Schubert polynomials and pipe dreams

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Topology of Torus Actions and Applications to Geometry and Combinatorics
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Outline

1. General definitions
   - Flag varieties
   - Schubert varieties and Schubert polynomials
   - Pipe dreams and Fomin–Kirillov theorem

2. Numerology of Schubert polynomials
   - Permutations with many pipe dreams
   - Catalan numbers and Catalan–Hankel determinants

3. Combinatorics of Schubert polynomials
   - Pipe dream complexes
   - Generalizations for other Weyl groups

4. Open questions
Flag varieties

- $G = \text{GL}_n(\mathbb{C})$
- $B \subset G$ upper-triangular matrices
- $Fl(n) = \{ V_0 \subset V_1 \subset \cdots \subset V_n \mid \text{dim } V_i = i \} \cong G/B$

**Theorem (Borel, 1953)**

$$\mathbb{Z}[x_1, \ldots, x_n]/(x_1 + \cdots + x_n, \ldots, x_1 \ldots x_n) \cong H^*(G/B, \mathbb{Z}).$$

This isomorphism is constructed as follows:

- $\mathcal{V}_1, \ldots, \mathcal{V}_n$ tautological vector bundles over $G/B$;
- $\mathcal{L}_i = \mathcal{V}_i/\mathcal{V}_{i-1}$ ($1 \leq i \leq n$);
- $x_i \mapsto -c_1(\mathcal{L}_i)$;
- The kernel is generated by the symmetric polynomials without constant term.
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Schubert varieties

- \( G/B = \bigsqcup_{w \in S_n} B^- wB/B \) — Schubert decomposition;
- \( X^w = B^- wB/B \), where \( B^- \) is the opposite Borel subgroup;
- \( H^*(G/B, \mathbb{Z}) \cong \bigoplus_{w \in S_n} \mathbb{Z} \cdot [X^w] \) as abelian groups.

**Question**

Are there any “nice” representatives of \([X^w]\) in \( \mathbb{Z}[x_1, \ldots, x_n]\)?

**Answer: Schubert polynomials**

- \( w \in S_n \leadsto \circ_{w}(x_1, \ldots, x_{n-1}) \in \mathbb{Z}[x_1, \ldots, x_n]; \)
- \( \circ_{w} \mapsto [X^w] \in H^*(G/B, \mathbb{Z}) \) under the Borel isomorphism;
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Question

Are there any “nice” representatives of $[X^w]$ in $\mathbb{Z}[x_1, \ldots, x_n]$?

Answer: Schubert polynomials

- $w \in S_n \mapsto \mathcal{S}_w(x_1, \ldots, x_{n-1}) \in \mathbb{Z}[x_1, \ldots, x_n]$;
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- \( w \in S_n \; \mapsto \; S_w(x_1, \ldots, x_{n-1}) \in \mathbb{Z}[x_1, \ldots, x_n] \);
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Pipe dreams

Let \( w \in S_n \). Consider a triangular table filled by \( \uparrow \) and \( \nearrow \), such that:

- the strands intertwine as prescribed by \( w \);
- no two strands cross more than once (reduced pipe dream).

Pipe dreams for \( w = (1432) \)

Pipe dream \( P \) \( \mapsto \) monomial \( x^{d(P)} = x_1^{d_1} x_2^{d_2} \ldots x_{n-1}^{d_{n-1}} \),

\( d_i = \#\{\uparrow\}'s in the \( i \)-th row\)

\[ x_2^2 x_3 \quad x_1 x_2 x_3 \quad x_1^2 x_3 \quad x_1 x_2^2 \quad x_1^2 x_2 \]
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\end{align*}
\]

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\[
\mathcal{G}_w(x_1, \ldots, x_{n-1}) = \sum_{w(P) = w} x^{d(P)},
\]

where the sum is taken over all reduced pipe dreams \( P \) corresponding to \( w \).

Example

\[
\mathcal{G}_{1432}(x_1, x_2, x_3) = x_2^2 x_3 + x_1 x_2 x_3 + x_1^2 x_3 + x_1 x_2^2 + x_1^2 x_2.
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Corollary

\[
\mathcal{G}_w(1, \ldots, 1) = \#\{P \mid \text{pipe dream } P \text{ corresponds to } w\}.
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Pipe dreams and Schubert polynomials


Let $w \in S_n$. Then

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Pipe dreams and torus actions

Toric degeneration of a flag variety (N. Gonciulea, V. Lakshmibai)

\[ Fl(n) \to \tilde{Fl}(n) \]

- \( \tilde{Fl}(n) \) is a \textit{singular} (but still irreducible!) toric variety.
- It corresponds to Gelfand-Zetlin polytope \( GZ(n) \).

Degenerate Schubert varieties (A. Knutson, M. Kogan, E. Miller)

\[ X^w \to \tilde{X}^w \subset \tilde{Fl}(n) \]

- \( \tilde{X}^w \) may be reducible!
- Its irreducible components are indexed by the pipe dreams corresponding to \( w \).
- \( \mathcal{G}_w(1, \ldots, 1) \) “measures how singular \( X^w \) is”.

Evgeny Smirnov (HSE & Labo Poncelet)
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Permutations with the maximal number of pipe dreams

How many pipe dreams can a permutation have?

Find \( w \in S_n \), such that \( G_w(1, \ldots, 1) \) is maximal.

Answers for small \( n \)

- \( n = 3 \): \( w = (132), G_w(1) = 2 \);
- \( n = 4 \): \( w = (1432), G_w(1) = 5 \);
- \( n = 5 \): \( w = (15432) \) and \( w = (12543), G_w(1) = 14 \);
- \( n = 6 \): \( w = (126543), G_w(1) = 84 \);
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Definition

\( w \in S_n \) is a Richardson permutation, if for \((k_1, \ldots, k_r), \sum k_i = n\),

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    w = \begin{pmatrix}
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Counting pipe dreams of Richardson permutations

Let $w_{k,m}^0 = \begin{pmatrix} 1 & 2 & \ldots & k & k+1 & \ldots & k+m \\ 1 & 2 & \ldots & k & k+m & \ldots & k+1 \end{pmatrix}$.

**Theorem (A. Woo)**

Let $w = w_{1,m}^0$. Then $G_w(1) = Cat(m)$.

**Theorem (S. Fomin, An. Kirillov)**

Let $w = w_{k,m}^0$. Then $G_w(1)$ is equal to the number of “Dyck plane partitions of height $k$”, i.e., subdiagrams of the prism of height $k$ and side length $m$. 
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Let $w^0_{k,m} = \begin{pmatrix} 1 & 2 & \ldots & k & k+1 & \ldots & k+m \\ 1 & 2 & \ldots & k & k+m & \ldots & k+1 \end{pmatrix}$.

**Theorem (A. Woo)**

Let $w = w^0_{1,m}$. Then $\mathcal{G}_w(1) = \text{Cat}(m)$.

**Theorem (S. Fomin, An. Kirillov)**

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Determinantal formulas for Schubert polynomials

**Theorem (G. Merzon, E. S.)**

Let $w = w_{k,m}^0$. Then the following “Jacobi–Trudi type” formula holds:

$$
\frac{\mathcal{S}_w(x_1, \ldots, x_{m+k-1})}{x_1^m \cdots x_k^m x_{k+1}^{m-1} \cdots x_{m+k-1}} = \det \left( \frac{\mathcal{S}_{w_1, m+i+j}^0(x_{i+1}, \ldots, x_{m+i+j-1})}{x_i^{m+j-1} x_{i+1}^{m+j-2} \cdots x_{m+i+j-1}} \right)_{i,j=0}^{k-1}
$$

**Corollary**

$\mathcal{S}_w(1)$ is equal to a $(k \times k)$ Catalan–Hankel determinant:

$$
\mathcal{S}_w(1) = \det \begin{pmatrix}
\text{Cat}(m) & \text{Cat}(m+1) & \cdots & \text{Cat}(m+k-1) \\
\text{Cat}(m+1) & \text{Cat}(m+2) & \cdots & \text{Cat}(m+k) \\
\vdots & \vdots & \ddots & \vdots \\
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$\mathcal{S}_w(1)$ is equal to a $(k \times k)$ Catalan–Hankel determinant:

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To each permutation $w \in S_n$ one can associate a shellable CW-complex $PD(w)$;

- 0-dimensional cells $\leftrightarrow$ reduced pipe dreams for $w$;
- higher-dimensional cells $\leftrightarrow$ non-reduced pipe dreams for $w$;
- $PD(w) \cong B^\ell$ or $S^\ell$, where $\ell = \ell(w)$. 
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Theorem (probably folklore? also cf. V. Pilaud)

Let $w = w_{1,n}^0 = (1, n+1, n, \ldots, 3, 2) \in S_{n+1}$ be as in Woo’s theorem. Then $PD(w)$ is the Stasheff associahedron.
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What about $PD(w)$ for other Richardson elements $w$?

- $w = w_{1,n}^0 = (1, n + 1, n, \ldots, 3, 2)$ associahedron;
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- $w = w_{n,3}^0 = (1, 2, \ldots, n, n + 3, n + 2, n + 1)$ dual cyclic polytope $(C(2n + 3, 2n))^\vee$.
- $w = w_{k,n}^0$ ???
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$C(n, d) = \text{Conv}((t_i, t_i^2, \ldots, t_i^d))_{i=1}^n \subset \mathbb{R}^d$. 
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Generalization: other Weyl groups

- $G$ semisimple group, $W$ its Weyl group;
- The longest element in $W$ is denoted by $w^0$;
- $P \subset G$ parabolic subgroup, $P = L \rtimes U$ its Levi decomposition.
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- For $W = S_n$, that is exactly our previous definition of Richardson elements.
- Fix a reduced decomposition $w^0$ of the longest element $w^0 \in W$.
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Theorem

Let $W$ be of type $C_n$, generated by $s_1, \ldots, s_n$, where $s_1$ corresponds to the longest root $\alpha_1$. Consider a Richardson element $w = (s_1 s_2 \ldots s_{n-1})^{n-1}$. Then $PD(w)$ is a cyclohedron.
Questions about $PD(w)$

- Is it true that $PD(w)$ is always a polytope?
- At least, is it true when $w$ is a Richardson element?
- If yes, what is the combinatorial meaning of this polytope?
- Possible answer: associahedra, cyclohedra etc. are examples of 2-truncated cubes (cf. V. Buchstaber’s works). Is it true that $PD(w)$ are 2-truncated cubes?
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