Non-commutative Tsfasman-Vladuţ formula

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Setup

Fix a finite field $\mathbb{F}_q$. Let $\mathcal{X} = \{X_i\}_{i \in \mathbb{N}}$ be a sequence of geometrically connected smooth proper curves over $\mathbb{F}_q$ and assume that $g_{X_i} \to \infty$.

For such a curve X over $\mathbb{F}_q$ consider the abelian variety $\mathrm{Pic}^0(X)$. The group $\mathrm{Pic}^0(X)(\mathbb{F}_q)$ is the group of line bundles of degree 0 on X. It is a finite group and we define *the class number* $h_X := |\mathrm{Pic}^0(X)(\mathbb{F}_q)|$

Question

Given $\mathcal{X}$ how fast does h_{X_i} grow compared to g_{X_i}?

Weil's bound

The Grothendieck-Lefschetz fixed point formula

Let Y be a smooth proper scheme over $\mathbb{F}_q$. Then

$$|Y(\mathbb{F}_q)| = \sum_{i=0}^{2\dim Y} (-1)^i \cdot \mathrm{tr}(F_Y^* | H^i_{\text{ét}}(Y_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l))$$

One has $H^*_{\text{ét}}(\mathrm{Pic}^0(X)_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l) \simeq \bigwedge^*_{\mathbb{Q}_l}(H^1_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l))$.

Partial answer: Weil's bound

$$(\sqrt{q}-1)^{2g_{X_i}} \leq h_{X_i} \leq (\sqrt{q}+1)^{2g_{X_i}} \Rightarrow$$

$$\Rightarrow 2\log_q(\sqrt{q}-1) \leq \frac{\log_q h_{X_i}}{g_{X_i}} \leq 2\log_q(\sqrt{q}+1)$$

Asymptotically exact sequences

We get that $\frac{\log_q h_X}{g_X}$ lies in the interval $[2\log_q(\sqrt{q}-1), 2\log_q(\sqrt{q}+1)]$.

Question

Given a sequence $\mathcal{X} = \{X_i\}_{i\in\mathbb{N}}$ does the limit $\frac{\log_q h_{X_i}}{g_{X_i}}$ exist?

For a curve X let $B_n(X) = \{x \in |X| \mid x \simeq \operatorname{Spec} \mathbb{F}_{q^n}\}$ be the set of points of degree n on X.

Definition

A sequence $\mathcal{X}$ with $g_{X_i} \to \infty$ is called *asymptotically exact* if all of the limits

$$\beta_n(\mathcal{X}) := \lim_{i\to\infty} \frac{|B_n(X_i)|}{g(X_i)}$$

exist. The numbers $\beta_n(\mathcal{X}) \in \mathbb{R}_{\geq 0}$ are called *the Tsfasman-Vladuţ invariants* of $\mathcal{X}$.

Tsfasman-Vladuţ formula

Theorem (Tsfasman-Vladuţ)

Let $\mathcal{X} = \{X_i\}_{i \in \mathbb{N}}$ *be an asymptotically exact sequence of curves. Then*

$$\lim_{i \to \infty} \frac{\log_q h_{X_i}}{g_{X_i}} = 1 + \sum_{n \geq 1} \beta_n(\mathcal{X}) \log_q \left(\frac{q^n}{q^n - 1} \right)$$

Example (Tower of curves)

Let $\mathcal{X} = \{X_i\}_{i \in \mathbb{N}}$ be given by a tower of curves

$$\ldots \to X_3 \to X_2 \to X_1 \to X_0$$

such that $g_{X_i} \to \infty$. Then $\mathcal{X}$ is asymptotically exact.

Tsfasman-Vladuţ invariants

Theorem (Drinfeld-Vladuţ bound)

For any asymptotically exact sequence $\mathcal{X}$ *one has*

$$\sum_{n=1}^{\infty} \frac{n\beta_n(\mathcal{X})}{q^{n/2}-1} \leq 1 \implies \beta_n(\mathcal{X}) \leq \frac{q^{n/2}-1}{n} \text{ for each } n > 0.$$

Example (Garcia-Stichtenoth's tower)

Assume q is a square. $\mathsf{GS}(q,n) = \cdots \to X_i \to X_{i-1} \to \cdots \to X_0 \simeq \mathbb{P}^1$ corresponds to a tower $\mathbb{F}_q(x_0) \subset \mathbb{F}_q(x_0, x_1) \subset \mathbb{F}_q(x_0, x_1, x_2) \subset \cdots$ of field extensions obtained as a sequence of Artin-Schreier extensions:

$$x_{i+1}^{q^{n/2}} + x_{i+1} = \frac{x_i^{q^{n/2}}}{x_i^{q^{n/2-1}} - 1}$$

Then $\beta_n(\mathsf{GS}(q,n)) = \frac{q^{n/2-1}}{n}$.

Nonabelian version: setup

Let G be a split reductive group over $\mathbb{F}_q$. Let $\mathrm{Bun}_{G,X}$ be the stack of G-bundles on X and let $\mathrm{Bun}^0_{G,X} \subset \mathrm{Bun}_{G,X}$ be the substack of G-bundles with the generalised determinant 0.

Definition

The *G-mass of X* is defined as

$$M_{G,X} := \sum_{\mathcal{E} \in \mathrm{Bun}^0_{G,X}(\mathbb{F}_q)} \frac{1}{|\mathrm{Aut}(\mathcal{E})|}$$

Example ($G = \mathbb{G}_m$)

$$M_{\mathbb{G}_m,X} = \sum_{x \in \mathrm{Pic}^0_X(\mathbb{F}_q)} \frac{1}{|\mathbb{G}_m(\mathbb{F}_q)|} = \frac{h_X}{q-1}$$

Nonabelian version: question

Question

Given an asymptotically exact sequence $\mathcal{X}$, does the limit of $\frac{\log_q M_{G,X_i}}{g_{X_i}}$ exist? Is it still expressed through $\beta_n(\mathcal{X})$?

Example ($G = \mathbb{G}_m^n$)

$$M_{\mathbb{G}_m^n,X} = \sum_{x\in \mathrm{Pic}_X^0(\mathbb{F}_q)^n} \frac{1}{|\mathbb{G}_m^n(\mathbb{F}_q)|} = \left(\frac{h_X}{q-1}\right)^n$$

Then for an asymptotically exact $\mathcal{X} = \{X_i\}_{i\in\mathbb{N}}$:

$$\lim_{i\to\infty} \frac{\log_q M_{\mathbb{G}_m^n,X}}{g_{X_i}} = \lim_{i\to\infty} \frac{\log_q h_X^n}{g_{X_i}} = n\cdot \left(1+\sum_{n\geq 1}\beta_n(\mathcal{X})\log_q\left(\frac{q^n}{q^n-1}\right)\right)$$

The quasi-residue of $L(X, s)$ at $s = 1$

One defines *the L-function of X*:

$$L(X, s) := \prod_{x \in |X|} (1 - q^{-s \cdot \deg x})^{-1}$$

By Weil conjectures

$$L(X, s) = \frac{\det(1 - F_X^* \cdot q^{-s} | H^1_{\text{ét}}(X, \mathbb{Q}_l))}{(1 - q^{-s})(1 - q \cdot q^{-s})}.$$

It has a simple pole of degree 1 at $s = 1$ and we define *the quasi-residue*

$$\rho_X := \lim_{s \to 1} \left(L(X, s)(1 - q^{1-s}) \right).$$

In fact one can explicitly compute

$$\rho_X = q^{(1 - g_X)} \cdot M_{\mathbb{G}_m, X}.$$

Siegel's mass formula

Let $\mathfrak{g}$ be the Lie algebra of G and let $Z(\mathfrak{g}) \subset \mathfrak{g}$ be the center. Let $S^*(\mathfrak{g})^G_{\mathbb{Q}} \subset S^*(\mathfrak{g})_{\mathbb{Q}}$ be the subalgebra of G-invariant polynomials. Then

$$S^*(\mathfrak{g})^G_{\mathbb{Q}} \simeq S^*(Z(\mathfrak{g}))_{\mathbb{Q}} \otimes \mathbb{Q}[e_{d_1}, e_{d_2} \ldots, e_{h(G)}]$$

with $d_i := \deg e_i \geq 2$ being the *fundamental degrees of* G.

Theorem (Siegel's mass formula)

$$M_{G,X} = q^{\dim G(g_X - 1)} \cdot \tau_{G,X} \cdot \rho_X^{\dim Z(\mathfrak{g})} \cdot \prod_i L(X, d_i)$$

with $\tau_{G,X}$ *being the Tamagawa number of* G.

Example ($G = GL_n$)

$$M_{GL_n,X} = q^{n^2(g_X - 1)} \cdot \rho_X \cdot L(X,2) \cdot \ldots \cdot L(X,n)$$

Nonabelian Tsfasman-Vladuţ formula

Theorem (K.)

Let $\mathcal{X} = \{X_i\}_{i\in\mathbb{N}}$ be an asymptotically exact sequence. Then

$$\lim_{i\to\infty} \frac{\log_q M_{G,X_i}}{g_{X_i}} = \dim G + \sum_{n\geq 1} \beta_n(\mathcal{X}) \log_q \left(\frac{q^{n \dim G}}{|G(\mathbb{F}_{q^n})|}\right)$$

The proof is easy after we assume Weil's conjecture on Tamagawa numbers (proved fully by Gaitsgory-Lurie). Idea in the semisimple case: for a given $x \simeq \operatorname{Spec} \mathbb{F}_{q^n} \in |X|$ the product of the corresponding local factors in $\prod_i L(X, d_i)$ is given exactly by $q^{n \dim G} \cdot |G(\mathbb{F}_{q^n})|^{-1}$. In general mix this with the proof of the Tsfasman-Vladuţ formula.

The semistable G-mass

When G is noncommutative the number of G-bundles on X is usually infinite. However for any G there is a nice substack $\mathrm{Bun}^{0,ss}_{G,X} \subset \mathrm{Bun}^0_{G,X}$ which is finite type and so its set of $\mathbb{F}_q$-points is finite. Let

$$M^{ss}_{G,X} := \sum_{\mathcal{E} \in \mathrm{Bun}^{0,ss}_{G,X}(\mathbb{F}_q)} \frac{1}{|\mathrm{Aut}(\mathcal{E})|}$$

be *the semistable G-mass of X*.

Question

Will the asymptotic formula change if we replace $M_{G,X}$ with $M^{ss}_{G,X}$?

Asymptotic formula for $M^{ss}_{G,X}$

Theorem (K.)

Let $\mathcal{X} = \{X_i\}_{i\in\mathbb{N}}$ *be an asymptotically exact sequence. Then*

$$\lim_{i\to\infty} \frac{\log_q M^{ss}_{G,X_i}}{g_{X_i}} = \lim_{i\to\infty} \frac{\log_q M_{G,X_i}}{g_{X_i}}$$

The proof uses certain complicated inversion formula to express $M^{ss}_{G,X}$ through $M_{L,X}$ for all Levi subgroups of G containing a fixed maximal torus: $T \subset G$.

Example ($G = GL_n$)

$$M^{ss}_{GL_n,X} = \sum_{\substack{n_1\geq\ldots\geq n_k>0\\ n_1+\ldots+n_k=n}} q^{(g_X-1)\sum_{i<j} n_i n_j} \cdot \Psi_{n_*}(q) \cdot M_{GL_{n_1},X} \cdot \ldots \cdot M_{GL_{n_k},X}$$

with $\Psi_{n_*}(q) = \prod_{i=1}^{k-1}(1-q^{n_i+n_{i+1}})^{-1}$.

Stacky count vs actual count

Question

Can we replace $M^{ss}_{G,X}$ by the actual count of semistable bundles on X in the asymptotic formula?

Let p be a prime such that $q = p^k$.

Theorem (K.)

Let X be a smooth geometrically connected proper curve over $\mathbb{F}_q$ and assume that $X(\mathbb{F}_{q^n}) \neq \{\varnothing\}$. Assume also that $p \geq h(G)$ where $h(G)$ is the Coxeter number. Then for any semistable G-bundle $\mathcal{E}$ we have

$$|\mathrm{Aut}(\mathcal{E})| < |G(\mathbb{F}_{q^n})|$$

Asymptotics for the number of semistable G-bundles

Let $SS(X) = \{\text{semistable } G\text{-bundles on } X\}$.

Corollary

Assume $p \geq h(G)$. Let $\mathcal{X} = \{X_i\}_{i \in \mathbb{N}}$ be an asymptotically exact sequence and assume that there exists n such that $X_i(\mathbb{F}_{q^n}) \neq \{\varnothing\}$ for $i \gg 0$. Then

$$\lim_{i \to \infty} \frac{\log_q |SS(X_i)|}{g_{X_i}} = \dim G + \sum_{n \geq 1} \beta_n(\mathcal{X}) \log_q \left(\frac{q^{n \dim G}}{|G(\mathbb{F}_{q^n})|} \right).$$

Question

Does one really need these extra assumptions?

Thank you!