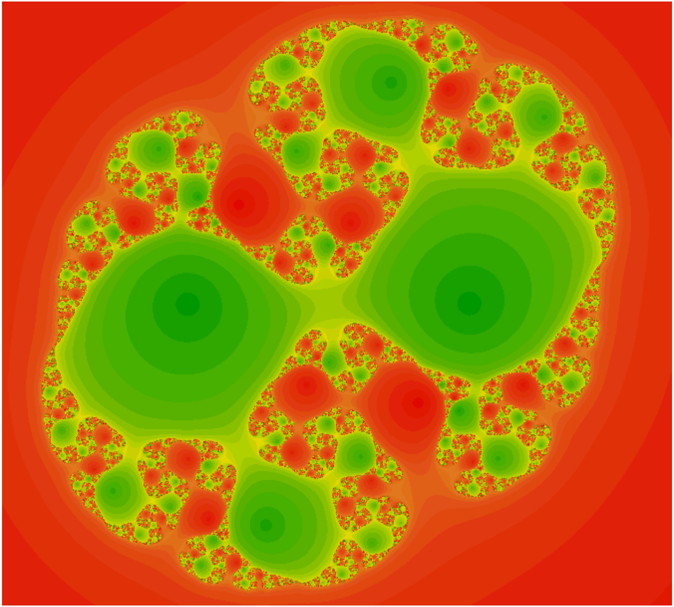

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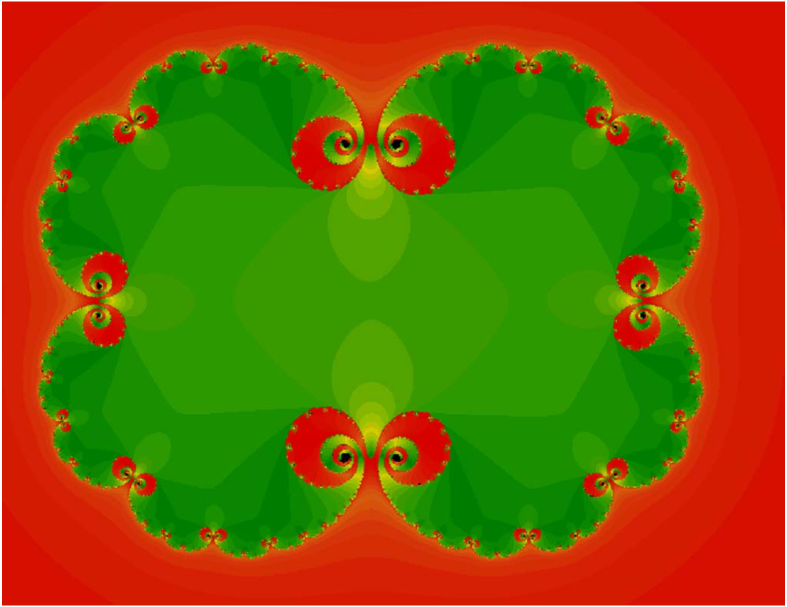

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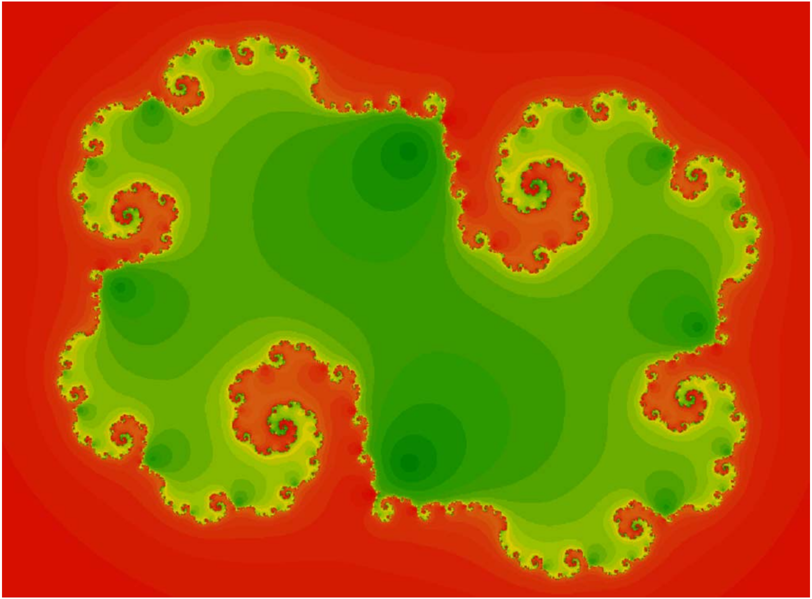

Ray combinatorics

- Let f be a map from the exterior hyperbolic component.
- Consider *rays* for $f^{\circ 2}$ emanating from iterated preimages of 0 and ∞. Recall that rays are the gradient lines of the Green function
$$G(z) = \lim_{n \to \infty} \frac{\log |f^{\circ 2n}(z)|}{2^n}$$
(which is positive near infinity and negative near zero).
- Some of these rays crash into an iterated pre-image of -1 and split.
- If the parameter ray containing f is not periodic, then the dynamic rays can not split more than once.

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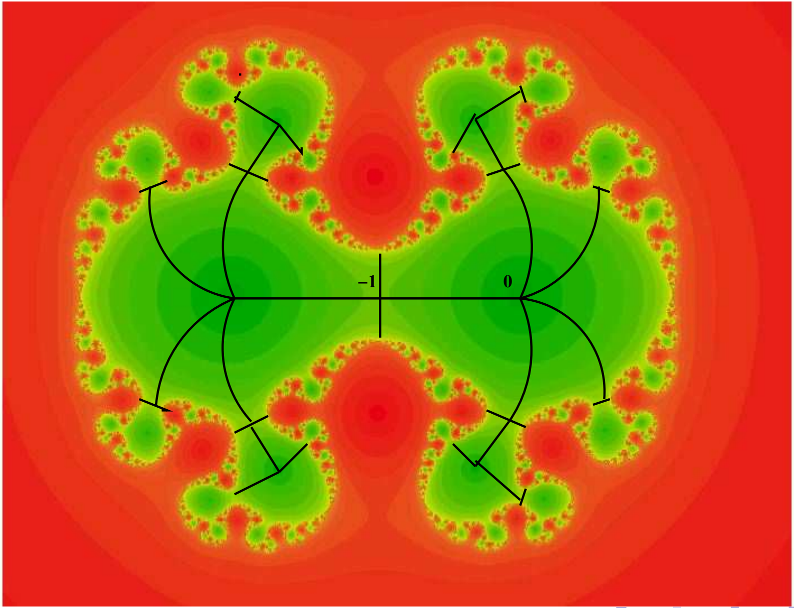

Ray laminations

- Consider a ray that splits.
- *A ray leaf* = the union of the two branches (that appear after the splitting).
- This is a curve going from one point in the Julia set to another point.
- Ray leaves live both inside and outside the Julia set.

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2-sided laminations

- The Julia set is *canonically* identified with the unit circle.
- Thus we have 2 Thurston laminations defined inside and outside of the unit circle. The outside leaves correspond to inside leaves under the map $z \mapsto 1/z^2$.
- We call this pair of laminations a *2-sided lamination*.
- This 2-sided lamination is the same along any parameter ray.
- For different parameter rays, the corresponding 2-sided laminations are not equivalent.

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Shrinking of ray leaves

- As one approaches to the exterior boundary along a parameter ray, the ray leaves become shorter and shorter.
- In the limit, they define identifications on the unit circle.
- *Rigidity*: a map from the exterior boundary is not topologically conjugate to any other rational map.

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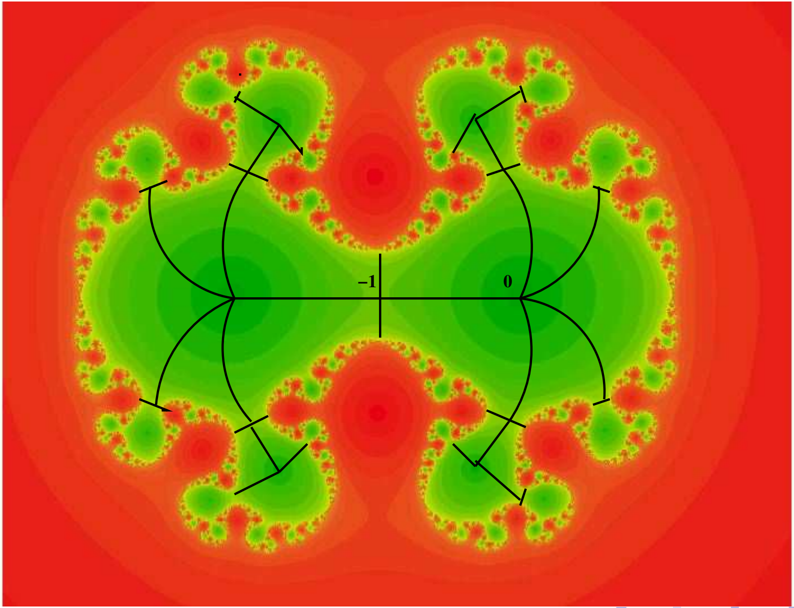
A map from the exterior hyperbolic component



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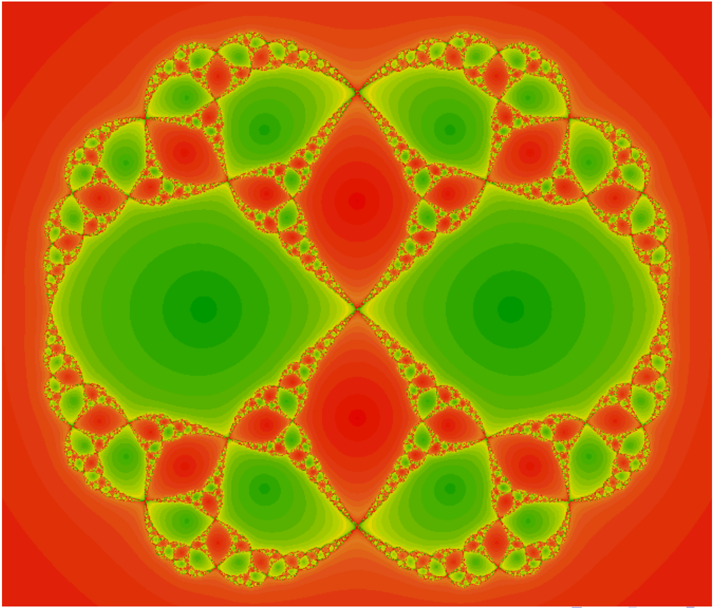

Combinatorial model: the measure μ

Let z_0 be any point on the unit circle. There is a unique probability measure μ on the unit circle with the following properties:

- The measure μ is concentrated on countably many points, namely, on all iterated preimages of the point z_0 under the map $z \mapsto z^2$ (the point z_0 itself is also regarded as an iterated preimage of z_0).
- For any point z on the unit circle different from z_0, we have $\mu\{z^2\} = 4\mu\{z\}$.

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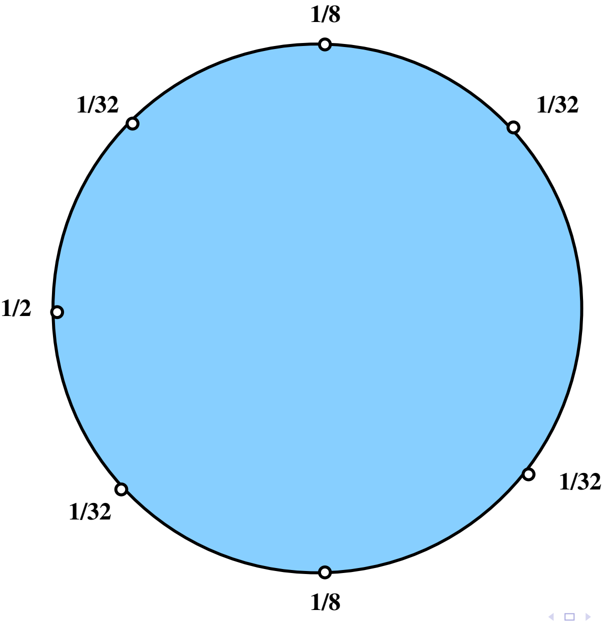
the measure μ

The measure μ can be given by the following formula

$$\mu\{z\} = \sum_{m:\ z^{2^m}=z_0} \frac{1}{2 \cdot 4^m}.$$

The summation is over all nonnegative integers m such that $z^{2^m} = z_0$. In particular, if the point z_0 is not periodic under the map $z \mapsto z^2$, then there is at most one summand.

The measure μ



The map h_0

There is a unique continuous map $h_0 : S^1 \to S^1$ with the following properties:

- $h_0(1) = 1$, and 1 is in the center of $h_0^{-1}(1)$.
- the push-forward of the uniform probability measure under the map h_0 is the measure μ,
- the map h_0 has topological degree 1.

Then h_0 *almost* semi-conjugates $z \mapsto z^4$ with $z \mapsto z^4$:

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The forward invariant lamination L_0

- Define the set $L_0 = L_0(z_0)$ of geodesics in the unit disk as follows: x and y are connected with a geodesic iff the arc between x and y is the full pre-image of some point under h_0.
- These geodesics do not intersect — they form a *geodesic lamination*.
- The lamination L_0 is *forward invariant* under $z \mapsto z^4$: endpoints of any leaf are mapped to endpoints of a leaf or to the same point.

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The invariant lamination L

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- take arcs subtended by all leaves of L_0,
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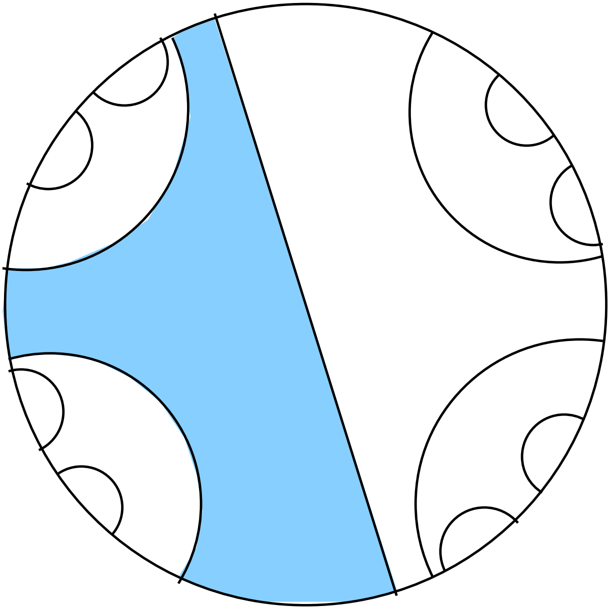

The two-sided lamination $2L$

- The lamination L is symmetric with respect to the antipodal map $z \mapsto -z$.
- Therefore, the image of L under $z \mapsto 1/z^2$ is a lamination in the exterior of the unit circle.
- lamination $2L(z_0)$ represents the ray splitting of maps from the exterior hyperbolic component.

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- Therefore, the image of L under $z \mapsto 1/z^2$ is a lamination in the exterior of the unit circle.
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Topological models — non-periodic case

Theorem 1. *For any map from the exterior hyperbolic component not lying on a periodic ray, the 2-sided ray lamination coincides with* $2L(z_0)$ *for some* z_0 *on the unit circle.*

Theorem 2. *For any map on the exterior boundary that is not a landing point of a periodic ray, the Julia set is the quotient of the unit circle by the equivalence relation generated by the 2-sided lamination* $2L(z_0)$ *for some* z_0.

Periodic parameter rays

- For a map f from a periodic parameter ray, the dynamic rays split infinitely many times.
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The pseudo-lamination corresponding to the 0-ray.