Berkovich spaces over $\mathbf{Z}$ and Schottky spaces

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Outline

1. Uniformization of curves
2. Berkovich spaces over $\mathbf{Z}$
3. Schottky spaces over $\mathbf{Z}$
4. Applications

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3. Schottky spaces over $\mathbf{Z}$
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Koebe's theorem

Theorem (Koebe, 1907)

Up to isomorphism, there are exactly three possibilities for the universal cover of a compact Riemann surface:

- *the projective line;*
- *the affine line;*
- *the open unit disc.*

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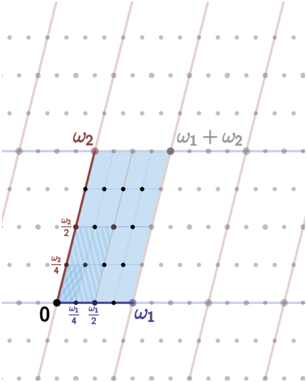
What happens in the p-adic setting?

Elliptic curves

Over **C**,

$$E(\mathbf{C}) \simeq \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau)$$

with $\mathrm{Im}(\tau) > 0$.



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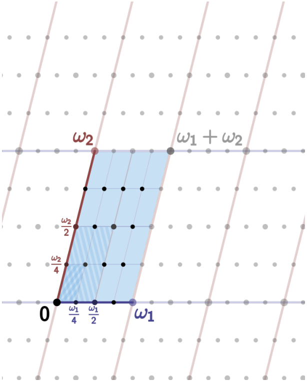

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J. Tate's idea: use a partial uniformization

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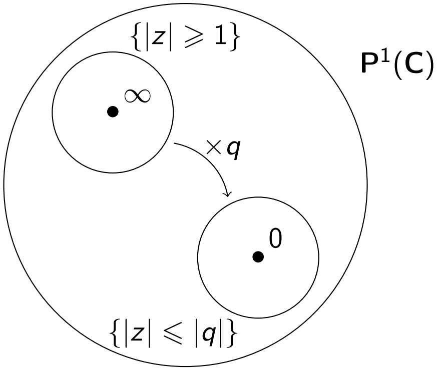

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Remark

Over $\mathbf{Q}_p$, not all elliptic curves arise this way: only those with split multiplicative reduction (Tate curves).

Schottky uniformization: setting

Let $g \geqslant 1$. Let $D_{\pm 1}, \ldots, D_{\pm g}$ be disjoint open discs in $\mathbf{P}^1(\mathbf{C})$.
Let $\gamma_1, \ldots, \gamma_g \in \mathrm{PGL}_2(\mathbf{C})$ such that, setting $\gamma_{-i} := \gamma_i^{-1}$, we have

$$\forall i,\ \gamma_i(\mathbf{P}^1(\mathbf{C}) - D_{-i}) = \overline{D_i}.$$

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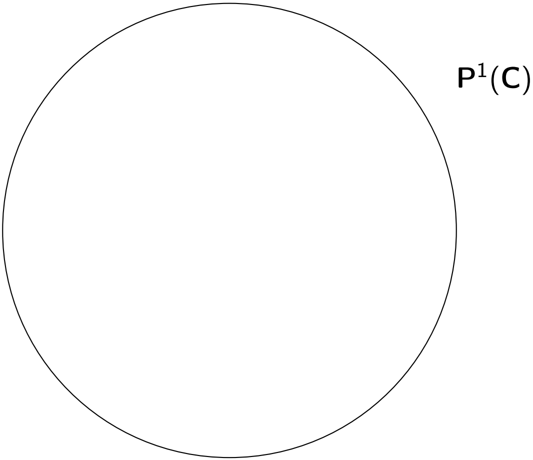

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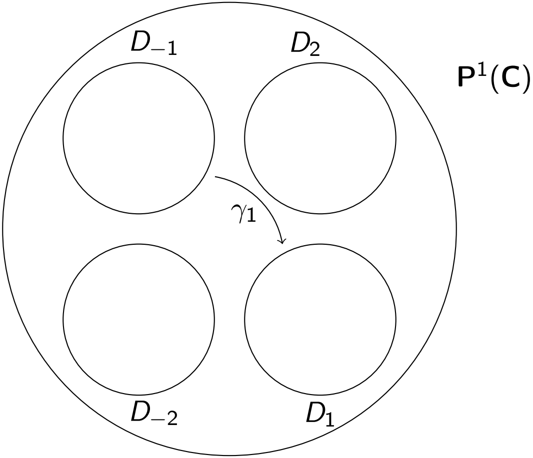

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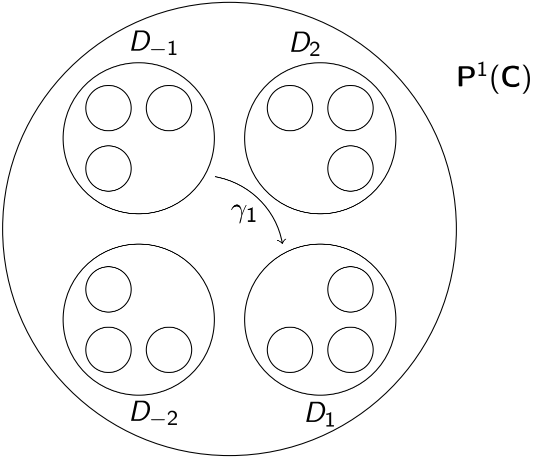

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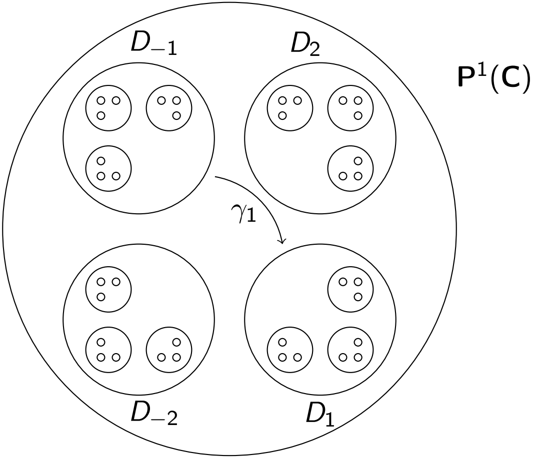

Schottky uniformization: properties

Set $\Gamma := \langle \gamma_1, \ldots, \gamma_g \rangle$. It is a free group of rank g, called

Schottky group.

There exists a compact subset $\mathcal{L}$ of $\mathbf{P}^1(\mathbf{C})$ such that

1. the action of Γ on $\mathbf{P}^1(\mathbf{C}) - \mathcal{L}$ is properly discontinuous;
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- Every compact Riemann surface of genus g may be obtained this way, possibly replacing the discs by domains bounded by Jordan curves.
- D. Mumford (1972) adapted the theory to the non-archimedean setting. The resulting curves are called Mumford curves.

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The Berkovich analytic space $\mathbf{A}_{\mathbf{Z}}^{n,\mathrm{an}}$: definition

Definition

The analytic space $\mathbf{A}_{\mathbf{Z}}^{n,\mathrm{an}}$ is the set of multiplicative seminorms

$$|\cdot|_x : \mathbf{Z}[T_1, \ldots, T_n] \to \mathbf{R}_{\geqslant 0}.$$

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It is endowed with the topology generated by the subsets of the form

$$\{x \in \mathbf{A}^{n,\mathrm{an}}_{\mathbf{Z}} \ : \ r < |P|_x < s\},$$

for $P \in \mathbf{Z}[T_1, \dots, T_n]$ and $r, s \in \mathbf{R}$.

The Berkovich analytic space $\mathbf{A}_{\mathbf{Z}}^{n,\mathrm{an}}$: structure sheaf

To each $x \in \mathbf{A}_{\mathbf{Z}}^{n,\mathrm{an}}$, we associate a complete residue field

$\mathcal{H}(x) :=$ completion of the fraction field of $\mathbf{Z}[T_1, \dots, T_n]/\mathrm{Ker}(|\cdot|_x)$

and an evaluation map

$$\chi_x \colon \mathbf{Z}[T_1, \dots, T_n] \to \mathcal{H}(x).$$

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For every open subset U of $\mathbf{A}_{\mathbf{Z}}^{n,\mathrm{an}}$, $\mathcal{O}(U)$ is the set of maps

$$f \colon U \to \bigsqcup_{x \in U} \mathcal{H}(x)$$

such that

- $\forall x \in U,\ f(x) \in \mathcal{H}(x)$;
- f is locally a uniform limit of rational functions without poles.

The Berkovich analytic space $\mathbf{A}_{\mathbf{Z}}^{n,\mathrm{an}}$: examples of points

$$\mathbf{A}_{\mathbf{Z}}^{n,\mathrm{an}} = \{|\cdot|_x \colon \mathbf{Z}[T_1, \ldots, T_n] \to \mathbf{R}_{\geqslant 0}\}$$

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$$P(\mathbf{T}) \in \mathbf{Z}[\mathbf{T}] \mapsto |P(\mathbf{t})|_\infty.$$

Note that $\mathbf{t}$ and $\bar{\mathbf{t}}$ give rise to the same seminorm.

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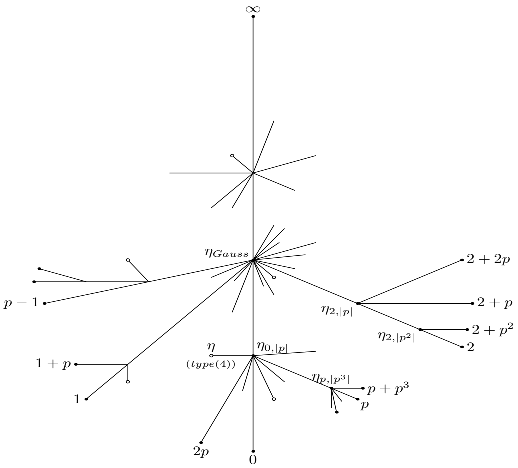
3. For $\mathbf{v} \in \mathbf{F}_p^n$,
$$P(\mathbf{T}) \in \mathbf{Z}[\mathbf{T}] \mapsto |P(\mathbf{v})|_0.$$

The Berkovich analytic space $\mathbf{A}_{\mathbf{Z}}^{n,\mathrm{an}}$: the $\mathbf{Q}_p$-points

$\mathbf{A}_{\mathbf{Q}_p}^{1,\mathrm{an}}$



The Berkovich analytic space $\mathbf{A}^{n,\mathrm{an}}_{\mathbf{Z}}$: picture

$$\mathcal{M}(\mathbf{Z}) := \mathbf{A}^{0,\mathrm{an}}_{\mathbf{Z}} = \{|\cdot|_x : \mathbf{Z} \to \mathbf{R}_{\geqslant 0}\}$$

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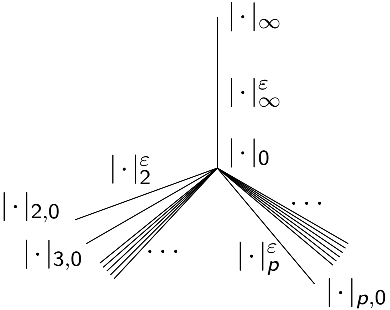

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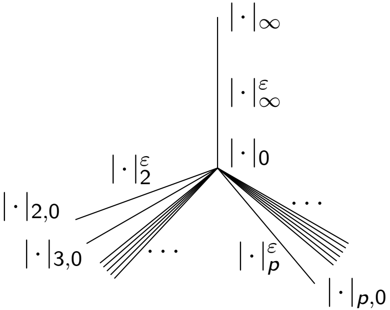

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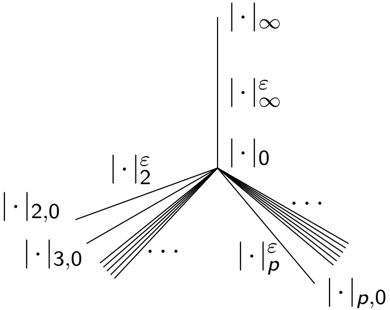

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- $\pi^{-1}(|\cdot|_\infty) = \mathbf{C}^n/\mathrm{Gal}(\mathbf{C}/\mathbf{R})$
- $\pi^{-1}(|\cdot|_p) = \mathbf{A}_{\mathbf{Q}_p}^{n,\mathrm{an}}$ usual Berkovich analytic space

The Berkovich analytic space $\mathbf{A}_{\mathbf{Z}}^{n,\mathrm{an}}$: functions

Let $\mathbf{D}$ be the open unit disk in $\mathbf{A}_{\mathbf{Z}}^{n,\mathrm{an}}$. Then $H^0(\mathbf{D},\mathcal{O})$ is a ring of convergent arithmetic power series (D. Harbater):

$$\begin{aligned} H^0(\mathbf{D},\mathcal{O}) &= \mathbf{Z}[\![T_1,\dots,T_n]\!]_{1^-} \\ &= \{f \in \mathbf{Z}[\![T]\!] \text{ with complex radius of convergence } \geqslant 1\}. \end{aligned}$$

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The local ring at the point 0 over $|\cdot|_0$ is the subring of $\mathbf{Q}[\![T_1, \ldots, T_n]\!]$ consisting of the power series f such that

i) $\exists N \in \mathbf{N}^*$, $f \in \mathbf{Z}[\frac{1}{N}][\![T_1, \ldots, T_n]\!]$;

ii) the complex radius of convergence of f is > 0;

iii) for each $p|N$, the p-adic radius of convergence of f is > 0.

Properties of $\mathbf{A}_{\mathbf{Z}}^{n,\mathrm{an}}$

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The space $\mathbf{A}_{\mathbf{Z}}^{n,\mathrm{an}}$ is Hausdorff and locally compact.

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Theorem (J. P.)

- *For every x in $\mathbf{A}_{\mathbf{Z}}^{n,\mathrm{an}}$, the local ring $\mathcal{O}_x$ is Henselian, Noetherian, regular, excellent.*
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Theorem (T. Lemanissier - J. P.)

Relative closed and open discs over $\mathbf{Z}$ are Stein.

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For $\alpha, \alpha', \beta \in k$ with $|\beta| \in (0, 1)$, we have

$$M(\alpha, \alpha', \beta) = \begin{pmatrix} \alpha - \beta\alpha' & (\beta - 1)\alpha\alpha' \\ 1 - \beta & \beta\alpha - \alpha' \end{pmatrix}.$$

Schottky space

Definition

For $g \geqslant 2$, the Schottky space $\mathcal{S}_g$ is the subset of $\mathbf{A}_{\mathbf{Z}}^{3g-3,\mathrm{an}}$ consisting of the points

$$z = (x_3, \ldots, x_g, x_2', \ldots, x_g', y_1, \ldots, y_g)$$

such that the subgroup of $\mathrm{PGL}_2(\mathcal{H}(z))$ defined by

$$\Gamma_z := \langle M(0, \infty, y_1), M(1, x_2', y_2), M(x_3, x_3', y_3), \ldots, M(x_g, x_g', y_g)\rangle$$

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Theorem (J. P. - D. Turchetti)

The Schottky space $\mathcal{S}_g$ is a connected open subset of $\mathbf{A}_{\mathbf{Z}}^{3g-3,\mathrm{an}}$.

Universal Mumford curve

Denote by $(X_3, \ldots, X_g, X'_2, \ldots, X'_g, Y_1, \ldots, Y_g)$ the coordinates on $\mathbf{A}^{3g-3,\mathrm{an}}_{\mathbf{Z}}$ and consider the subgroup of $\mathrm{PGL}_2(\mathcal{O}(\mathcal{S}_g))$:

$$\Gamma = \langle M(0, \infty, Y_1), M(1, X'_2, Y_2), M(X_3, X'_3, Y_3), \ldots, M(X_g, X'_g, Y_g)\rangle.$$

There exists a closed subset $\mathcal{L}$ of $\mathcal{S}_g \times_{\mathcal{M}(\mathbf{Z})} \mathbf{P}^{1,\mathrm{an}}_{\mathbf{Z}}$ such that

1. for each $z \in \mathcal{S}_g$, $\mathcal{L} \cap \mathrm{pr}_1^{-1}(z)$ is the limit set of Γ_z;
2. we have a commutative diagram of analytic morphisms

$$\begin{array}{ccc}
\mathcal{S}_g \times_{\mathcal{M}(\mathbf{Z})} \mathbf{P}^{1,\mathrm{an}}_{\mathbf{Z}} - \mathcal{L} & \longrightarrow & (\mathcal{S}_g \times_{\mathcal{M}(\mathbf{Z})} \mathbf{P}^{1,\mathrm{an}}_{\mathbf{Z}} - \mathcal{L})/\Gamma \, . \\
\downarrow & \swarrow & \\
\mathcal{S}_g & &
\end{array}$$

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Teichmüller modular forms

M_g moduli space of smooth and proper curves of genus g
$\pi\colon C_g \to M_g$ universal curve over M_g
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The Torelli map τ gives rise to

$$\tau^*\colon S_{g,h}(R) \to T_{g,h}(R),$$

where $S_{g,h}(R)$ denotes Siegel modular forms.

Expansions

T. Ichikawa (1994) defined an expansion map

$$\kappa_R \colon T_{g,h}(R) \to R\left[x_{\pm 1}, \ldots, x_{\pm g}, \frac{1}{x_i - x_j}\right][\![y_1, \ldots, y_g]\!].$$

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- could be upgraded to

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- may be helpful for the Schottky problem (characterizing Jacobian varieties among Abelian varieties)

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Let $k \subset \mathbf{C}$. Let A/k be a principally polarized indecomposable Abelian threefold that is isomorphic to a Jacobian over $\mathbf{C}$. Then, A is isomorphic to a Jacobian over k if, and only if,

$$\chi_{18}(A) \in k^2.$$

Schottky groups

Let $(k, |\cdot|)$ be a complete valued field. We denote by $\mathbf{P}_k^{1,\mathrm{an}}$

- the Berkovich projective line if k is non-archimedean;
- $\mathbf{P}^1(\mathbf{C})$ if $k = \mathbf{C}$;
- $\mathbf{P}^1(\mathbf{C})/\mathrm{Gal}(\mathbf{C}/\mathbf{R})$ if $k = \mathbf{R}$.

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A Schottky group over k is a finitely generated free subgroup of $PGL_2(k)$ containing only hyperbolic elements and with a nonempty discontinuity locus.

Action of $Out(F_g)$

Let $\sigma \in Aut(F_g)$ act on the generators of Γ_z as on those of F_g.

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- the whole M_g on the Archimedean part.

Relationship with the Outer Space

Definition (M. Culler - K. Vogtmann, 1986)

The Outer Space CV_g is a space of metric graphs X of genus g endowed with a marking (isomorphism $F_g \xrightarrow{\sim} \pi_1(X)$).

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We have a continuous surjective map

$$\mathcal{S}_{g,k} \to CV_g \times_{M_g^{\mathrm{trop}}} \mathrm{Mumf}_{g,k}.$$

See also M. Ulirsch "Non-Archimedean Schottky Space and its Tropicalization", 2020