Analogs of Hodge–Riemann relations

in algebraic geometry, convex geometry and linear algebra

V. Timorin*

*State University of New York
at Stony Brook

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Hodge–Riemann form

Let X be a compact Kähler n-manifold with a Kähler form ω.

Example

X is an algebraic manifold in projective space, and ω is a differential form defining the cohomology class Poincaré dual to a hyperplane section.

Let α be a (p, q)-form: the space of such forms is locally spanned by products of p holomorphic and q anti-holomorphic differentials. The *Hodge–Riemann form* is

$$q(\alpha) = C \int_X \alpha \wedge \overline{\alpha} \wedge \omega^{n-p-q}.$$

The coefficient C is ± 1 or $\pm i$, and depends only on p and q. (if α is dual to an analytic cycle σ, then this is the "projective degree" of the self-intersection σ^2)

Hodge–Riemann relations

- Suppose that $[\alpha] \neq 0$ is a *primitive class*, i.e.

$$[\alpha \wedge \omega^{n-p-q+1}] = 0.$$

- Then

$$q(\alpha) > 0.$$

- In particular, if a divisor σ on a projective surface has "degree" 0, then its self-intersection is negative (the *index theorem*).

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Hard Lefschetz theorem

- The map
$$\wedge[\omega]^{n-p-q} : H^{p,q} \longrightarrow H^{n-q,n-p}$$
is an *isomorphism* of vector spaces ($H^{p,q}$ is the (p,q)-cohomology space: closed (p,q)-forms modulo exact (p,q)-forms)
- The Hodge–Riemann relations imply the Hard Lefschetz theorem.
- There is a close relation between the Hodge–Riemann relations and some inequalities from *convex geometry*.

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Minkowski addition

- Let A and B be convex sets in $\mathbb{R}^n$. The *Minkowsi sum* $A + B$ is defined as

$$A + B := \{a + b \mid a \in A, \quad b \in B\}.$$

- We can think of $A + B$ as the union of copies of B attached to every point of A.
- E.g. if B is the ball of radius ε, then $A + B$ is the ε-neighborhood of A.

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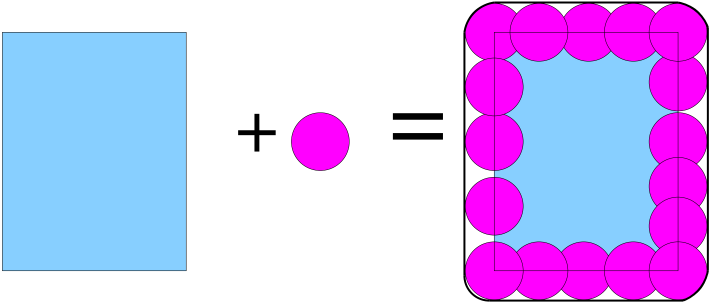

The Brunn theorem

- Let A and B be convex bodies in $\mathbb{R}^n$, then
$$\mathrm{Vol}(A+B)^{\frac{1}{n}} \geq \mathrm{Vol}(A)^{\frac{1}{n}} + \mathrm{Vol}(B)^{\frac{1}{n}}.$$
- (i.e. the function $\mathrm{Vol}^{\frac{1}{n}}$ is *concave*).
- If B is the unit ball, this is a variant of the *isoperimetric inequality*: note that the surface area is the growth rate of volume of the ε-neighborhood.
- In general: isoperimetric inequality in *Minkowski geometry* (where B plays the role of the unit ball).

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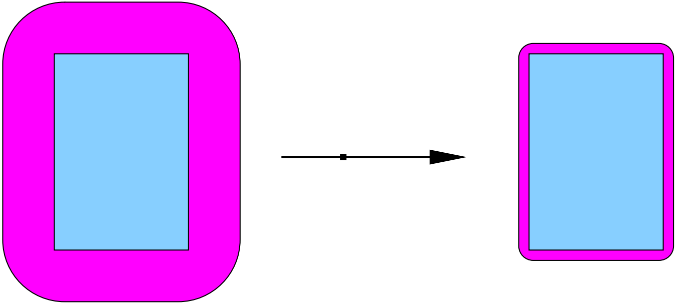
Minkowski addition and the surface area

If we differentiate the volume of ε-neighborhood by ε at $\varepsilon = 0$, then we obtain the surface area (surface volume):



Analogs in algebraic geometry

"Algebro-geometric" Brunn theorem (Khovanskii and Teissier, 70s):

$$V \subset \mathbb{P}_1 \times \mathbb{P}_2 \quad \text{irreducible algebraic submanifold.}$$

Then

$$\mathrm{Vol}(V)^{\frac{1}{n}} \geq \mathrm{Vol}(V_1)^{\frac{1}{n}} + \mathrm{Vol}(V_2)^{\frac{1}{n}},$$

where V_1 and V_2 are projections of V on $\mathbb{P}_1$ and $\mathbb{P}_2$, respectively.

Mixed volumes

The function of Volume is a homogeneous polynomial on convex sets in $\mathbb{R}^n$, i.e.

$$\mathrm{Vol}(\lambda_1 A_1 + \cdots + \lambda_n A_n)$$

is a homogeneous polynomial in $\lambda_1, \ldots, \lambda_n$ of degree n, for $\lambda_1, \ldots, \lambda_n \geq 0$.
Mixed volume $\mathrm{Vol}(A_1, \ldots, A_n)$ is the *polarization* of Vol (symmetric n-linear form s.t. $\mathrm{Vol}(A, \ldots, A) = \mathrm{Vol}(A)$).

Polarization

- The main idea of polarization is:

$$ab = \frac{(a+b)^2 - a^2 - b^2}{2}$$

(product can be expressed through squares).

- Similarly, for the mixed area:

$$\mathrm{Vol}_2(A, B) = \frac{\mathrm{Vol}_2(A+B) - \mathrm{Vol}_2(A) - \mathrm{Vol}_2(B)}{2}.$$

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Minkowski theorem

The Brunn theorem is equivalent to the following inequality on mixed volumes:

$$\mathrm{Vol}(A, B, \underbrace{B, \ldots, B}_{n-2 \ \textit{times}})^2 \geq \mathrm{Vol}(A, A, \underbrace{B, \ldots, B}_{n-2 \ \textit{times}}) \cdot \mathrm{Vol}(B).$$

In other words, the bilinear form

$$\mathrm{Vol}(\cdot, \cdot, \underbrace{B, \ldots, B}_{n-2 \ \textit{times}})$$

satisfies the reversed Cauchy-Schwartz inequality $\Leftrightarrow$
it has positive index 1.

Hodge index theorem

This is related to the *Hodge index theorem*:
the intersection form for divisors on a smooth compact projective surface has positive index 1, i.e.

$$(D_1 \cdot D_2)^2 \geq (D_1 \cdot D_1)(D_2 \cdot D_2), \quad D_1 > 0$$

More generally, for positive divisors $D_1, \ldots, D_n$ on a compact projective manifold,

$$(D_1 \cdot D_2 \cdot D_3 \cdots D_n)^2 \geq (D_1 \cdot D_1 \cdot D_3 \cdots D_n)(D_2 \cdot D_2 \cdot D_3 \cdots D_n).$$

(mixed Hodge index theorem — A. Khovanskii, B. Teissier)

Aleksandrov–Fenchel inequalities

Let $A_1, \ldots, A_n$ be convex bodies. Then

$$\mathrm{Vol}(A_1, A_2, A_3, \ldots, A_n)^2 \geq$$

$$\mathrm{Vol}(A_1, A_1, A_3, \ldots, A_n)\mathrm{Vol}(A_2, A_2, A_3, \ldots, A_n)$$

[A. Aleksandrov, 1937] Aleksandrov gave two proofs of this theorem:

1. through convex polytopes ("intersection theory on toric varieties"),
2. through elliptic differential operators ("Hodge theory")

Table of correspondence (beginning)

Minkowski inequality	Hodge index theorem
Aleksandrov–Fenchel inequality	mixed Hodge index theorem (A. Khovanskii, B. Teissier)
"Hodge–Riemann relations" for simple polytopes (P. McMullen, V.T.)	Hodge–Riemann relations
"mixed Hodge–Riemann relations" for simple polytopes (P. McMullen, V.T.)	mixed Hodge–Riemann relations (T. Dinh and V. Nguyên)

Inequalities from Linear Algebra

Mixed discriminant $\det(A_1, \ldots, A_n)$ of symmetric matrices $A_1, \ldots, A_n$ is the polarization of the discriminant (=determinant). The Aleksandrov inequality for mixed discriminants:

$$\det(A_1, A_2, A_3, \ldots, A_n)^2 \geq$$

$$\det(A_1, A_1, A_3, \ldots, A_n) \det(A_2, A_2, A_3, \ldots, A_n)$$

for positive definite matrices $A_1, \ldots, A_n$.
This is an infinitesimal version of both Aleksandrov–Fenchel inequality and the mixed Hodge index theorem.

Van der Waerden problem

The Aleksandrov inequality for mixed discriminants was used in a proof of the Van der Waerden conjecture on permanents of doubly stochastic matrices (Falikman, Egorychev).
(an inequality for the probability of a random self-map of a finite set to be a permutation).

Inequalities from Linear Algebra

Let $\omega_1, \ldots, \omega_{n-p-q+1}$ be positive $(1,1)$-forms on $\mathbb{C}^n$. Set $\Omega = \omega_1 \wedge \cdots \wedge \omega_{n-p-q}$. For any (p,q)-form $\alpha \neq 0$ such that $\alpha \wedge \Omega \wedge \omega_{n-p-q+1} = 0$,

$$C * (\alpha \wedge \overline{\alpha} \wedge \Omega) > 0$$

(V.T., 1998). This is a generalization of the Aleksandrov inequality for mixed discriminants.

Mixed Hodge–Riemann relations

Let $\omega_1, \ldots, \omega_{n-p-q+1}$ be Kähler forms on a compact complex n-manifold X. Set $\Omega = \omega_1 \wedge \cdots \wedge \omega_{n-p-q}$. For any closed (but not exact) (p, q)-form α such that $[\alpha \wedge \Omega \wedge \omega_{n-p-q+1}] = 0$,

$$C \int_X \alpha \wedge \overline{\alpha} \wedge \Omega > 0$$

(T. Dinh and V. Nguyên, 2005).
This result was motivated by dynamics of holomorphic automorphisms.

Analogs of Hodge–Riemann relations for simple convex polytopes

- A convex polytope $\Delta \in \mathbb{R}^n$ is said to be *simple* if exactly n facets meet at each vertex. Simple polytopes are stable under small perturbations of facets.
- A polytope *analogous to* Δ is any polytope obtained from Δ by parallel translations of facets, without changing the combinatorial type.

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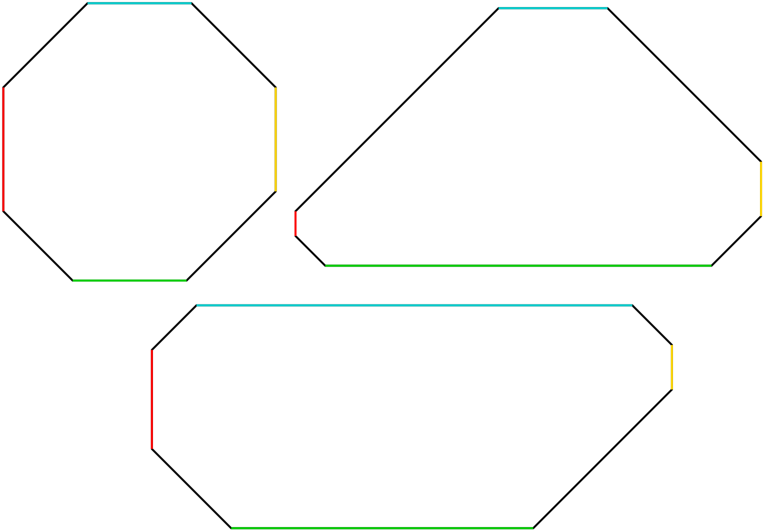
Analogous polytopes



Support numbers

- The set of polytopes analogous to a given one, is stable under Minkowski addition and multiplication by positive numbers. This is a convex cone, which can be extended to a finite dimensional vector space.
- *Coordinates* in this space are *support numbers* (signed distances from the origin to support hyperplanes containing the facets).
- The space of all "virtual" polytopes analogous to a given one has the dimension $N =$ the number of all facets.

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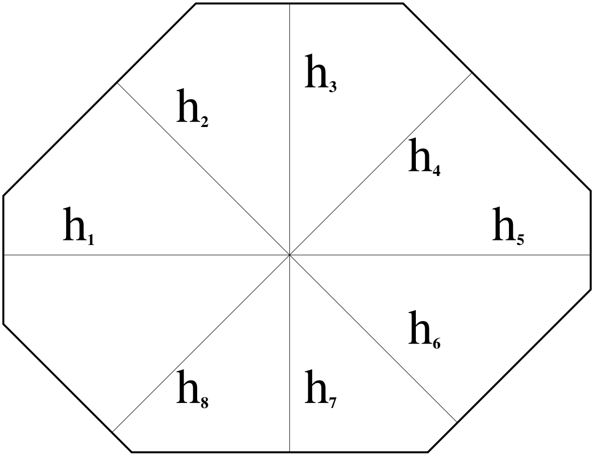

Volume polynomial

The restriction of the volume polynomial to the space of polytopes analogous to a simple polytope Δ can be expressed as a polynomial

$$\mathrm{Vol}_\Delta(h_1, \dots, h_N)$$

in support numbers $h_1, \dots, h_N$. Let $H_1, \dots, H_N$ be the support numbers of Δ. Introduce the operator

$$L_\Delta = \sum_{k=1}^{N} H_k \frac{\partial}{\partial h_k}$$

of differentiation along Δ. Then

$$\mathrm{Vol}_\Delta = \frac{1}{n!} L_\Delta^n(\mathrm{Vol}_\Delta).$$

Description of the "cohomology ring"

With each simple polytope Δ, one associates a "cohomology algebra". For lattice polytopes, it is isomorphic to the cohomology algebra of the corresponding toric variety:

$$A_\Delta = \frac{\text{Diff. operators with const. coefficients } \alpha}{\alpha \mathrm{Vol}_\Delta = 0}.$$

(A. Khovanskii and A. Pukhlikov, unpublished)
Poincaré duality and the Lefschetz operator look especially simple in this description.

“Hodge–Riemann relations” for simple polytopes

Consider a differential operator α of order k with constant coefficients. The operator α is said to be *primitive*, if

$$L_{\Delta}^{n-2k+1}\alpha(\mathrm{Vol}_{\Delta}) = 0.$$

Equivalently, the polynomial $\alpha\mathrm{Vol}_{\Delta}$ has zero of order $\geq k$ at point $(H_1, \ldots, H_N)$. If $\alpha \neq 0$ is primitive, then

$$(-1)^k \alpha^2 L_{\Delta}^{n-2k}(\mathrm{Vol}_{\Delta}) > 0.$$

(P. McMullen 1993; in this form V.T. 1999)

The g-theorem

- "Hodge–Riemann relations" for convex simple polytopes imply an analog of Hard Lefschetz theorem, which can be expressed as certain inequalities on combinatorial parameters of the polytope.
- For a simple convex n-polytope Δ, let $f_k(\Delta)$ be the number of all k-faces in Δ. The vector $(f_0(\Delta), \ldots, f_n(\Delta))$ is called the *f-vector of Δ*.
- **The g-theorem:** *An integer vector $(f_0, \ldots, f_n)$ is the f-vector of a simple n-polytope if and only if it satisfies certain explicit inequalities.*
- An algebro-geometric proof: R. Stanley, 1980 (intersection cohomology for toric varieties), a convex-geometric proof: P. McMullen, 1993.

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"Mixed Hodge–Riemann relations" for simple polytopes

Consider a differential operator α of order k with constant coefficients. The operator α is said to be *primitive* with respect to simple analogous polytopes $\Delta_1, \dots, \Delta_{n-2k+1}$, if

$$\Lambda L_{n-2k+1}\alpha(\mathrm{Vol}_\Delta) = 0,$$

where $\Lambda = L_{\Delta_1} \cdots L_{\Delta_{n-2k}}$. If $\alpha \neq 0$ is primitive, then

$$(-1)^k \alpha^2 \Lambda(\mathrm{Vol}_\Delta) > 0.$$

(P. McMullen 1993; in this form V.T. 1999)
This is a generalization of the Aleksandrov–Fenchel inequality.

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“Hodge–Riemann relations” for arbitrary polytopes (K.Karu, . . .)	Hodge–Riemann relations in intersection cohomology (M. Saito, . . .)

"Hodge–Riemann relations" for non-simple polytopes

- *Combinatorial intersection cohomology* (P. Bressler and V. Lunts; G. Barthel, J.-P. Brasselet, K.-H. Fiesler and L. Kaup): for rational polytopes, isomorphic to intersection cohomology of the corresponding toric varieties.
- First, one defines a sheaf on the dual fan (regarded as a finite topological space), which is an analog of the *equivariant intersection cohomology sheaf* (in the sense of J. Bernstein and V. Lunts).
- The combinatorial intersection cohomology is a certain quotient of the space of global sections.
- K. Karu (2003) proved Hodge–Riemann relations in combinatorial intersection cohomology of arbitrary polytopes.

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