Schubert calculus and Gelfand-Zetlin polytopes

(joint work with Valentina Kiritchenko and Vladlen Timorin)

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Outline

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   - Toric varieties and polytopes: a dictionary
   - Pukhlikov–Khovanskii ring
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   - Schubert varieties and Demazure modules
   - Gelfand–Zetlin polytopes
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   - Representing Schubert classes by faces of GZ-polytopes
   - Example of computation in $H^*(GL_3/B, \mathbb{Z})$

Toric varieties and polytopes: a dictionary

Polarized projective toric varieties

$T = (\mathbb{C}^*)^n \overset{\varphi}{\hookrightarrow} X \hookrightarrow \mathbb{P}^N$ X normal, $\operatorname{Im}\varphi$ dense in X	$\leftrightarrow$	integral polytope $P \subset \mathbb{R}^n$, $\#(P \cap \mathbb{Z}^n) = N + 1$
k-dimensional T-orbits in X	$\leftrightarrow$	k-dimensional faces of P
X is smooth	$\leftrightarrow$	P is integrally simple (in each vertex, the primitive vectors form a basis of $\mathbb{Z}^n$)
product in $H^*(X, \mathbb{Z})$ (for X smooth)	$\leftrightarrow$	intersection of faces of P

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Pukhlikov–Khovanskii ring

- Let $P \subset \mathbb{R}^n$ be an integral polytope with r facets $F_1, \dots, F_r$.
- $h_1, \dots, h_r$ — *support numbers*: $h_i = dist(0, F_i)$.
- P is uniquely determined by its normal fan and the support numbers $h_1, \dots, h_r$.
- $vol(P)$ is a polynomial in $h_1, \dots, h_r$.

Definition

$$R_P = \mathbb{Z}[\partial/\partial h_1, \dots, \partial/\partial h_r]/Ann\ vol(P).$$

is called *the Pukhlikov–Khovanskii ring* of P.

Theorem (Pukhlikov–Khovanskii, 1992)

Let $X = X_P$ be a smooth toric variety. Then

$$H^*(X, \mathbb{Z}) \cong R_P$$

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Schubert varieties: definitions

- $G = GL(V)$, $\dim V = n$;
- $B \subset G$ — the group of upper-triangular matrices;
- G/B — full flag variety;
- $G/B = \bigsqcup_{w \in S_n} BwB/B$ — *Schubert decomposition*;
- $X_w = \overline{BwB/B}$ — *Schubert varieties*;
- X_w have many important properties; in particular, the *Schubert cycles* $[X_w]$ form a basis in $H^*(G/B, \mathbb{Z})$.

Main problem of Schubert calculus

How to describe the multiplication in $H^*(G/B, \mathbb{Z})$?

$$[X_v] \cdot [X_w] = \sum c_{vw}^u [X_u]; \qquad c_{vw}^u = ?$$

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More definitions: Demazure modules

- $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}^n$, $\lambda_1 < \cdots < \lambda_n$;
- $V(\lambda)$ — the irreducible G-module with highest weight λ;
- $v_0 \in V(\lambda)$ — highest weight vector.
- $G/B \cong \mathbb{P}(\overline{G \cdot v_0}) \hookrightarrow \mathbb{P}(V(\lambda))$;
- For a given w,

$$X_w = \mathbb{P}(\overline{B \cdot wv_0}) = (G/B) \cap \mathbb{P}(Span\,(B \cdot wv_0)).$$

Definition

A B-module

$$D_w(\lambda) := Span\,(B \cdot wv_0) \subset V(\lambda)$$

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Gelfand–Zetlin polytopes

Consider the following table:

$$\begin{array}{ccccccccccc}
\lambda_1 & & \lambda_2 & & & & \dots & & \lambda_{n-1} & & \lambda_n \\
& x_{11} & & x_{12} & & & \dots & & & x_{1,n-1} & \\
& & x_{21} & & x_{22} & & \dots & & x_{2,n-2} & & \\
& & & \ddots & & & \dots & & & & \\
& & & & & & x_{n-1,1} & & & &
\end{array}$$

where the notation $\begin{array}{ccc} a & & b \\ & c & \end{array}$ means $a \leq c \leq b$.

Definition

For a given $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$, this system of inequalities defines a (bounded) polytope $GZ(\lambda) \subset \mathbb{R}^{n(n-1)/2}$, called the *Gelfand–Zetlin polytope associated with* λ.

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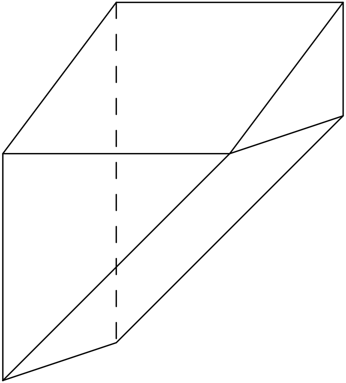
Gelfand-Zetlin polytope for $n = 3$



$$\begin{matrix} a & & b & & c \\ & x & & y & \\ & & z & & \end{matrix}$$

Properties of GZ-polytopes

- $GZ(\lambda)$ can be projected to the weight polytope of $V(\lambda)$:

$$\pi : (x_{11}, \ldots, x_{n-1,1}) \mapsto (x_{11} + \cdots + x_{1,n-1}, x_{21} + \cdots + x_{2,n-2}, \ldots, x_{n-1,1});$$

$$\pi : GZ(\lambda) \to wt(V(\lambda));$$

$$\sum_{x \in GZ(\lambda) \cap \mathbb{Z}^{n(n-1)/2}} \exp \pi(x) = ch\, V(\lambda).$$

- $vol(GZ(\lambda))$ is proportional to the van der Monde determinant:

$$vol(GZ(\lambda)) = const \cdot \prod_{i>j} (\lambda_i - \lambda_j).$$

- Now consider the Pukhlikov–Khovanskii ring of $GZ(\lambda)$.

Theorem (Borel)

$$R_{GZ(\lambda)} = \mathbb{Z}[\partial/\partial\lambda_1, \ldots, \partial/\partial\lambda_n]/Ann\, vol\, GZ(\lambda) \cong H^*(G/B, \mathbb{Z}).$$

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Schubert calculus and Gelfand–Zetlin polytopes

- For toric varieties, products in $H^*(X_P, \mathbb{Z})$ can be computed by intersecting faces in P.
- Goal: compute products in $R_{GZ} \cong H^*(G/B, \mathbb{Z})$ by intersecting faces of $GZ(\lambda)$.
- Problem: $GZ(\lambda)$ is not simple.
- Solution: define an R_{GZ}-module

$$M_{GZ} = \langle [\Gamma_i] \mid \Gamma \text{ is a face of } GZ \rangle / (\text{relations}),$$

  such that

$$R_{GZ} \hookrightarrow M_{GZ}.$$

- Regard $[X_w]$ as elements of M_{GZ}:

$$\text{Schubert cycle } [X_w] \rightsquigarrow \text{set of faces } \Gamma \text{ of } GZ(\lambda).$$

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Main results

- There is a combinatorial procedure mapping *certain* faces of the GZ-polytope (called *rc-faces*) to permutations:

$$\Gamma \mapsto w(\Gamma) \in S_n.$$

Theorem

The following identities hold in M_{GZ}:

$$[X_w] = \sum_{w(\Gamma)=w} [\Gamma],$$

where the sum is taken over all rc-faces corresponding to $w \in S_n$.

Remark

This theorem is formally equivalent to (but not implied by) the theorem of Fomin and An. Kirillov on Schubert polynomials and "pipe dreams" (1994).

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This generalizes a result of Postnikov and Stanley (2009), who showed this for 312-avoiding permutations.

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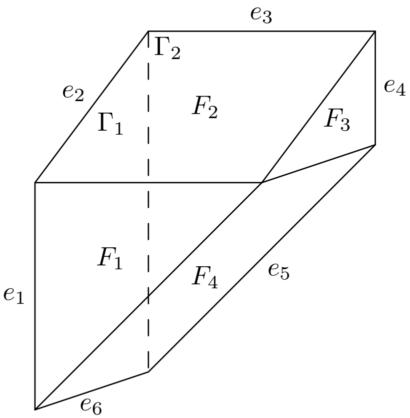
Example of computation in $H^*(GL_3/B, \mathbb{Z})$



Relations in M_{GZ}:

$$[\Gamma_1] = [F_2] + [F_3] = [F_3] + [F_4];$$
$$[\Gamma_2] = [F_1] + [F_2] = [F_1] + [F_4];$$
$$[e_1] = [e_3] = [e_5];$$
$$[e_2] = [e_4] = [e_6].$$

Schubert cycles:

$$[X_{s_2 s_1}] = [\Gamma_1];$$
$$[X_{s_1 s_2}] = [F_1] + [F_4];$$
$$[X_{s_1}] = [e_1];$$
$$[X_{s_2}] = [e_6].$$

$$[X_{s_2 s_1}] \cdot [X_{s_1 s_2}] = [\Gamma_1] \cdot ([F_1] + [F_4]) = [e_1] + [e_6] = [X_{s_1}] + [X_{s_2}].$$

$$[X_{s_2 s_1}]^2 = [\Gamma_1] \cdot ([F_3] + [F_4]) = [e_6] = [X_{s_2}].$$

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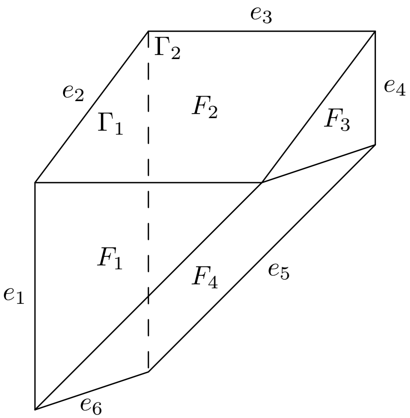

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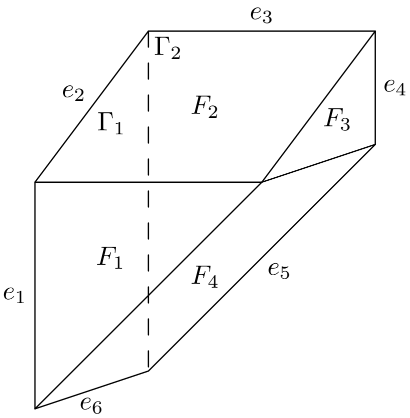
$$[X_{s_2 s_1}] \cdot [X_{s_1 s_2}] = [\Gamma_1] \cdot ([F_1] + [F_4]) = [e_1] + [e_6] = [X_{s_1}] + [X_{s_2}].$$

$$[X_{s_2 s_1}]^2 = [\Gamma_1] \cdot ([F_3] + [F_4]) = [e_6] = [X_{s_2}].$$



Relations in M_{GZ}:

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Schubert cycles:

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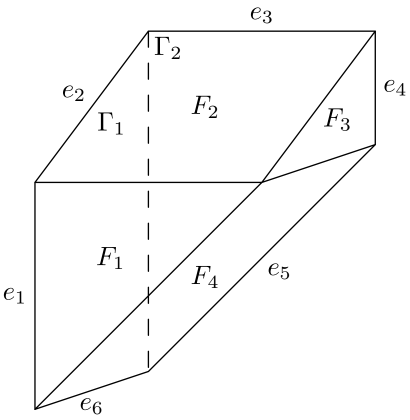
Example of computation in $H^*(GL_3/B, \mathbb{Z})$



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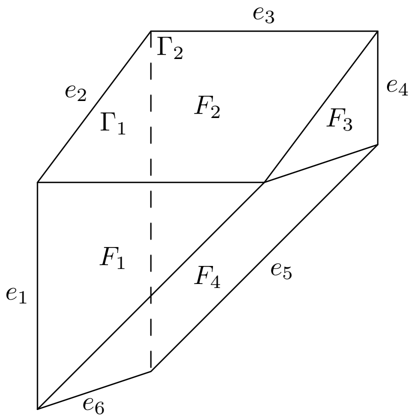

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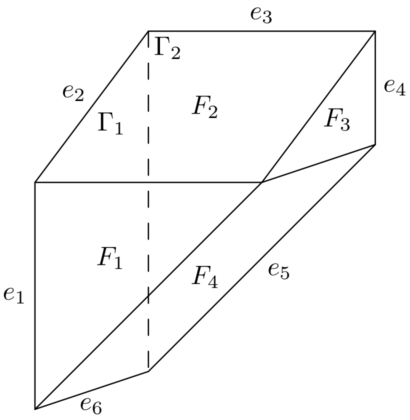

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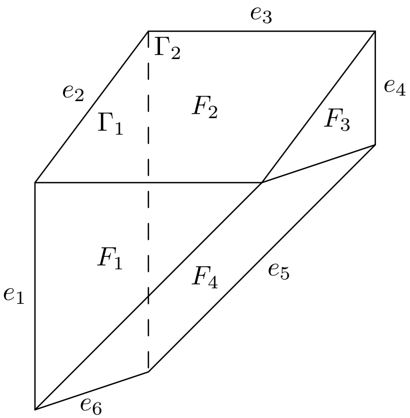

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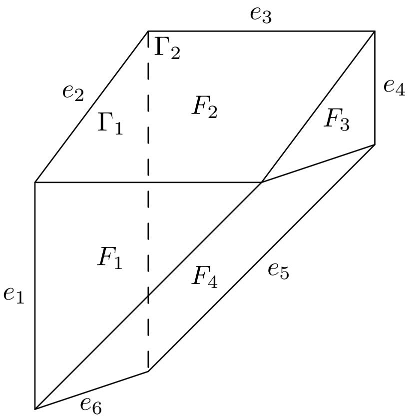

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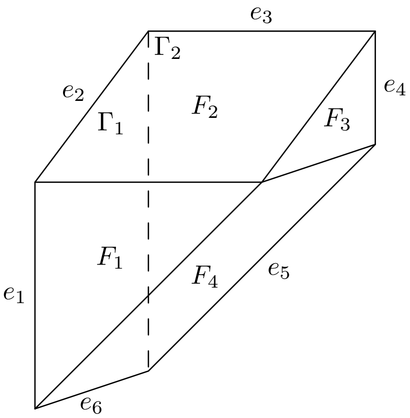

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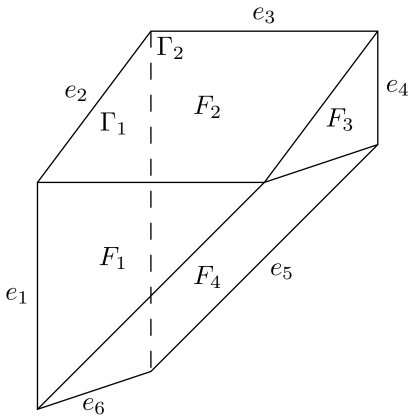

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Slogan

Multiplying Schubert cycles

=

intersecting faces of the GZ-polytope