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# ORBITES D'UN SOUS-GROUPE DE BOREL DANS LE PRODUIT DE DEUX GRASSMANNIENNES

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*Soutenance à Grenoble le 29 octobre 2007 devant le jury :*

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# Contents

# Chapitre 1

# Introduction

Dans la présente thèse, nous étudions les orbites d'un sous-groupe de Borel qui agit dans le produit direct de deux variétés grassmanniennes.

Soit $V$ un espace vectoriel de dimension $n$ sur un corps algébriquement clos. L'ensemble des sous-espaces vectoriels de $V$ de dimension fixée $k < n$ peut être muni d'une structure de variété algébrique. Une telle variété est dite *grassmannienne* ; on va la noter par $\mathrm{Gr}(k, V)$. Elle est homogène pour l'action du groupe linéaire général $\mathrm{GL}(V)$ ; on peut également s'intéresser à l'action d'un sous-groupe de Borel $B \subset \mathrm{GL}(V)$.

La décomposition de $\mathrm{Gr}(k, V)$ en orbites de $B$, ou *décomposition de Schubert*, a beaucoup de propriétés intéressantes. Elle est apparue à la fin du 19ième siècle, pour les besoins de la géométrie énumérative. On sait paramétrer les orbites (*cellules de Schubert*) et décrire leurs adhérences (*variétés de Schubert*). La géométrie de ces dernières a été beaucoup étudiée également. En particulier, on peut démontrer que ces adhérences sont normales et de Cohen–Macaulay, et que leurs singularités sont rationnelles (voir, par exemple, [Br2] et [BrKu]). On connaît aussi une résolution de leurs singularités, apparue dans les travaux de Bott–Samelson, Demazure et Hansen ([BS], [Dem], [Han]), et on sait décrire leur lieu singulier.

La notion de décomposition de Schubert admet une généralisation directe aux *variétés de drapeaux* $G/P$, où $G$ est un groupe algébrique réductif et connexe, et $P$ est un sous-groupe parabolique. Le cas « extrême » (et en un certain sens le plus compliqué) est celui où $P = B$ ; dans ce cas, la variété $G/B$ s'appelle *la variété de drapeaux complets*. Pour les variétés de drapeaux, on peut accomplir le même programme que pour les grassmanniennes ; cependant, la réponse à la question sur le lieu singulier d'une variété de Schubert n'a été obtenue qu'en 2000, presque simultanément, dans les articles de L. Manivel [Man2], A. Cortez [Cor], S. Billey et G. Warrington [BiW], et, enfin, de C. Kassel, A. Lascoux et C. Reutenauer [KLR].

La motivation pour le présent travail est de double nature. Tout d'abord, en 1998, P. Magyar, J. Weyman et A. Zelevinsky ont considéré le problème suivant (cf. [MWZ]). Soit $G = \mathrm{GL}(V)$ ; ils prennent une *variété de drapeaux multiple*, c'est-à-dire, le produit direct d'un certain nombre $r$ de variétés de drapeaux (complets ou partiels) $G/P_i$, et considèrent l'action diagonale de $G$ dans

$$G/P_1 \times \cdots \times G/P_r.$$

Dans quelles situations (sous quelles conditions portant sur $r$ et $P_1, \ldots, P_r$) cette action n'a-t-elle qu'un nombre fini d'orbites ? (De telles variétés sont dites *de type fini*).

Evidemment, ceci équivaut au fait que l'action de $P_1$ dans $G/P_2 \times \cdots \times G/P_r$ n'a qu'un nombre fini d'orbites. Si $P_1 = B$, c'est la définition d'une variété sphérique. Pour $G$ semisimple quelconque, la réponse à cette question est connue dans le cas où $P_2$, $P_3$ sont paraboliques maximaux ; ce résultat est due à P. Littelmann dans [Lit].

Dans le cas où $G = \mathrm{GL}(V)$ et $P_1$ est quelconque, [MWZ] démontrent que c'est possible seulement lorsque $r \leq 3$, et ils obtiennent une réponse complète en termes de certains carquois, dont la classification est très proche (mais différente) de la classification des carquois de type de représentation fini (cf. [Kac]). En outre, dans ces situations ils donnent une description combinatoire de ces orbites, et obtiennent certains résultats partiels sur leurs adhérences.

En particulier, la classification des variétés de drapeaux multiples de type fini comprend des séries $A$, $D$ et $E$. La série $A$ correspond au cas $r = 2$, c'est-à-dire à l'action de $P_1$ dans $G/P_2$. Comme cas particulier, on obtient l'action de $B$ dans $G/P$ : c'est la situation classique de la décomposition de Schubert d'une variété de drapeaux.

La série $D$ (qui est dans un certain sens le cas « suivant ») correspond à l'action de $G$ dans $G/P_1 \times G/P_2 \times G/P_3$, où $P_2$, $P_3$ sont des sous-groupes paraboliques maximaux, ou encore à l'action de $P_1$ dans $\mathrm{Gr}(k, V) \times \mathrm{Gr}(l, V)$. En particulier, cela contient le cas le plus intéressant, où $P_1 = B$.

Ainsi, il serait intéressant de connaître les réponses aux questions analogues à celles sur les variétés de drapeaux. Comment peut-on paramétrer les orbites de $B$ agissant dans $\mathrm{Gr}(k, V) \times \mathrm{Gr}(l, V)$ ? Quelles sont les orbites contenues dans l'adhérence d'une orbite donnée ? Que peut-on dire sur la géométrie de ces adhérences ? Quand sont-elles lisses ? Si elles sont singulières, comment peut-on résoudre leurs singularités ? Comment caractériser leur lieu singulier, leurs singularités génériques ?... Dans la présente thèse, nous répondrons à une partie de ces questions.

Une autre série de questions concerne les anneaux de coordonnées des variétés de Schubert et de leurs analogues. Hodge [Hod] a montré que, si on prend le plongement de Plücker

$$\mathrm{Gr}(k, V) \hookrightarrow \mathbb{P} \bigwedge^k V,$$

on peut obtenir les variétés de Schubert comme les sections de la grassmannienne par certains sous-espaces projectifs de $\mathbb{P} \bigwedge^k V$. Ceci est vrai même au sens « schématique », c'est-à-dire, les équations de ces sous-espaces engendrent les idéaux des variétés de Schubert dans l'anneau des coordonnées homogènes de la grassmannienne. Cet énoncé est essentiel pour décrire ces anneaux de coordonnées homogènes. Il serait intéressant de généraliser ces résultats à notre cas. Conjecturalement, les adhérences des $B$-orbites peuvent être obtenues comme sections linéaires du produit de deux grassmanniennes, plongé dans l'espace projectif par Plücker–Segre :

$$\mathrm{Gr}(k, V) \times \mathrm{Gr}(l, V) \hookrightarrow \mathbb{P} \bigwedge^k V \times \mathbb{P} \bigwedge^l V \hookrightarrow \mathbb{P}(\bigwedge^k V \otimes \bigwedge^l V).$$

Par contre, la description des anneaux des coordonnées homogènes de ces adhérences nous paraît un problème assez difficile.

Une motivation supplémentaire pour l'étude des $B$-orbites dans $\mathrm{Gr}(k, V) \times \mathrm{Gr}(l, V)$ vient des travaux récents de G. Bobiński et G. Zwara. Ils ont découvert des liens intéressants entre représentations des carquois et variétés de Schubert. Dans leurs articles [BZ1], [BZ2], ils démontrent les résultats suivants. Les singularités des adhérences des orbites dans les représentations des carquois de type $A_n$ sont équivalentes aux singularités des variétés de Schubert dans la variété des drapeaux complets ; et les singularités des adhérences des orbites dans les représentations des carquois de type $A_n$ et $D_n$ sont équivalentes aux singularités des variétés de Schubert dans les produits de deux grassmanniennes.

Cette thèse est formée de deux parties. Dans la première, nous considérons le groupe $\mathrm{GL}(V)$ qui agit dans une variété de drapeaux multiple de type $D$,

$$X = \mathrm{GL}(V)/P \times \mathrm{Gr}(k, V) \times \mathrm{Gr}(l, V),$$

où $P \subset \mathrm{GL}(V)$ est un sous-groupe parabolique. Autrement dit, nous considérons les triplets qui consistent en deux sous-espaces et un drapeau partiel dans $V$, à l'action de $\mathrm{GL}(V)$ près. On s'intéresse à la question suivante : quand un tel triplet peut-il être dégénéré en un autre triplet ? Nous obtenons un critère pour cela, en termes des dimensions d'intersections et de sommes de certains sous-espaces. En outre, nous décrivons les dégénérescences minimales.

Nos méthodes n'utilisent que de l'algèbre lineaire et de la combinatoire des graphes. Notre description combinatoire de ces triplets utilise le langage des carquois d'Auslander–Reiten.

Dans la deuxième partie, nous nous intéressons au cas où $P_1 = B$, c'est-à-dire, à l'action d'un sous-groupe de Borel dans le produit de deux grassmanniennes. Nous introduisons une autre description combinatoire des orbites de cette action, qui nous paraît être mieux adaptée à ce cas particulier. Cela nous permet de décrire l'ordre partiel sur ces orbites, obtenu par l'action des sous-groupes paraboliques minimaux (« ordre faible »). Cet ordre partiel nous permet de construire des résolutions des singularités des adhérences des orbites « à la Bott–Samelson ».

À la fin de la deuxième partie, nous mettons en lumière certains liens intéressants et inattendus entre la combinatoire de l'ordre donné par les adhérences des $B$-orbites dans $\mathrm{Gr}(k, V) \times \mathrm{Gr}(l, V)$, et l'ordre analogue sur les orbites du sous-groupe de Borel $B \subset \mathrm{GL}(V)$ qui agit par conjugaison dans les matrices triangulaires supérieures $A \in \mathrm{Mat}(V)$, telles que $A^2 = 0$. Ce dernier ordre apparaît dans des travaux récents d'Anna Melnikov ([Mel1], [Mel2]) : ces $B$-orbites sont parametrées par les permutations involutives. Dans notre cas, les $B$-orbites dans une $(B \times B)$-orbite donnée peuvent aussi être paramétrées par un certain sous-ensemble des permutations involutives ; on obtient ainsi un ordre partiel sur ce sous-ensemble. Miraculeusement, cet ordre coïncide avec celui de Melnikov, bien qu'ils apparaissent dans des situations très différentes.

# Chapter 2

# Bruhat order for two subspaces and a flag

## 2.1 Introduction

In this chapter we will consider certain configurations of subspaces in an $n$-dimensional vector space $V$ over an algebraically closed field $\mathbb{K}$. These configurations $(U, W, V_\bullet)$ consist of two subspaces $U$ and $W$ of $V$ of fixed dimensions $k$ and $l$, and a partial flag $V_\bullet = (V_{d_1} \subset V_{d_2} \subset \dots \subset V_{d_m} = V)$, where $\dim V_{d_i} = d_i$.

Our goal is to describe such configurations up to a linear change of coordinates in $V$ and the ways how configurations degenerate. In other words, we consider the direct product $X = \mathrm{Gr}(k, V) \times \mathrm{Gr}(l, V) \times \mathrm{Fl}_{\mathbf{d}}(V)$ of two Grassmannians and a flag variety of type $\mathbf{d} = (d_1, \dots, d_m)$ in $V$, the group $\mathrm{GL}(V)$ acting diagonally on this variety, and describe orbits of this action and the inclusion relations between their closures.

One can easily show that the number of these orbits is finite. Such a product $X$ of flag varieties is said to be a *multiple flag variety of finite type*. In the paper [MWZ] Magyar, Weyman and Zelevinsky list all such varieties and describe a way of indexing the orbits of the general linear group acting on them.

They also obtain a necessary condition for the closure of a $\mathrm{GL}(V)$-orbit on such a variety to contain another $\mathrm{GL}(V)$-orbit. This condition comes from the results by C. Riedtmann [Rie] on degenerations of representations of quivers.

It is not always clear whether this condition provides a criterion. As is mentioned in [MWZ], this is so in several cases, as follows from some general results on quivers due to K. Bongartz ([Bon1, §4], [Bon2, §5.2]). One more case is treated in the paper [Mag] by P. Magyar, where a similar criterion is obtained for configurations of two flags and a line. Magyar's approach is elementary; it uses only combinatorics and linear algebra.

The case $X = \mathrm{Gr}(k, V) \times \mathrm{Gr}(l, V) \times \mathrm{Fl}_{\mathbf{d}}(V)$ we are interested in is covered by the results of Bongartz. However, in this case we provide a simpler criterion for a configuration to degenerate to another one, in terms of dimensions of certain subspaces obtained from $U$, $W$, and $V_\bullet$ by taking sums and intersections, and we give a completely elementary proof of this result. Along with this criterion, we obtain an explicit description of minimal degenerations.

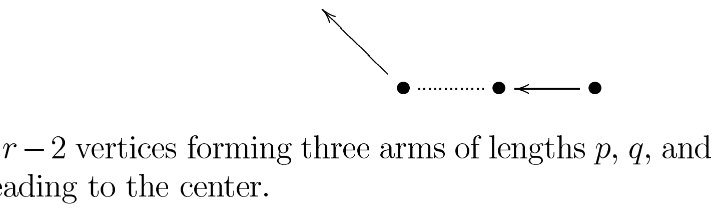
For this, we follow in general the approach of [Mag]. But the combinatorics we use for indexing the orbits in $X$ is quite different.

**The structure of this chapter** is as follows. In Section 2.2, we recall some results from [MWZ] concerning classification of orbits in an arbitrary multiple flag variety of finite type. In Section 2.3, we introduce an indexing of orbits of $\mathrm{GL}(V)$ in $X$ by multisets of vertices of a certain quiver. Section 2.4 is devoted to defining three partial orders on this set of orbits: the first order is given by degenerations of orbits, the second one is given by conditions on dimensions of certain subspaces, and the definition of the third order is purely combinatorial, involving the description of orbits from Section 2.3. The principal result of this chapter states that the three orders are the same; this is proved in Section 2.5. The last subsection of Section 2.5 is devoted to the proof of the theorem on minimal degenerations, also stated in the beginning of this section.

## 2.2 Orbits and representations: a general approach

In this section, we consider the problem of classifying orbits of the general linear group in a multiple flag variety in its general setting, after [MWZ].

Let $V$ be an $n$-dimensional vector space over a field $\mathbb{K}$, which we suppose to be arbitrary throughout this and the next Section. Let $Q_{p,q,r}$ be the three-arm star-like quiver of the following form:



with $p+q+r-2$ vertices forming three arms of lengths $p$, $q$, and $r$, and with all arrows leading to the center.

Let $\mathcal{R}\mathrm{ep}(Q_{p,q,r})$ denote the category of representations of this quiver. Magyar, Weyman, and Zelevinsky [MWZ] consider the full subcategory $\mathcal{R}\mathrm{ep}\mathcal{E}\mathrm{mb}(Q_{p,q,r})$ in $\mathcal{R}\mathrm{ep}(Q_{p,q,r})$ whose objects are those representations such that all the linear maps corresponding to the arrows are embeddings. The subcategory $\mathcal{R}\mathrm{ep}\mathcal{E}\mathrm{mb}(Q_{p,q,r})$ is closed under taking direct sums and subobjects (but not quotients!), so one can introduce the notions of decomposition into direct sums and indecomposable objects. The uniqueness of a decomposition into a sum of indecomposables is guaranteed by the Krull–Schmidt theorem (see, for instance, [Ba]).

In particular, the set of indecomposables $\mathrm{Ind}(\mathcal{R}\mathrm{ep}\mathcal{E}\mathrm{mb}(Q_{p,q,r}))$ forms a subset of $\mathrm{Ind}(\mathcal{R}\mathrm{ep}(Q_{p,q,r}))$, since it is closed under taking subobjects.

Fix a dimension vector $(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (a_1, \dots, a_p; b_1, \dots, b_q; c_1, \dots, c_r)$, where $a_p = b_q = c_r$, and take a representation

$$\underline{V} = (V_1, \dots, V_p; V_1', \dots, V_q'; V_1'', \dots, V_r'') \in \mathcal{R}\mathrm{ep}\mathcal{E}\mathrm{mb}(Q_{p,q,r})$$

with dimension vector $(\mathbf{a}, \mathbf{b}, \mathbf{c})$. This representation can be considered as a triple of partial flags in $V = V_p = V_q' = V_r''$ with the given depths and dimension vectors, defined up to $\mathrm{GL}(V)$-action. And, vice versa, any such triple of flags provides a representation from $\mathcal{R}\mathrm{ep}\mathcal{E}\mathrm{mb}(Q_{p,q,r})$. So, the orbits of the diagonal action of $\mathrm{GL}(V)$ on the direct product of three partial flag varieties

$$\mathrm{Fl}_{(\mathbf{a},\mathbf{b},\mathbf{c})}(V) = \mathrm{Fl}_{\mathbf{a}}(V) \times \mathrm{Fl}_{\mathbf{b}}(V) \times \mathrm{Fl}_{\mathbf{c}}(V)$$

are in one-to-one correspondence with the elements of $\mathcal{R}\mathrm{ep}\mathcal{E}\mathrm{mb}(Q_{p,q,r})$ with dimension vector $(\mathbf{a}, \mathbf{b}, \mathbf{c})$.

Kac's theorem on indecomposable representations of a quiver (cf. [Kac]) implies that the category $\mathcal{R}\mathrm{ep}\mathcal{E}\mathrm{mb}(Q_{p,q,r})$ has the following property: there exists at most one indecomposable object with a given dimension vector. This means that the $\mathrm{GL}(V)$-orbits in $\mathrm{Fl}_{(\mathbf{a},\mathbf{b},\mathbf{c})}(V)$ correspond to the possible decompositions of the dimension vector $(\mathbf{a}, \mathbf{b}, \mathbf{c})$:

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \sum \underline{\dim} I_\alpha,$$

where $I_\alpha$ are indecomposable objects. So, if the number of $\mathrm{GL}(V)$-orbits in $\mathrm{Fl}_{(\mathbf{a},\mathbf{b},\mathbf{c})}(V)$ is finite (in this case this multiple flag variety is said to be *of finite type*), the classification of orbits is thus reduced to a purely combinatorial problem.

So, knowing all the indecomposable objects in the category $\mathcal{R}\mathrm{ep}\mathcal{E}\mathrm{mb}(Q_{p,q,r})$ for a given quiver $Q_{p,q,r}$ allows us to describe the $\mathrm{GL}(V)$-orbits in the multiple flag variety $\mathrm{Fl}_{(\mathbf{a},\mathbf{b},\mathbf{c})}(V)$ for an arbitrary dimension vector $(\mathbf{a}, \mathbf{b}, \mathbf{c})$. The complete list of all multiple flag varieties of finite type and indecomposable objects in the corresponding categories is given in [MWZ, Theorem 2.3].

In particular, this list includes quivers $Q_{p,q,1}$ (type $A$) and $Q_{p,2,2}$ (type $D$). The multiple flag varieties corresponding to these two series of quivers will be the main objects of our interest throughout this paper.

## 2.3 Combinatorial enumeration of objects with a specific dimension vector

Consider the Auslander–Reiten quiver (AR-quiver) for the category $\mathcal{R}\mathrm{ep}(Q)$. Its vertices correspond to indecomposable objects, and arrows represent "minimal" morphisms between indecomposables — i.e., morphisms

$$f\colon I \to I'$$

that cannot be presented as a composition of two morphisms

$$f = g \circ h\colon I \xrightarrow{h} I'' \xrightarrow{g} I',$$

where $I$, $I'$ and $I''$ are pairwise non-isomorphic indecomposables.

Having the AR-quiver for $\mathcal{R}\mathrm{ep}(Q)$, consider its subquiver defined as follows: we take all vertices that correspond to indecomposable objects from $\mathcal{R}\mathrm{ep}\mathcal{E}\mathrm{mb}(Q)$ and all arrows between these vertices. This is the Auslander–Reiten quiver for the category $\mathcal{R}\mathrm{ep}\mathcal{E}\mathrm{mb}(Q)$. We will refer to *the latter* quiver (not to the former) as to the AR-quiver for the quiver $Q$; it will be denoted by $AR(Q)$.

For background on Auslander–Reiten quivers, see the book [ARS].

Now let us pass to the explicit study of cases $A$ and $D$.

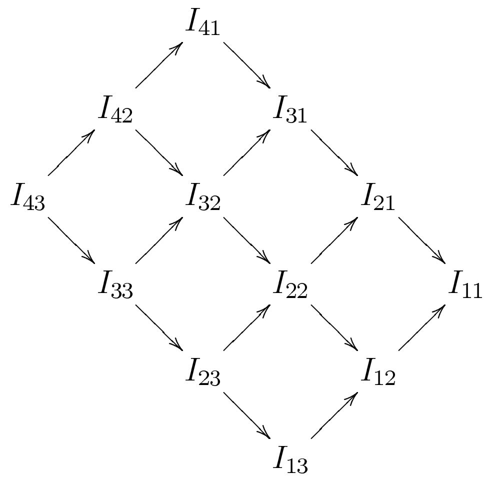
### 2.3.1 Case A: two flags

Let $Q$ equal $Q_{p,q,1}$. That is, $Q$ is a linear quiver with $p+q-1$ vertices and arrows oriented as follows: $\bullet \to \bullet \cdots \bullet \to \overset{p}{\bullet} \leftarrow \bullet \cdots \bullet \leftarrow \bullet$

For all the indecomposable injective representations of this quiver, the dimension of each subspace occuring in them equals 0 or 1. These representations are as follows:

$$I_{ij} = \left( 0 \cdots 0 \to \mathbb{K} \cdots \mathbb{K} \to \mathbb{K} \leftarrow \mathbb{K} \cdots \mathbb{K} \leftarrow 0 \cdots 0 \right),$$

where the first nonzero space has number $i$, the last — the number $p+q-j$, and $i \in [1,p]$, $j \in [1,q]$. So, there are $pq$ non-isomorphic indecomposable objects.




$(A)$

Figure 2.1: AR-quiver for $Q_{4,3,1}$ (type $A$)

The AR-quiver for such a quiver is a rectangle of size $(p \times q)$. The example where $p = 4$, $q = 3$ is given on Figure 2.1.

Given an object $F \in \mathcal{R}\text{ep}\mathcal{E}\text{mb}(Q)$, we will say that an indecomposable object $I$ *occurs* in $F$, if it occurs with nonzero multiplicity in the decomposition of $F$ into indecomposables.

**Proposition 2.1.** *Let $F$ be an object in $\mathcal{R}\text{ep}\mathcal{E}\text{mb}(Q_{p,q,1})$ corresponding to a configuration of two flags, such that $\underline{\dim} F = (a_1, \dots, a_p; b_1, \dots, b_q)$, $a_p = b_q = n$, and let $F = \bigoplus I_{ij}$ be its decomposition into a sum of indecomposable objects. Then there are $n$ summands. On each path formed by the elements $I_{i\alpha}$ with $i$ fixed, there are exactly $a_i - a_{i-1}$ indecomposable objects, counted with multiplicities, occuring in $F$. On each path formed by the elements $I_{\alpha j}$ with $j$ fixed, there are exactly $b_j - b_{j-1}$ indecomposable objects occuring in $F$. (We set formally $a_0 = b_0 = 0$).*

*Proof.* Since all the indecomposable summands are one-dimensional, there are exactly $n$ of them. As we have seen before,

$$\underline{\dim} I_{ij} = (\underbrace{0, \dots, 0,}_{i-1 \text{ entry}} 1, \dots, 1, \dots, 1, \underbrace{0, \dots, 0,}_{j-1 \text{ entry}}).$$

The resulting dimension is the sum of dimensions of the indecomposable objects occuring in $F$:

$$\underline{\dim} F = \sum \underline{\dim} I_{ij}.$$

Denote the dimension vector of a representation by

$$(\mathbf{a}', \mathbf{b}') = (a'_1, \dots, a'_p; b'_1, \dots, b'_q).$$

For a given $i$, the objects $I_{ij}$ are characterized by the equality $a'_i = a'_{i-1} + 1$. For all other indecomposable objects, $a'_i = a'_{i-1}$. This means that there are exactly $a_i - a_{i-1}$ objects of the form $I_{ij}$ occuring in $F$.

The fact that $F$ contains exactly $b_j - b_{j-1}$ summands of the form $I_{ij}$ for a given $j$ is proved similarly. □

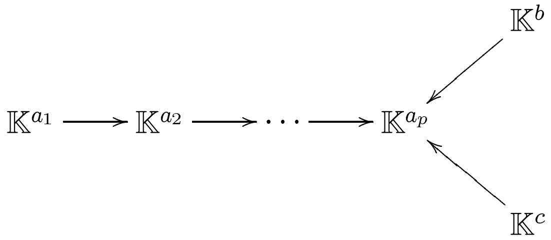
**Corollary 2.2.** *Consider the particular case* $p = q = n$, $(\mathbf{a}, \mathbf{b}) = (1, 2 \dots, n; 1, 2 \dots, n)$. *Then for any two summands* $I_{ij}$ *and* $I_{i'j'}$ *occuring in* $F$*, we have* $i \neq i'$ *and* $j \neq j'$*. So, objects with such dimension vector are in one-to-one correspondence with the permutations of the set of* $n$ *elements. In particular, there are* $n!$ *such non-isomorphic objects.*

We will see in Section 3.3.2 that this description coincides with the well-known indexing of $B$-orbits in a full flag variety by permutations.

### 2.3.2 Case D: two subspaces and a flag

Now let $Q$ be the quiver $D_{p+2}$ with all arrows mapping to the center.

Having a representation

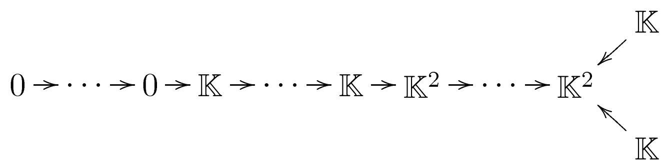



we denote its dimension vector by $(a_1, \dots, a_p; b; c)$.

Here is the complete list of indecomposable objects in $\mathcal{R}\text{ep}\mathcal{E}\text{mb}(Q)$, taken from [MWZ, Theorem 2.3]. There are four series with one-dimensional middle spaces, which we present in the table below together with their dimension vectors:

| | |
|---|---|
| $I_i^+$ | $(0, \dots, 0, 1, \dots, 1; 1; 0)$ |
| $I_i^-$ | $(0, \dots, 0, 1, \dots, 1; 0; 1)$ |
| $I_{i\infty}$ | $(0, \dots, 0, 1, \dots, 1; 0; 0)$ |
| $I_{0i}$ | $(0, \dots, 0, 1, \dots, 1; 1; 1)$ |

(all the maps between one-dimensional spaces are nonzero, the dimension jumps at the $i$-th step, $i \in [1, p]$), and one family of the following form:



where all the images of the three maps $\mathbb{K} \to \mathbb{K}^2$ are distinct (this guarantees indecomposability), and the dimension within the longest arm jumps at the $i$-th and the $j$-th steps, $i < j$. Denote these objects by $I_{ij}$.

On Figure 2.2, we give an example of AR-quiver of type $D$, for $p = 5$. Knowing the AR-quiver for $\mathcal{R}\mathrm{ep}(D_{p+2})$ with arrows oriented to the center,

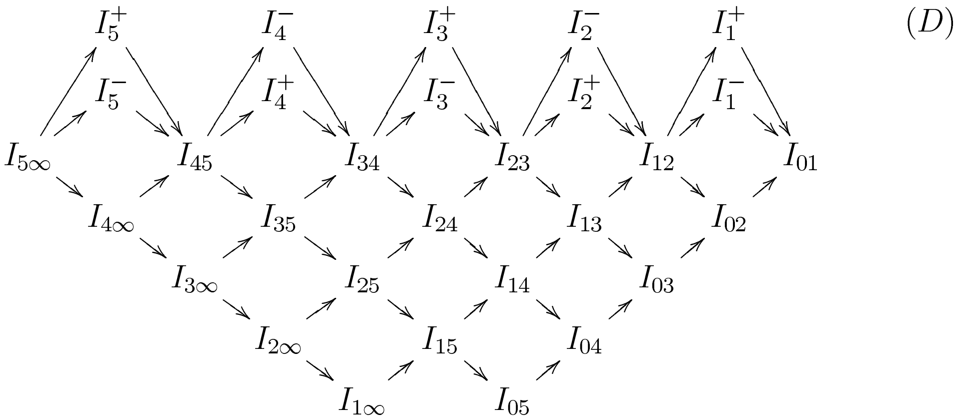



Figure 2.2: AR-quiver $Q_{5,2,2}$ (type $D$)

we restrict ourselves to its vertices corresponding to indecomposable objects from $\mathcal{R}\mathrm{ep}\mathcal{E}\mathrm{mb}(D_{p+2})$. Construction of the AR-quiver for $\mathcal{R}\mathrm{ep}(Q)$ with $Q$ arbitrary is discussed, for instance, in [ARS, Chap. VII]

**Notation.** The two subsets of vertices $\{I_1^+, \ldots, I_n^+\}$ and $\{I_1^-, \ldots, I_n^-\}$ are called *zigzags*. On the figure below the two zigzags are shown as follows: they are formed by the vertices situated on the dashed and on the dotted lines, respectively. Subsets of vertices of the following form, represented by white circles on Figure 2.3, are said to be *roads*:

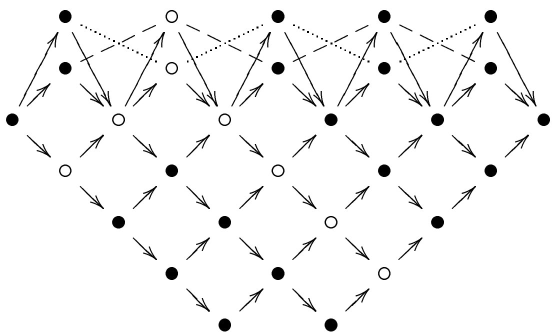


Figure 2.3: Roads and zigzags

They are formed by the objects $I_{i\infty}, \dots, I_{i,i+1}, I_i^+, I_i^-, I_{i-1,i}, \dots, I_{0i}$ for a given $i$. Each road starts on the left edge of the AR-quiver, at an object $I_{i\infty}$, goes up, then passes through the "mountain range" formed by two upper rows, bifurcates there and then goes down to the right edge, ending at the object $I_{0i}$. This road is said to be the $i$-th one. So, there are exactly 2 different zigzags and $p$ different roads.

**Proposition 2.3.** *Let $F$ be an object in $\mathcal{R}\mathrm{ep}\mathcal{E}\mathrm{mb}(Q_{p,2,2})$, such that*

$$\underline{\dim}F = (a_1, a_2, \dots, a_p; k; l),$$

*and let $F = \bigoplus I_\alpha$ be its decomposition into a sum of indecomposables. Then:*

*(i) For the i-th road in $AR(Q_{p,2,2})$ there are exactly $a_i - a_{i-1}$ objects occuring in $F$ situated on this road (as before, $a_0$ is set to be equal to 0);*

*(ii) The total number of $I_\alpha$ of the form $I_{ij}$, $1 \le i < j \le n$, and $I_i^+$, equals $k$;*

*(iii) The total number of $I_\alpha$ of the form $I_{ij}$, $1 \le i < j \le n$, and $I_i^-$, equals $l$.*

*Proof.* Fix a road; let $I_{i\infty}$ be its first element. From the description of indecomposable objects given on Page 14, it follows that the dimension vectors $(\mathbf{a}'; b'; c')$ of the indecomposable objects situated on this road are characterized by the equality $a'_i = a'_{i-1} + 1$. For all other elements, $a'_i = a'_{i-1}$. So, $F$ contains exactly $a_i - a_{i-1}$ indecomposable objects with dimension jump on the $i$-th step. This proves the first part of the proposition.

(ii) and (iii) are proved similarly. □

So, an object with dimension vector $(a_1, \dots, a_p; k; l)$ gives us a set of vertices in $AR(D_{p+2})$, satisfying the properties (i)–(iii). Obviously, the converse is also true: each set of vertices determines an object, namely, the direct sum of the corresponding indecomposables, and the properties (i)–(iii) guarantee that the dimension vector of this object equals $(a_1, \dots, a_p; k; l)$.

## 2.4 Three orders

Throughout this section, $Q$ is either the quiver $A_{p+q-1} = Q_{p,q,1}$ or the quiver $D_{p+2} = Q_{p,2,2}$. Recall that throughout the rest of this paper, the ground field $\mathbb{K}$ is supposed to be algebraically closed.

In this section we present three different ways to turn the set of objects $F \in \mathcal{R}\mathrm{ep}\mathcal{E}\mathrm{mb}(Q)$ with a given dimension vector into a partially ordered set (or shortly *poset*). We will show that these three orders are the same in the next section.

### 2.4.1 Degeneration order

The first definition uses the bijection between objects with dimension vector $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ and orbits in the corresponding multiple flag variety $\mathrm{Fl}_{(\mathbf{a},\mathbf{b},\mathbf{c})}(V)$. Given an object $F$, we denote the corresponding orbit by $\mathcal{O}_F$.

**Definition.** We say that $F$ is less or equal than $F'$ w.r.t. the *degeneration order*, if there is an inclusion of the corresponding orbit closures (in the Zariski topology):

$$F \overset{\deg}{\leq} F' \quad \Leftrightarrow \quad \mathcal{O}_F \subseteq \bar{\mathcal{O}}_{F'}.$$

### 2.4.2 Rank order

Another partial order is defined by means of dimensions of the homomorphism spaces between objects in the category $\mathcal{R}\mathrm{ep}\mathcal{E}\mathrm{mb}(Q)$. For short, for two elements $F, G \in \mathcal{R}\mathrm{ep}\mathcal{E}\mathrm{mb}(Q)$ we denote the dimension $\dim \mathrm{Hom}(F, G)$ by $\langle F, G \rangle$.

**Definition.** $F$ is less or equal than $F'$ w.r.t. the *rank order* (notation: $F \overset{\mathrm{rk}}{\leq} F'$), if for each indecomposable object $I \in \mathcal{R}\mathrm{ep}\mathcal{E}\mathrm{mb}(Q)$

$$\langle I, F \rangle \geq \langle I, F' \rangle.$$

(NB: the inequality is reversed!)

In our cases ($A_{p+q-1}$ and $D_{p+2}$) we shall give a simple geometric interpretation of the numbers $\langle I, F \rangle$. In general, this interpretation also exists (see [MWZ, Prop. 4.1]), but it is not evident at all.

**Proposition 2.4.** *1. Let $Q$ equal $Q_{p,q,1}$, and let $V_\bullet = (V_{a_1} \subseteq \cdots \subseteq V_{a_p} = V)$ and $V'_\bullet = (V'_{b_1} \subseteq \cdots \subseteq V'_{b_q} = V)$ be two flags of the same depth in a vector space $V$. Then for the object $F$ corresponding to the configuration $(V_\bullet, V'_\bullet)$ the following equalities hold:*

$$\langle I_{ij}, F \rangle = \dim V_{a_i} \cap V'_{b_j}$$

*for each $i \in [1, p]$, $j \in [1, q]$. (A description of the $I_{ij}$ is given on Page 12.)*

*2. Let $Q$ equal $Q_{p,2,2}$, and let $V_\bullet = (V_{a_1} \subseteq \cdots \subseteq V_{a_p} = V)$, $U$ and $W$ be a flag and two subspaces in $V$. Then for the object $F$ corresponding to the configuration $(U, W, V_\bullet)$ the following equalities hold:*

$$\begin{aligned} \langle I_{i\infty}, F\rangle &= \dim V_{a_i} = a_i; \\ \langle I_i^+, F\rangle &= \dim V_{a_i} \cap U; \\ \langle I_i^-, F\rangle &= \dim V_{a_i} \cap W; \\ \langle I_{0i}, F\rangle &= \dim V_{a_i} \cap U \cap W; \\ \langle I_{ij}, F\rangle &= \dim V_{a_j} \cap U \cap W + \dim V_{a_i} \cap ((V_{a_j} \cap U) + (V_{a_j} \cap W)). \end{aligned} \tag{2.1}$$

*Proof.* A first observation: these formulas are additive under taking direct sums of objects and componentwise direct sums of corresponding configurations of subspaces.

Next, the bracket $\langle \cdot, \cdot \rangle$ is bilinear, so

$$\langle I, F \oplus F' \rangle = \langle I, F \rangle + \langle I, F' \rangle.$$

Thus, it only suffices to prove these formulas for an indecomposable $F$. And this is done by a direct verification. □

**Definition.** The numbers $\langle I, F\rangle$ are called *rank numbers*.

### 2.4.3 Move order

In the previous section we have obtained a combinatorial description of objects in $\mathcal{R}\mathrm{ep}\mathcal{E}\mathrm{mb}(Q)$ with a given dimension vector. Objects are encoded by multisets of vertices of a certain quiver, satisfying the properties (i)–(iii) from Prop. 2.3.

To introduce the third partial order, we define some operations, called *elementary moves*, that bring these subsets of vertices into other ones.

As usual, we begin with type $A$. In this case the definition of elementary move is quite simple.

Take the decomposition of $F$ into indecomposables: $F = \bigoplus I_\alpha$. Suppose that among these $I_\alpha$'s there are two objects $I_{ij}$ and $I_{i'j'}$ occuring in $F$ (probably with multiplicities), such that $i < i'$ and $j < j'$. Let us also suppose that there is no other $I_{i''j''}$, such that $i \le i'' \le i'$ and $j \le j'' \le j'$. Graphically, this can be reformulated as follows: there is no other vertex occuring in $F$ and situated in the rectangle shown on Figure 2.4. If this is the case, this rectangle is called *admissible*.

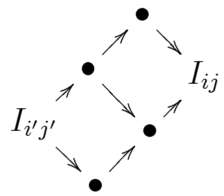



Figure 2.4: An admissible rectangle

Having this, we construct an object $F'$ by replacing this pair of indecomposables $I_{ij} \oplus I_{i'j'}$ with the pair $I_{ij'} \oplus I_{i'j}$. This means that the multiplicities $\mathrm{mult}_{F'} I$ of occurences of indecomposable objects $I$ in $F'$ are obtained from $\mathrm{mult}_F I$ according to the following rule:

$$\begin{aligned} \mathrm{mult}_{F'} I_{ij} &= \mathrm{mult}_F I_{ij} - 1; \\ \mathrm{mult}_{F'} I_{i'j'} &= \mathrm{mult}_F I_{i'j'} - 1; \\ \mathrm{mult}_{F'} I_{i'j} &= \mathrm{mult}_F I_{i'j} + 1; \\ \mathrm{mult}_{F'} I_{ij'} &= \mathrm{mult}_F I_{ij'} + 1; \\ \mathrm{mult}_{F'} I &= \mathrm{mult}_F I \qquad \text{otherwise.} \end{aligned}$$

Informally, can be described as flipping the rectangle, whose "corners" $I_{ij}$ and $I_{i'j'}$ occuring in $F$ are replaced by $I_{i'j}$ and $I_{ij'}$, see Figure 2.5

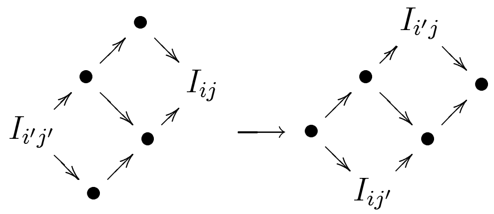



Figure 2.5: An elementary move

Let $F'$ be obtained from $F$ by an elementary move. We denote this as follows: $F \lessdot F'$.

Now we are ready to give the definition of the third order.

**Definition.** An object $F$ is said to be less or equal than an object $F'$ w.r.t. the *move order*, if there exists a sequence of objects $F_0, F_1, \ldots, F_s$, such that

$$F = F_0 \lessdot F_1 \lessdot \cdots \lessdot F_s = F'.$$

This is denoted as follows: $F \overset{\mathrm{mv}}{\leq} F'$.

**Remark.** Of course, each element is less or equal than itself. This corresponds to the empty sequence.

So, given two vertices of the AR-quiver, we have at most one possibility to perform an elementary move affecting them. As a result of this move, this pair of vertices is replaced with another pair.

In type $D$ everything is more complicated. As above, elementary moves consist in replacing a pair of marked vertices, but now they can be replaced by one, two or three other vertices. Moreover, the choice of an initial pair does not uniquely define the move any more; there may be up to three different possibilities.

To begin with, we introduce some convention that allows us to make the description of elementary moves less bulky. Let us add a "fake vertex" in the missing lowest corner, and the corresponding fake indecomposable object $I_{0\infty}$, equal to zero. So, the resulting quiver will be as shown on Figure 2.6

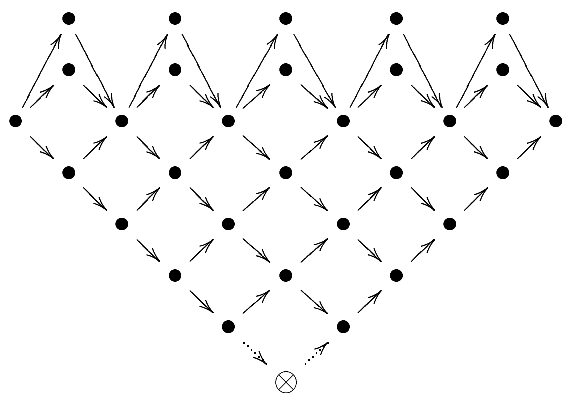


Figure 2.6: An AR-quiver of type $D$ with the "fake vertex"

Now let us describe the moves explicitly.

Our general strategy will be as follows: first, we define *regions*, which are analogues of rectangles in the case $A_n$.

A *region* is a triple $(\mathfrak{A}, \operatorname{Init}\mathfrak{A}, \operatorname{Term}\mathfrak{A})$, where $\mathfrak{A}$ is a subquiver in our AR-quiver of a certain form, described below. Each $\mathfrak{A}$ has exactly one source (vertex of incoming degree 0) and one sink (vertex of outcoming degree 0). These two vertices are called *initial vertices*; we denote this two-elementary set by $\operatorname{Init}\mathfrak{A}$. There are also at least one and at most three vertices marked as *terminal* ones, denoted $\operatorname{Term}\mathfrak{A}$ (they will be defined below in an *ad hoc* way).

**Remark.** The uniqueness of a source and a sink implies, in particular, that $\mathfrak{A}$ is connected and that there exists an (oriented) path joining the initial vertices.

Now let us describe regions explicitly. We distinguish between the following six cases, denoted I.a)-I.e) and II.

The cases I.a)–I.e) are characterized by the following property: $\mathfrak{A}$ consists of those vertices that are situated on the paths joining the source of $\mathfrak{A}$ with its sink.

**I.a)** The initial vertices of a region of type I.a) are of the form $I_1 = I_{i'j'}$, $I_2 = I_{ij}$, where $i < i' < j < j'$. In this case we define an admissible region $\mathfrak{A}$ of type I.a) as follows:

$$\mathfrak{A} = \{I_{\alpha\beta} \mid i \leq \alpha \leq i', j \leq \beta \leq j'\}.$$

It is a rectangle with corners in $I_1$ and $I_2$. We define the terminal vertices as the two other corners of this rectangle, $I_{ij'}$ and $I_{i'j}$:

A region of this type is shown on Figure 2.7. The initial vertices are outlined by squares, the terminal ones — by circles.

**I.b)** The initial vertices of regions of this type are of form $I_1 = I_{i'j'}$, $I_2 = I_{ij}$, such that $0 \leq i < j \leq i' < j' \leq \infty$. For each such pair of vertices, there are two regions of type I.b), defined as follows:

$$\mathfrak{A}^+ = \mathfrak{A}^- = \{I_{\alpha\beta} \mid i \leq \alpha \leq i', j \leq \beta \leq j'\} \cup \{I^+_\gamma, I^-_\gamma \mid j \leq \gamma \leq i'\}$$

Each such region has three terminal vertices, defined by

$$\begin{aligned} \operatorname{Term}\mathfrak{A}^+ &= \{I_{ij'}, I^+_j, I^-_{i'}\}; \\ \operatorname{Term}\mathfrak{A}^- &= \{I_{ij'}, I^-_j, I^+_{i'}\}. \end{aligned}$$

These two regions are shown on Figure 2.8.

**I.c)** For regions of this type, the initial vertices are of the form $I_1 = I_{i'j'}$, $I_2 = I^\pm_i$, such that $i < i' < j'$. In this case, we define $\mathfrak{A}$ to be

$$\mathfrak{A} = \{I_{\alpha\beta} \mid i \leq \alpha \leq i', \beta \leq j'\} \cup \{I^+_\gamma, I^-_\gamma \mid i \leq \gamma \leq i'\} \cup \{I^\pm_{i'}\},$$

and $\operatorname{Term}\mathfrak{A} = \{I^\pm_{i'}, I_{ij'}\}$, as shown on Figure 2.9.

**I.d)** The initial vertices are of the form $I_1 = I^\pm_{j'}$, $I_2 = I_{ij}$, and $i < j < j'$. Then

$$\mathfrak{A} = \{I_{\alpha\beta} \mid i \leq \alpha, j < \beta \leq j'\} \cup \{I^+_\gamma, I^-_\gamma \mid j \leq \gamma \leq j'\} \cup \{I^\pm_{j'}\},$$

and $\operatorname{Term}\mathfrak{A} = \{I^\pm_j, I_{ij'}\}$, see Figure 2.10.

**I.e)** The initial vertices are of the form $I^\pm_i$ and $I^\mp_{i'}$ (signs are different), $i < i'$. Then

$$\mathfrak{A} = \{I_{\alpha\beta} \mid i \leq \alpha < \beta \leq i'\} \cup \{I^+_\gamma, I^-_\gamma \mid i < \gamma < i'\} \cup \{I^\pm_i, I^\mp_{i'}\}.$$

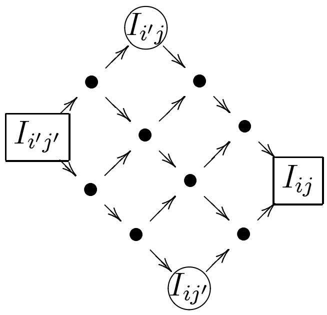



Figure 2.7: Region of type I.a)

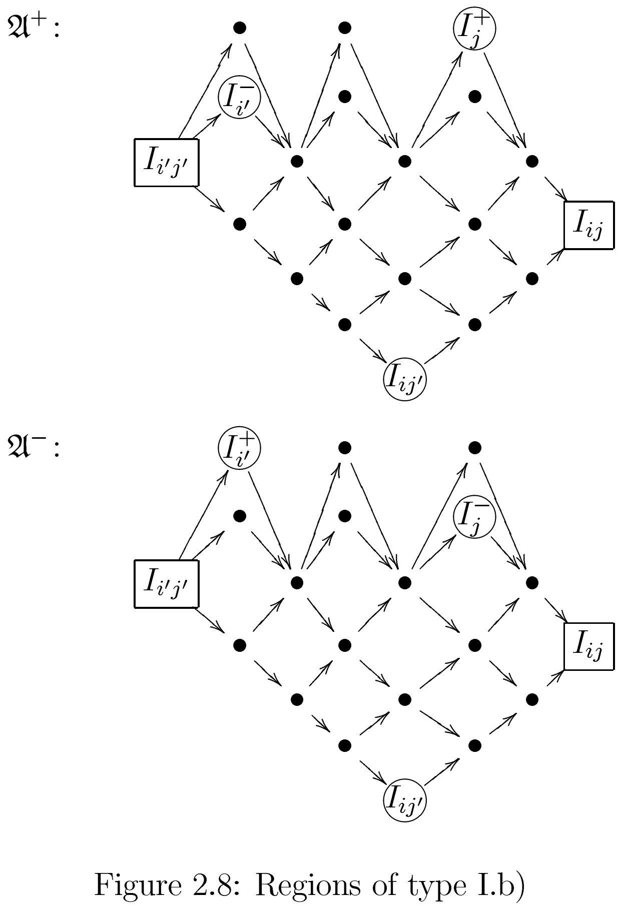



Figure 2.8: Regions of type I.b)

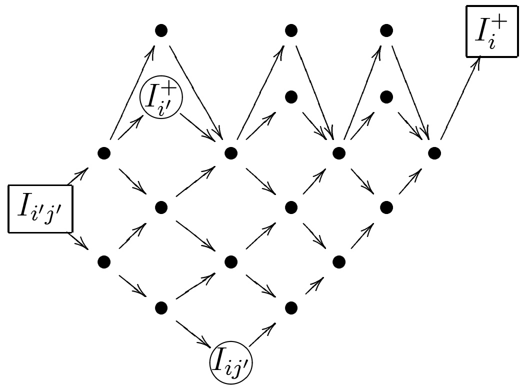



Figure 2.9: Region of type I.c)

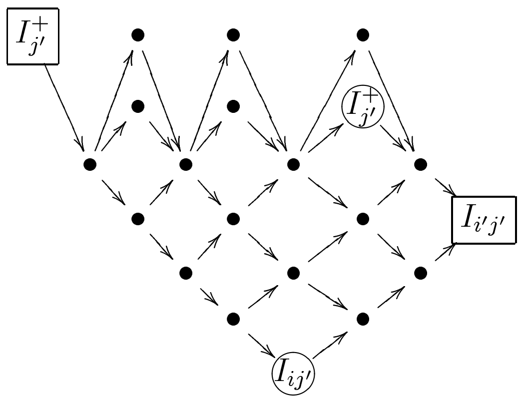



Figure 2.10: Region of type I.d)

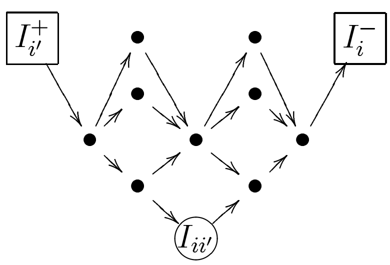



Figure 2.11: Region of type I.e)

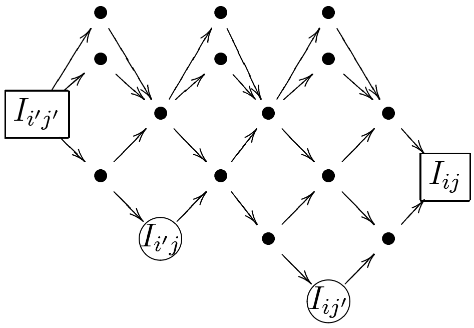



Figure 2.12: Region of type II

Then there is a unique terminal vertex: $\operatorname{Term}\mathfrak{A} = \{I_{ii'}\}$, see Figure 2.11

**II.** In this case, the initial vertices are of the form $I_{ij}$ and $I_{i'j'}$, where $i < j < i' < j'$. The corresponding subquiver $\mathfrak{A}$ is shown on Figure 2.12. It is given by

$$\mathfrak{A} = \{I_{\alpha\beta} \mid i \le \alpha \le i', j \le \beta \le j'\} \cup \{I_\gamma^+, I_\gamma^- \mid j \le \gamma \le i'\},$$

$I_{i'j}$ and $I_{ij'}$ are its terminal vertices.

One can think of the obtained set of vertices as a "folded rectangle", with corners in the initial and the terminal vertices.

After having defined regions, we can go further and pass to the definition of the move order. For the following definition, we fix an object $F \in \mathcal{R}\mathrm{ep}\mathcal{E}\mathrm{mb}(Q_{p,2,2})$.

**Definition.** A region $\mathfrak{A}$ is called *admissible* w.r.t. an object $F$, if for both initial vertices of $\mathfrak{A}$, the corresponding indecomposable objects occur in $F$ with nonzero multiplicities. An admissible region $\mathfrak{A}$ is called *minimal*, if any non-initial vertex from $\mathfrak{A}$ occurs in $F$ with multiplicity 0.

As in the case $A$, elementary moves that can be performed with an object $F$ correspond to the minimal admissible regions:

**Definition.** We say that $F'$ is obtained from $F$ by an *elementary move* (notation: $F \lessdot F'$), if there is a minimal admissible region $\mathfrak{A}$ w.r.t. $F$, such that

$$\begin{aligned} \operatorname{mult}_{F'} I &= \operatorname{mult}_F I - 1 \quad \text{for } I \in \operatorname{Init}\mathfrak{A}; \\ \operatorname{mult}_{F'} I &= \operatorname{mult}_F I + 1 \quad \text{for } I \in \operatorname{Term}\mathfrak{A}; \\ \operatorname{mult}_{F'} I &= \operatorname{mult}_F I \qquad \text{otherwise.} \end{aligned}$$

This means that, as a result of an elementary move, a pair of indecomposable objects is replaced by one, two or three other indecomposable objects.

Now the *move order* is defined as follows: $F$ is said to be less or equal than $F'$ (notation: $F \overset{\text{mv}}{\leq} F'$), if $F'$ is obtained from $F$ by a sequence of elementary moves.

It remains to show that the move order is well-defined, i.e., the graph of the move order does not contain oriented cycles. This follows from the fact that the move order implies the degeneration order (see Lemma 2.6 below) and the latter order is well-defined.

## 2.5 The main result

**Theorem 2.5.** *Let $Q$ equal $Q_{p,2,2}$. Then for all $F, F' \in \mathcal{R}\text{ep}\mathcal{E}\text{mb}(Q)$, such that $\underline{\dim}F = \underline{\dim}F'$,*

$$F \overset{\text{deg}}{\leq} F' \quad \Leftrightarrow \quad F \overset{\text{rk}}{\leq} F' \quad \Leftrightarrow \quad F \overset{\text{mv}}{\leq} F'.$$

*So, all the three orders are the same.*

This is proved in [Mag] for $Q = Q_{p,q,1}$. We follow the same strategy and split the proof into three lemmas, corresponding to [Mag, Lemmas 5,6,7].

**Lemma 2.6.** $F \overset{\text{mv}}{\leq} F' \Longrightarrow F \overset{\text{deg}}{\leq} F'$.

This will be proved in Subsection 2.5.1 by constructing an explicit degeneration of the larger of the corresponding orbits to the smaller one.

**Lemma 2.7.** $F \overset{\text{deg}}{\leq} F' \Longrightarrow F \overset{\text{rk}}{\leq} F'$.

This is a particular case of [Rie, Prop. 2.1]. However, in Subsection 2.5.2 we present an elementary geometric proof of this result.

**Lemma 2.8.** $F \overset{\text{rk}}{\leq} F' \Longrightarrow F \overset{\text{mv}}{\leq} F'$.

This will be proved in Subsection 2.5.3 as follows: given $F \overset{\text{rk}}{\leq} F'$, we find an object $\tilde{F}$, such that $F \overset{\text{mv}}{\lessdot} \tilde{F} \overset{\text{rk}}{\leq} F'$.

Another important result is that the elementary moves correspond to the covers in the poset of orbits. More precisely, the following theorem holds.

**Theorem 2.9** (Minimality Theorem). *With the notation of Theorem 2.5, the relation $F \overset{\text{deg}}{\leq} F'$ is a cover iff $F \overset{\text{mv}}{\lessdot} F'$*

Its proof will be given in Subsection 2.5.4.

### 2.5.1 Move order implies degeneration order

First let us recall the description of "standard" representatives in $\mathrm{GL}(V)$-orbits, taken from [MWZ, Def. 2.8, Prop. 2.9]. As usual, this is described on orbits $\mathcal{O}_I$ corresponding to indecomposable objects $I$, and then extended via taking direct sums.

Let $(U, W, V_\bullet)$ be a triple corresponding to an indecomposable object. This means that $V = V_{a_p}$ is of dimension 1 or 2. If $\dim V = 1$, each of $U$ and $W$ is either equal to $V$ or to zero.

If $I = I_{ij}$, $0 < i < j < \infty$, then $\dim V = 2$. Let $(e_i, e_j)$ be an ordered basis of $V$, such that $V_i = \cdots = V_{j-1} = \langle e_i \rangle$. Then the triple $(U, W, V_\bullet)$ with $U = \langle e_j \rangle$, $W = \langle e_i + e_j \rangle$ is called the standard representative of the orbit $\mathcal{O}_{I_{ij}}$.

Later on, we will deal with certain deformations of bases in our subspaces. For this, the following notational convention will be useful. Introduce two more "vectors": $e_0$ and $e_\infty$. Set formally $e_0 = 0$ and each linear combination of vectors involving $e_\infty$ be also equal to 0. Note that with this convention, the definition of standard representatives for $I_{ij}$, $0 < i < j < \infty$, is extended to the cases of $I_{0i}$ and $I_{i\infty}$, so later we will consider these three cases simultaneously.

Now we pass to the proof of Lemma 2.6.

*Proof of Lemma 2.6.* The main idea is as follows: for any two objects $F$ and $F'$, such that $F \lessdot F'$, we take a specific representative $(U, W, V_\bullet)$ of the orbit $\mathcal{O}_F$ and present a one-parameter family $(U(\tau), W(\tau), V_\bullet(\tau))$ of subspace configurations ($\tau$ runs over the ground field), such that $(U(0), W(0), V_\bullet(0)) = (U, W, V_\bullet)$, and $(U(\tau), W(\tau), V_\bullet(\tau)) \in \mathcal{O}_{F'}$ when $\tau \neq 0$.

Since $F'$ is obtained from $F$ by replacing exactly two indecomposable summands with some other object (consisting of one, two or three indecomposables), and all the other summands in $F$ remain unchanged, we can assume that $F$ consists only of these two objects. It turns out to be convenient to take the representative $(U, W, V_\bullet)$ in its standard form, as indicated in the beginning of this subsection.

Now consider all the cases listed in Section 2.4.3. We will consider an initial pair of objects depending on numbers $i, j, i', j \in [0, n] \cup \{\infty\}$, where $n = \dim V$; when we need to speak about linear combinations of vectors involving $e_0$ or $e_\infty$, we follow the convention from the beginning of this subsection. By $V_\bullet$ we always denote the flag whose components are spanned by basis vectors $\{e_1, \ldots, e_n\}$, such that $\dim V_{a-\alpha}/V_{a_{\alpha-1}} = 1$ iff $\alpha \in \{i, j, i', j'\}$, and 0 otherwise. This flag will always be invariant along the curves we are going to construct: $V_\bullet(\tau) = V_\bullet$.

**I.a)** $F = I_{ij} \oplus I_{i'j'}$, $F' = I_{i'j} \oplus I_{ij'}$, where $i' < i < j' < j$.

$$(U, W) = (\langle e_j, e_{j'} \rangle, \langle e_i + e_j, e_{i'} + e_{j'} \rangle),$$

$$(U(\tau), W(\tau)) = (\langle e_j, e_{j'} \rangle, \langle e_i + e_j, e_{i'} + e_{j'} + \tau e_j \rangle).$$

The triple $(U(\tau), W(\tau), V_\bullet)$ for each nonzero $\tau$ corresponds to the object $F' = I_{i'j} \oplus I_{ij'}$, as may be seen by calculating its rank numbers, or by the decomposition of this configuration into a direct sum of two indecomposables.

Note that this deformation also works for the case when $i' = 0$ or/and $j = \infty$.

**I.b)** $F = I_{ij} \oplus I_{i'j'}$, $F' = I_{i'j} \oplus I_i^+ \oplus I_{j'}^-$ or $F' = I_{i'j} \oplus I_i^- \oplus I_{j'}^+$ where $i' < j' \leq i < j$. In the first case the initial configuration

$$(U, W) = (\langle e_{j'}, e_j \rangle, \langle e_{i'} + e_{j'}, e_i + e_j \rangle),$$

is deformed to

$$(U(\tau), W(\tau)) = (\langle e_{j'} + \tau e_i, e_j \rangle, \langle e_{i'} + e_{j'}, e_i + e_j \rangle).$$

and in the second one — to

$$(U(\tau), W(\tau)) = (\langle e_{j'}, e_j \rangle, \langle e_{i'} + e_{j'} + \tau e_i, e_i + e_j \rangle).$$

**I.c)** $F = I_{ij} \oplus I_{i'}^+$, $F' = I_i^+ \oplus I_{i'j}$, where $i' < i < j$.

$$(U, W) = (\langle e_{i'}, e_j \rangle, \langle e_i + e_j \rangle),$$

$$(U(\tau), W(\tau)) = (\langle e_{i'} + \tau e_i, e_j \rangle, \langle e_i + e_j \rangle).$$

Similarly, if $F = I_{ij} \oplus I_{i'}^-$ for $i' < i < j$, this object is transformed to $F' = I_i^- \oplus I_{i'j}$: for the representative

$$(U, W) = (\langle e_j \rangle, \langle e_{i'}, e_i + e_j \rangle)$$

there is a curve

$$(U(\tau), W(\tau)) = (\langle e_j \rangle, \langle e_{i'} + \tau e_i, e_i + e_j \rangle),$$

having the configuration type $F'$.

**I.d)** $F = I_{i'j'} \oplus I_j^+$ for $i' < j' < j$, and $F' = I_{j'}^+ \oplus I_{i'j}$. Similarly,

$$(U, W) = (\langle e_{j'}, e_j \rangle, \langle e_{i'} + e_{j'} \rangle),$$

and

$$(U(\tau), W(\tau)) = (\langle e_{j'}, e_j \rangle, \langle e_{i'} + e_{j'} + \tau e_j \rangle).$$

For $F = I_{i'j'} \oplus I_j^-$ for $i' < j' < j$, and $F' = I_{j'}^- \oplus I_{i'j}$, we have

$$(U, W) = (\langle e_{j'} \rangle, \langle e_{i'} + e_{j'}, e_j \rangle),$$

$$(U(\tau), W(\tau)) = (\langle e_{j'} + \tau e_j \rangle, \langle e_{i'} + e_{j'}, e_j \rangle).$$

**I.e)** $F = I_i^+ \oplus I_{i'}^-$ for $i' < i$, $F' = I_{i'i}$.

$$(U, W) = (\langle e_i \rangle, \langle e_{i'} \rangle),$$

and

$$(U(\tau), W(\tau)) = (\langle e_i \rangle, \langle e_{i'} + \tau e_i \rangle).$$

The case $F = I_i^- \oplus I_{i'}^+$, $F' = I_{i'i}$ for $i' < i$ is completely analogous.

And here comes the last case:

**II.** $F = I_{ij} \oplus I_{i'j'}$, where $0 \le i' < j' < i < j \le \infty$, and $F' = I_{i'i} \oplus I_{j'j}$. Then

$$(U, W) = (\langle e_{j'}, e_j \rangle, \langle e_{i'} + e_{j'}, e_i + e_j \rangle),$$

and

$$(U(\tau), W(\tau)) = (\langle e_{j'} + \tau e_i, e_j \rangle, \langle e_{i'} + e_{j'} + \tau e_i, e_i + e_j \rangle)$$

So, for all the possible types of elementary moves we constructed curves that are contained in the closure of the "larger" orbit and that intersect the "smaller" orbit in exactly one point. This proves the lemma. □

### 2.5.2 Degeneration order implies rank order

*Proof of Lemma 2.7.* According to Proposition 2.4, it suffices to show that all the inequalities of the form

$$\begin{aligned} \dim V_{a_i} \cap U &\ge d; \\ \dim V_{a_i} \cap W &\ge d; \\ \dim V_{a_i} \cap U \cap W &\ge d; \\ \dim(((U \cap V_{a_j}) + (W \cap V_{a_j})) \cap V_{a_i}) + \dim(U \cap W \cap V_{a_j}) &\ge d \end{aligned} \qquad (2.2)$$

define closed conditions on $X = \mathrm{Gr}(k, V) \times \mathrm{Gr}(l, V) \times \mathrm{Fl}_{\mathbf{a}}(V)$.

For the first three families of inequalities this is clear — these conditions define closed subvarieties in $X$ cut out by vanishing of certain determinants in the homogeneous coordinates on $X$. Let us show this for the last family of inequalities.

Fix $i$ and $j$, $i < j$, and take a configuration of subspaces $(U, W, V_\bullet)$. Now define a linear map

$$\varphi_{ij} \colon (U \cap V_{a_j}) \times (W \cap V_{a_j}) \to V_{a_j}/V_{a_i}$$

by

$$(u,w) \mapsto u+w \mod V_{a_i}.$$

The dimension of its kernel equals $\dim(((U\cap V_{a_j})+(W\cap V_{a_j}))\cap V_{a_i})+\dim(U\cap W\cap V_{a_j})$. Indeed,

$$\begin{gathered}\dim\operatorname{Ker}(\varphi_{ij}) = \dim(U\cap V_{a_j})+\dim(W\cap V_{a_j})-\operatorname{rk}\varphi_{ij} = \\ \dim(U\cap V_{a_j})+\dim(W\cap V_{a_j})-\dim(((U\cap V_{a_j})+(W\cap V_{a_j}))/V_{a_i}) = \\ \dim(U\cap V_{a_j})+\dim(W\cap V_{a_j})-\dim((U\cap V_{a_j})+(W\cap V_{a_j}))+ \\ \dim(((U\cap V_{a_j})+(W\cap V_{a_j}))\cap V_{a_i}) = \\ \dim((U\cap V_{a_j})\cap(W\cap V_{a_j}))+\dim(((U\cap V_{a_j})+(W\cap V_{a_j}))\cap V_{a_i}) = \\ \dim(U\cap W\cap V_{a_j})+\dim(((U\cap V_{a_j})+(W\cap V_{a_j}))\cap V_{a_i}).\end{gathered}$$

Now let us prove that the condition $\dim\operatorname{Ker}\varphi_{ij}\geq d$ defines a closed condition on $X$. This will be done as follows. Consider the direct product $\tilde{X}$ of $X$ and three copies of $V=V_n$:

$$\tilde{X} = \operatorname{Gr}(k,V)\times\operatorname{Gr}(l,V)\times\operatorname{Fl}_{\mathbf{a}}(V)\times V\times V\times V,$$

and take the subset $Z_{ij}\subset\tilde{X}$ formed by the sixtuples $(U,W,V_\bullet,x,y,z)\in Y$ satisfying the following conditions:

$$\begin{aligned} x,y &\in V_{a_j};\\ x &\in U;\\ y &\in W;\\ z &\in V_{a_i};\\ x+y &= z \quad \text{(as vectors in } V).\end{aligned}$$

Obviously, $Z_{ij}$ is closed in $\tilde{X}$. Moreover, $\operatorname{Ker}\varphi_{ij}\simeq\pi_{ij}^{-1}((U,W,V_\bullet))$, where $\pi_{ij}$ is the projection $Z_{ij}\to X$.

This means that the condition 2.2 is equivalent to the condition

$$\dim\pi_{ij}^{-1}((U,W,V_\bullet))\geq d,$$

and the latter condition is closed on $X$. □

### 2.5.3 Rank order implies move order

Let us first establish two general facts about rank numbers.

**Proposition 2.10.** *The set of rank numbers uniquely defines the corresponding object.*

*Proof.* Assume the contrary: let $F$ and $F'$ correspond to the same set of rank numbers. This means that $\langle I, F\rangle = \langle I, F'\rangle$ for each indecomposable $I$.

Since the direct sums of objects correspond to the sums of their rank numbers, one can consider that no indecomposable objects appear in $F$ and $F'$ simultaneously. Now take two rightmost objects $I$ and $I'$ (in the sense of AR-quiver of type $D$) occuring in $F$ and $F'$. Without loss of generality suppose that $I$ is situated in the same column or to the right of $I'$, and, consequently, (non-strictly) to the right of all indecomposable objects appearing in $F'$. This means that $\langle I, F'\rangle = 0$. Similarly, $I$ is situated non-strictly to the right of all the indecomposables from $F$, except for $I$ itself. So $\langle I, F\rangle = \langle I, I\rangle = 1$, a contradiction. □

**Proposition 2.11.** *Let $\mathfrak{A}$ be a region with initial vertices $I_1$ (source) and $I_2$ (sink), and $J$ the sum of the indecomposable objects corresponding to the terminal vertices of $\mathfrak{A}$. Then for an arbitrary object $F$*

$$\langle I_1, F\rangle + \langle I_2, F\rangle \geq \langle J, F\rangle.$$

*Moreover, if $\mathfrak{A} \setminus I_2$ contains no indecomposable subobject of $F$, the inequality is an equality.*

*Proof.* By bilinearity of $\langle \cdot, \cdot\rangle$, one can assume $F$ to be indecomposable. So, suppose $F = I$.

Let $I'$ and $I''$ be two neighbor indecomposable objects in a horizontal line (that is, $I_{ij}$ and $I_{i+1,j+1}$, or $I_i^{\pm}$ and $I_{i+1}^{\mp}$). Also denote by $J$ the sum of the objects corresponding to vertices situated on the paths from $I'$ to $I''$ ($J$ may consist of at most three indecomposable objects). With (2.1) from Page 18, one can see that

$$\langle I', I\rangle + \langle I'', I\rangle \geq \langle J, I\rangle, \tag{2.3}$$

and the inequality is strict iff $I' = I$.

Now, taking the sum of the inequalities (2.3) over all pairs $(I', I'')$, where both $I'$ and $I''$ belong to $\mathfrak{A}$, we obtain the desired inequality. If all the inequalities (2.3) are equalities, the latter is equality as well. □

Next, we need notions of the *interior* and the *nucleus* of a region.

**Definition.** Let $\mathfrak{A}$ be a region. The *interior* and the *nucleus* of $\mathfrak{A}$ (denoted by $\operatorname{Int}\mathfrak{A}$ and $\operatorname{Nuc}\mathfrak{A}$, respectively) are sets of indecomposable objects, defined as follows:

$$\begin{aligned}\operatorname{Int}\mathfrak{A} &= \{I \mid \sum_{I'\in\operatorname{Term}\mathfrak{A}} \langle I, I'\rangle < \sum_{I'\in\operatorname{Init}\mathfrak{A}} \langle I, I'\rangle\};\\ \operatorname{Nuc}\mathfrak{A} &= \{I \mid \sum_{I'\in\operatorname{Term}\mathfrak{A}} \langle I, I'\rangle = \sum_{I'\in\operatorname{Init}\mathfrak{A}} \langle I, I'\rangle - 2\} \subset \operatorname{Int}\mathfrak{A};\end{aligned}$$

A simple verification shows that $\operatorname{Int}\mathfrak{A} \subset \mathfrak{A}$ and that the difference between $\langle I, F\rangle$ and $\langle I, \tilde{F}\rangle$ does not exceed 1 for regions of type I.a)-e) and 2 for regions of type II. (So, the nucleus is nonempty only for regions of type II).

On Figure 2.13 below, for a region of each type its nucleus is marked with stars, and the interior is formed by the union of the nucleus with the set of black dots. As before, the initial and terminal vertices are outlined by squares and circles, respectively.

Now let us pass to the proof of Lemma 2.8.

*Proof of Lemma 2.8.* Let $F$ and $F'$ be two objects, such that $\underline{\dim}F = \underline{\dim}F'$ and $F \overset{\mathrm{rk}}{\leq} F'$. We have $\langle I, F\rangle \geq \langle I, F'\rangle$ for all indecomposables $I$. For the "fake vertex" $I_{0\infty}$ we set $\langle I_{0\infty}, F\rangle = \langle I_{0\infty}, F'\rangle = 0$.

We begin with the following definition, which will be the last one in this chapter.

**Definition.** A region $\mathfrak{B}$ is said to be *dominant* w.r.t. $F$ and $F'$, if the following inequalities hold:

$$\langle I, F\rangle > \langle I, F'\rangle \qquad \forall I \in \operatorname{Int}\mathfrak{B};$$

$$\langle I, F\rangle > \langle I, F'\rangle + 1 \qquad \forall I \in \operatorname{Nuc}\mathfrak{B}.$$

(Of course, the second set of inequalities is trivial for regions of type I).

The following technical lemma is essential for the sequel.

**Lemma 2.12.** *With the notation as above, take a rightmost object $I$, such that the corresponding rank numbers for $F$ and $F'$ differ: $\langle I, F\rangle > \langle I, F'\rangle$. Then there exists a dominant region $\mathfrak{B}$ with sink $I$ and an indecomposable object $J \neq I$ situated in $\mathfrak{B}$ and occuring in $F$ as a direct summand.*

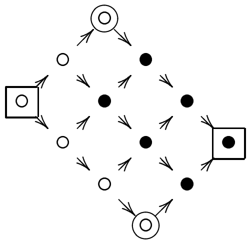
*Proof.* Take a maximal dominant region $\mathfrak{B}$ with sink $I$. Assume the contrary: no indecomposable summand of $F$ other than $I$ is situated in $\mathfrak{B}$.

1. First suppose that $\mathfrak{B}$ is of type II, with sink $I = I_{ij}$ and source $I' = I_{i'j'}$. We know that $i < j < i' < j'$.

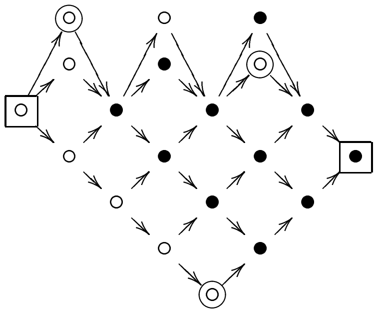
Since $\mathfrak{B}$ is maximal dominant, there must exist two objects $J_1$ and $J_2$ with the property

$$\langle J_{1,2}, F\rangle = \langle J_{1,2}, F'\rangle,$$

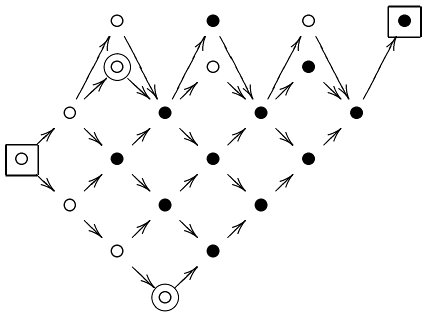
**I.a)**



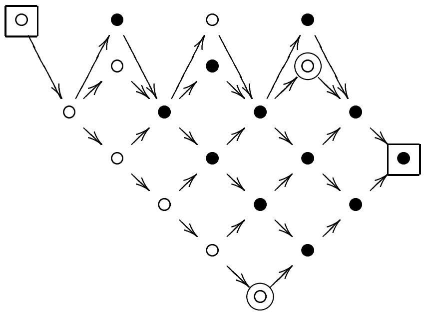
**I.b)**



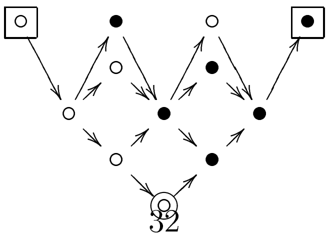
**I.c)**



**I.d)**



**I.e)**



**II.**

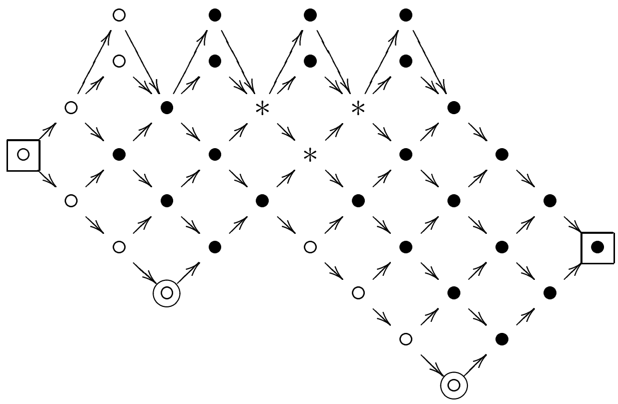


Figure 2.13: Interiors and nuclei of admissible regions

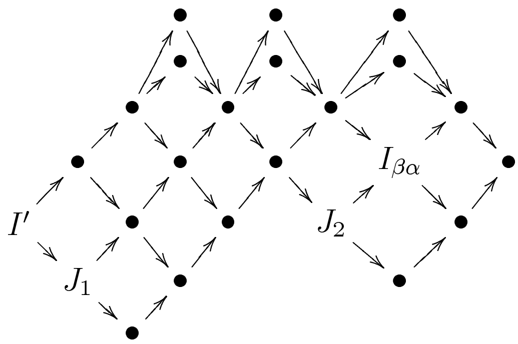



Figure 2.14: To the proof of Lemma 2.12: Case 1a

such that

$$J_1 \in \{I_{\alpha j'} \mid \alpha \in [j, i')\}$$

and

$$J_2 \in \{I_{\beta i'} \mid \beta \in (i, j]\} \cup \{I_{i'\gamma} \mid \gamma \in (i', j')\} \cup \{I^{\pm}_{i'}\}$$

(otherwise $\mathfrak{B}$ would be contained in a larger dominant region).

According to the position of $J_2$, three cases can occur:

1a. $J_1 = I_{\alpha j'}$, $J_2 = I_{\beta i'}$, where $\alpha \in [j, i')$, $\beta \in (i, j]$.

Consider also two objects $I_{i'j'}$ and $I_{\beta\alpha}$. These four objects determine a region of type II, as shown on Figure 2.14: Apply Prop. 2.11 twice to this region, taking into account that $I_{\beta\alpha} \in \operatorname{Int} \mathfrak{B}$:

$$\begin{aligned}\langle I_{i'j'}, F\rangle &= \langle J_1, F\rangle + \langle J_2, F\rangle - \langle I_{\beta\alpha}, F\rangle \\ &< \langle J_1, F'\rangle + \langle J_2, F'\rangle - \langle I_{\beta\alpha}, F'\rangle \leq \langle I_{i'j'}, F'\rangle,\end{aligned}$$

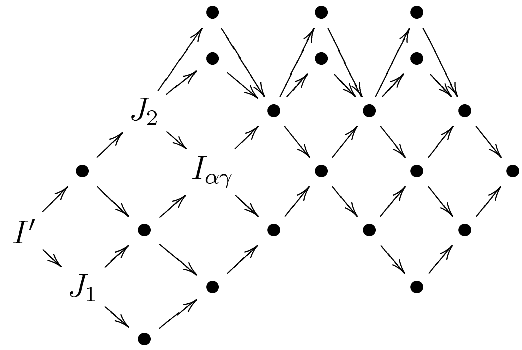



Figure 2.15: Case 1b

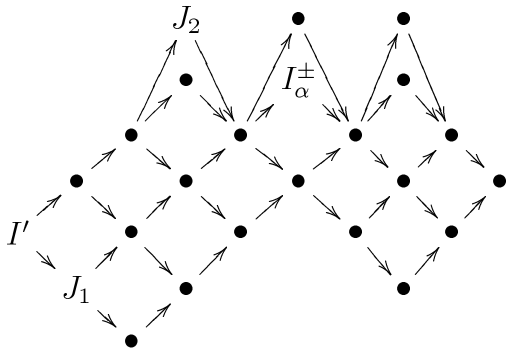



Figure 2.16: Case 1c

that gives us a contradiction. This means that this smaller region, and hence $\mathfrak{B}$, contain subobjects of $F$ different from $I$.

1b. $J_1 = I_{\alpha j'}$, $J_2 = I_{i'\gamma}$, where $\alpha \in [j, i')$, $\gamma \in (i', j')$.

In this case, we consider the objects $I_{i'j'}$ and $I_{\alpha\gamma}$, shown on Figure 2.15: and again apply the same Proposition:

$$\langle I_{i'j'}, F\rangle = \langle J_1, F\rangle + \langle J_2, F\rangle - \langle I_{\alpha\gamma}, F\rangle \\ < \langle J_1, F'\rangle + \langle J_2, F'\rangle - \langle I_{\alpha\gamma}, F'\rangle \leq \langle I_{i'j'}, F'\rangle,$$

obtaining a contradiction with our assumption.

1c. $J_1 = I_{\alpha j'}$, $\alpha \in [j, i')$, and $J_2 = I_{i'}^{\pm}$.

We consider the pair of objects $(I_{i'j'}, I_\alpha^\pm)$ and again apply the same procedure, see Figure 2.16.

2. The region $\mathfrak{B}$ is of type I.a)–I.c). This means that its source $I'$ is of the form $I_{ij}$.

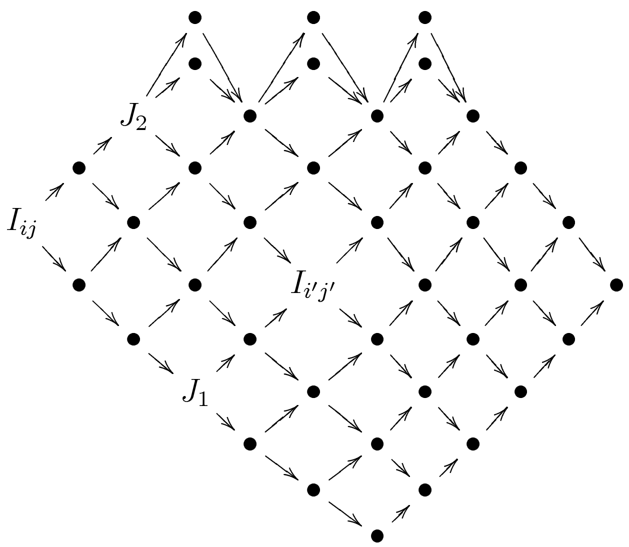



Figure 2.17: Case 2a

The maximality of $\mathfrak{B}$ implies the existence of at least two objects $J \in \mathfrak{B}$, such that $\langle J, F \rangle = \langle J, F' \rangle$. We distinguish between the following subcases:

2a. There are two such objects of the form $J_1 = I_{i'j}$ and $J_2 = I_{ij'}$, $j' \in (i, j)$. Then we can consider the objects $I_{ij}$ and $I_{i'j'}$ (see Figure 2.17) and apply Prop. 2.11 twice, writing

$$\langle I_{ij}, F \rangle = \langle J_1, F \rangle + \langle J_2, F \rangle - \langle I_{i'j'}, F \rangle \\ < \langle J_1, F' \rangle + \langle J_2, F' \rangle - \langle I_{i'j'}, F' \rangle \leq \langle I_{ij}, F' \rangle.$$

This gives us a contradiction.

2b. $J_1 = I_{i'j}$, but for all vertices $I_{ij'}$, where $i < j' < j$, the inequality

$$\langle I_{ij'}, F \rangle > \langle I_{ij'}, F' \rangle$$

holds. Then, by maximality of $\mathfrak{B}$, there exist two vertices $J_2 = I_i^{\pm}$ and $J_3 = I_{i''}^{\mp}$ (with different signs), such that $I_i^{\pm} \in \operatorname{Term} \mathfrak{B}$, and

$$\langle J_2, F \rangle = \langle J_2, F' \rangle$$
$$\langle J_3, F \rangle = \langle J_3, F' \rangle$$

Let us take for $J_3$ the leftmost element of form $I_{\bullet}^{\mp}$ situated in $\mathfrak{B}$ and satisfying the latter equality.

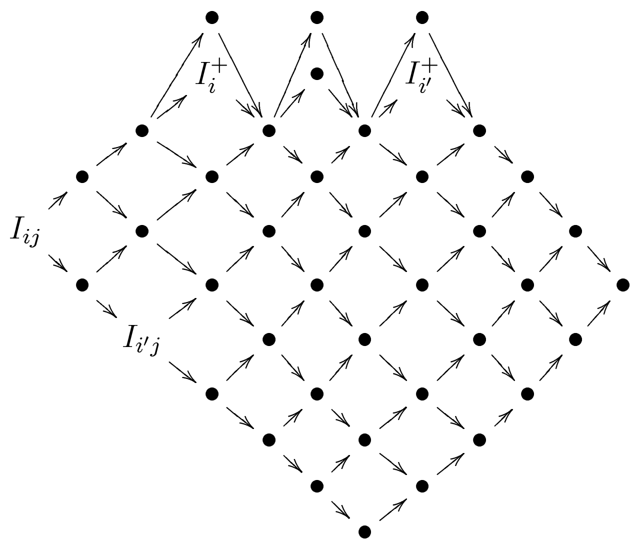



Figure 2.18: Case 2b

If $i'' \leq i'$, we can consider region $\mathfrak{C}$ of type I.c) with $\operatorname{Init}\mathfrak{C} = \{I_{ij}, I_{i'}^{\pm}\}$ and $\operatorname{Term}\mathfrak{C} = \{I_{i'j}, I_i^{\pm}\}$, see Figure 2.18. Then we can again apply Prop. 2.11 and obtain

$$\langle I_{ij}, F\rangle = \langle I_{i'j}, F\rangle + \langle I_i^{\pm}, F\rangle - \langle I_{i'}^{+}, F\rangle$$
$$< \langle I_{i'j}, F'\rangle + \langle I_i^{\pm}, F'\rangle - \langle I_{i'}^{+}, F'\rangle \leq \langle I_{ij}, F'\rangle.$$

2c. If $i'' > i'$, we consider the region $\mathfrak{C}'$ of type I.b), with $\operatorname{Init}\mathfrak{C}' = \{I_{ij}, I_{i'i''}\}$ and $\operatorname{Term}\mathfrak{C}' = \{I_{i'j}, I_i^{\pm}, I_{i''}^{\mp}\}$, as shown on Figure 2.19 Again we apply Prop. 2.11 to this region twice, obtaining

$$\langle I_{ij}, F\rangle = \langle I_{i'j}, F\rangle + \langle I_i^{\pm}, F\rangle + \langle I_{i''}^{\mp}, F\rangle - \langle I_{i'i''}, F\rangle$$
$$< \langle I_{i'j}, F'\rangle + \langle I_i^{\pm}, F'\rangle + \langle I_{i''}^{\mp}, F'\rangle - \langle I_{i'i''}, F'\rangle < \langle I_{ij}, F'\rangle.$$

3. The region $\mathfrak{B}$ is of type I.d) or I.e). Let its source be situated at the vertex $I = I_j^{\pm}$. The maximality of $\mathfrak{B}$ means that there exists at least one element $I_{ij}$, such that $\langle I_{ij}, F\rangle = \langle I_{ij}, F'\rangle$. Let $I_{ij}$ be the leftmost element with this property. We distinguish between the following subcases:

3a. There exists an element $I_{j'}^{\pm}$, such that $\langle I_{j'}^{\pm}, F\rangle = \langle I_{j'}^{\pm}, F'\rangle$, and $i \leq j'$. In this case, take a leftmost such element and consider region $\mathfrak{C}$ of type I.d), defined by $\operatorname{Init}\mathfrak{C} = \{I_j^{\pm}, I_{ij'}\}$ and $\operatorname{Term}\mathfrak{C} = \{I_{j'}^{\pm}, I_{ij}\}$, see Figure 2.20. It does

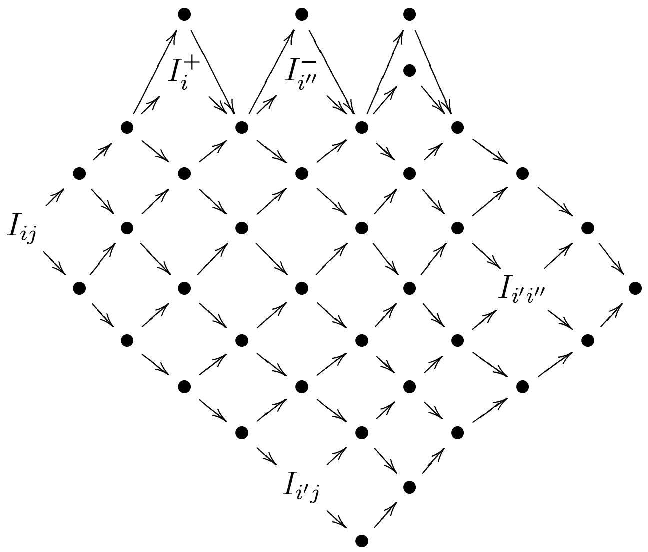



Figure 2.19: Case 2c

not contain objects occuring in $F$, so proceed as usual:

$$\langle I_j^\pm, F\rangle = \langle I_{ij'}, F\rangle - \langle I_{j'}^\pm, F\rangle - \langle I_{ij}, F\rangle \\ < \langle I_{ij'}, F'\rangle - \langle I_{j'}^\pm, F'\rangle - \langle I_{ij}, F'\rangle \leq \langle I_j^\pm, F'\rangle,$$

a contradiction.

3b. For all elements $I_{j'}^\pm$, such that $i < j' < j$, the inequality $\langle I_{j'}^\pm, F\rangle \geq \langle I_{j'}^\pm, F'\rangle$ is strict, and the element $I_i^\mp$ belongs to $\mathfrak{B}$. Then we consider $\mathfrak{C}$ of type I.e), as on Figure 2.21, with $\operatorname{Init}\mathfrak{C} = \{I_i^\mp, I_j^\pm\}$ and $\operatorname{Term}\mathfrak{C} = \{I_{ij}\}$, and

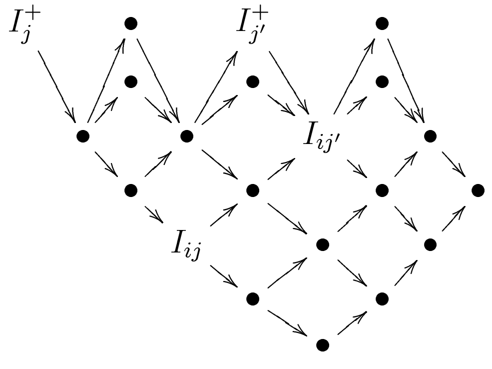



Figure 2.20: Case 3a

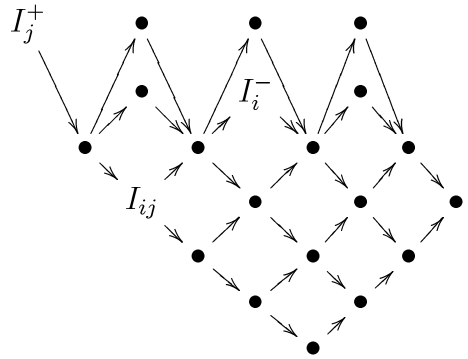



Figure 2.21: Case 3b

apply the same method:

$$\langle I_j^{\pm}, F\rangle = \langle I_i^{\mp}, F\rangle - \langle I_{ij}, F\rangle < \langle I_i^{\mp}, F'\rangle - \langle I_{ij}, F'\rangle \leq \langle I_j^{\pm}, F'\rangle.$$

3c. Here comes the last possibility: the equality of rank numbers holds in $I_{ij}$, but for all vertices $I_\alpha^{\pm} \in \mathfrak{B}$, $\alpha \neq j$, the inequality

$$\langle I_\alpha^{\pm}, F\rangle \geq \langle I_\alpha^{\pm}, F'\rangle$$

is strict, and the vertex $I_i^{\pm}$ does not belong to $\mathfrak{B}$. The latter means that $\mathfrak{B}$ is of type I.d) (not I.e)). Denote its sink by $I_{i_0 j_0}$.

In this case, we claim that region $\mathfrak{C}$ with $\operatorname{Init}\mathfrak{C} = \{I_{j,j+1}, I_{i_0 j_0}\}$ and $\operatorname{Term}\mathfrak{C} = \{I_{i_0 j}, I_{j+1}\}$ is dominant (see Figure 2.22).

Since $\mathfrak{B}$ is dominant and by the hypothesis of Case 3c, we see that for each $\tilde{I} \in \operatorname{Int}\mathfrak{C}$, $\langle I, F\rangle \geq \langle I, F'\rangle + 1$.

So, we have to show that for each vertex from $\operatorname{Nuc}\mathfrak{C}$, that is, for each vertex of the form $I_{\alpha\beta}$, where $j_0 \leq \alpha < \beta \leq j-1$, the inequality

$$\langle I_{\alpha\beta}, F\rangle \geq \langle I_{\alpha\beta}, F'\rangle + 1$$

is strict.

Let us prove this. Suppose that there exists an object $I_{\alpha_0\beta_0}$, where this inequality is an equality. Then we can apply Prop. 2.11, in a slightly different way than before:

$$\begin{aligned}\langle I_{\alpha_0 j}, F\rangle &= \langle I_{\alpha_0\beta_0}, F\rangle + \langle I_{\alpha_0 j}, F\rangle - \langle I_{i'\beta_0}, F\rangle \\ &< \langle I_{\alpha_0\beta_0}, F'\rangle + 1 + \langle I_{\alpha_0 j}, F'\rangle - \langle I_{i'\beta_0}, F'\rangle \leq \langle I_{\alpha_0 j}, F'\rangle + 1,\end{aligned}$$

that yields a contradiction.

So, having obtained a dominant region of type II, we proceed as in the case 1.

The lemma is proved. □

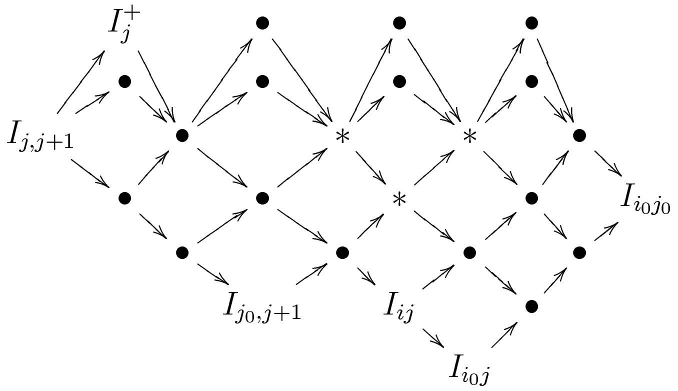



Figure 2.22: Case 3c

Having such a region $\mathfrak{B}$, let us take a *minimal* dominant region in it; that is, a dominant region $\mathfrak{C}$ satisfying the following properties:

1. The sink of $\mathfrak{C}$ equals $I$, and its source occurs in $F$ as a direct summand;
2. $\mathfrak{C}$ contains no subobjects of $F$ other that its source and its sink (minimality).

The properties 1 and 2 imply that such a region $\mathfrak{C}$ is minimal admissible. So we may perform the elementary move corresponding to $\mathfrak{C}$, thus obtaining an object $\tilde{F}$ from $F$. The property of $\mathfrak{C}$ to be dominant implies that $\langle I, \tilde{F}\rangle \geq \langle I, F'\rangle$ for each indecomposable object $I$. So, we have found the desired object $\tilde{F}$, such that

$$F \lessdot \tilde{F} \overset{\text{rk}}{\leq} F'.$$

This concludes the proof of Lemma 2.8. □

### 2.5.4 Proof of the Minimality Theorem

First note that for the "spherical case", when $\underline{\dim}F = \underline{\dim}F' = (1,\ldots,n;k;l)$, the Minimality Theorem is implied by the following fact: all degenerations that are given by an elementary move are of codimension 1, that is, $F \overset{\text{mv}}{\lessdot} F'$ implies $\dim \mathcal{O}'_F = \dim \mathcal{O}_F + 1$. However, in general this is not true. Now we shall prove this combinatorially in the general case.

Suppose we have a counterexample to the Minimality Theorem, an elementary move relation that is not a cover. This means that the relation $F \lessdot F'$ can be obtained as a longer sequence of elementary moves:

$$F = F_0 \lessdot F_1 \lessdot \cdots \lessdot F_r = F'.$$

Our aim is to show that $F_1 = F'$.

Let $\mathfrak{A}$ be the minimal admissible region which corresponds to the elementary move $F \lessdot F'$, and let $\mathfrak{B}$ be the region corresponding to the move $F \lessdot F_1$. Since $F \leq F_1 \leq F'$, then

$$\langle I, F\rangle \geq \langle I, \tilde{F}_1\rangle \geq \langle I, F'\rangle$$

for each indecomposable $I$. This means that

$$\operatorname{Int}\mathfrak{A} \supseteq \operatorname{Int}\mathfrak{B}, \qquad \operatorname{Nuc}\mathfrak{A} \supseteq \operatorname{Nuc}\mathfrak{B}. \tag{2.4}$$

So, $\mathfrak{A} \supseteq \mathfrak{B}$. This means that both initial vertices of $\mathfrak{B}$ belong to $\mathfrak{A}$. Due to the minimality of the admissible region $\mathfrak{A}$, this means that $\operatorname{Init}\mathfrak{A} = \operatorname{Init}\mathfrak{B}$.

So, the initial vertices of the regions $\mathfrak{A}$ and $\mathfrak{B}$ coincide, and $\mathfrak{A} \supseteq \mathfrak{B}$. Two cases may occur: either $\mathfrak{A}$ is of type I.b), and $\mathfrak{B}$ is of type II, or both $\mathfrak{A}$ and $\mathfrak{B}$ are of type I.b), as shown on Figure 2.8. However, in each of the both cases we obtain a contradiction with (2.4). In the first case, the inclusion $\operatorname{Nuc}\mathfrak{A} \supseteq \operatorname{Nuc}\mathfrak{B}$ does not hold. In the second case, $\operatorname{Int}\mathfrak{A} \not\supseteq \operatorname{Int}\mathfrak{B}$.

The Minimality Theorem is thus proved.

# Chapter 3

# Desingularizations of Schubert varieties in double Grassmannians

## 3.1 Introduction

In the previous chapter, we were considering orbits of $\mathrm{GL}(V)$ acting on the variety $X = \mathrm{GL}(V)/P \times \mathrm{Gr}(k, V) \times \mathrm{Gr}(l, V)$, or, in other words, orbits of a parabolic subgroup $P \subset \mathrm{GL}(V)$ acting on $Y = \mathrm{Gr}(k, V) \times \mathrm{Gr}(l, V)$. In this chapter we are going to focus on the case $P = B$.

For this case, we give another combinatorial description of orbits, which is rather close to the classical indexing of $B$-orbits in $\mathrm{Gr}(k, V)$ by means of Young diagrams. This is done in Section 3.2. It turns out that this description suits better for figuring out some combinatorial and geometric properties of $B$-orbits than the one that was considered in the previous chapter. It also does not refer to the results of [MWZ]; in the case $P = B$ everything can be done using only some elementary linear algebra. This is a generalization of the description of orbits in the symmetric space $\mathrm{GL}_{k+l}/(\mathrm{GL}_k \times \mathrm{GL}_l)$, that was obtained by Stéphane Pin in his thesis [Pin].

Further, we regard the closures of $B$-orbits in $Y$. They can be considered as analogues of Schubert varieties in Grassmannians. We are interested in their singularities. The singularities of Schubert varieties are well-known objects. They admit nice desingularizations, constructed by Bott and Samelson. They are normal, rational, their singular loci can be described explicitly. Good references on this topic are, for instance, [Br2] and [Man1]. So, it is natural to ask the same questions (resolutions of singularities, normality, rationality) for the case of $B$-orbit closures in $Y$. In this chapter we construct desingularizations of these varieties.

And finally, in Section 3.5 we provide a simple combinatorial criterion for inclusion of the $B$-orbit closures for a pair of orbits belonging to the same $B \times B$-orbit. This construction gives us a partial order on the set of involutive permutations. Quite unexpectedly, this order coincides with the order on $B$-orbits in strictly upper-triangular matrices of order 2, appearing in the recent papers [Mel1] and [Mel2] by A. Melnikov.

## 3.2 Description of orbits

### 3.2.1 Notation

As before, let $V$ be an $n$-dimensional vector space over a field $\mathbb{K}$. The results of Section 3.2 are valid over an arbitrary ground field; however, starting from Section 3.3 we assume $\mathbb{K}$ be algebraically closed.

Let $k, l < n$ be positive integers. The direct product $\mathrm{Gr}(k, V) \times \mathrm{Gr}(l, V)$ is denoted by $Y$. Usually we do not make any difference between points of $Y$ and the corresponding configurations of subspaces $(U, W)$, where $U, W \subset V$, $\dim U = k$, $\dim W = l$.

We fix a Borel subgroup $B$ in $\mathrm{GL}(V)$. Let $V_\bullet = (V_1, \ldots, V_n = V)$ be the complete flag in $V$ stabilized by $B$.

### 3.2.2 Combinatorial description

In this section we will introduce some combinatorial objects that parametrize pairs of subspaces up to $B$-action. Namely, orbits will be parametrized by triples consisting of two Young diagrams contained in the rectangles of size $k \times (n-k)$ and $l \times (n-l)$, respectively, and an involutive permutation of $S_n$.

Together with constructing these data we will also construct some "canonical" bases in subspaces $U$, $W$, and $V$, respectively.

**Proposition 3.1.** *(i). There exist ordered bases* $(u_1, \ldots, u_k)$, $(w_1, \ldots, w_l)$, *and* $(v_1, \ldots, v_n)$ *of* $U$, $W$, *and* $V$, *respectively, such that:*

- $V_i = \langle v_1, \ldots, v_i \rangle$ *for each* $i \in \{1, \ldots, n\}$ *(angle brackets stand for the linear span of vectors);*
- $u_i = v_{\alpha_i}$, *where* $i \in \{1, \ldots, k\}$, *and* $\{\alpha_1, \ldots, \alpha_k\} \subset \{1, \ldots, n\}$*;*
- *The* $w_i$ *are either basic vectors of* $V$ *or vectors with two-element "support":* $w_i = v_{\beta_i}$ *or* $w_i = v_{\gamma_i} + v_{\delta_i}$, *where* $\gamma_i > \delta_i$*; moreover, in the latter case* $v_{\gamma_i} \in U$ *(that is,* $\{\gamma_1, \ldots, \gamma_r\} \subset \{\alpha_1, \ldots, \alpha_k\}$*).*

- *All the $\beta_i$, $\gamma_i$ and $\delta_i$ are distinct; moreover, all the $\delta_i$ are distinct from the $\alpha_i$.*

*(ii). With the notation of (i), define a permutation $\sigma \in S_n$ as the product of all the transpositions $(\gamma_i, \delta_i)$. Their supports do not intersect, so this product does not depend of their order.*

*Then for the given pair $(U, W)$ the subsets $\bar{\alpha} = \{\alpha_1, \dots, \alpha_k\}$, $\bar{\beta} = \{\beta_1, \dots, \beta_{l-r}\}$, $\bar{\gamma} = \{\gamma_1, \dots, \gamma_r\}$ of $\{1, \dots, n\}$, and the permutation $\sigma$ are independent of the choice of bases in $U$, $W$, and $V$.*

*Proof.* (i) We will prove this by induction over $n$.

If $n = 1$, there is nothing to prove.

For arbitrary $n$, take a nonzero vector $v_1 \in V_1$, and consider the following cases:

- $v_1 \notin U + W$. Take the quotient $\bar{V} = V/\langle v_1 \rangle$ with the flag $\bar{V}_\bullet = \bar{V}_2 \subset \dots \subset \bar{V}_n$, consider the image of our configuration, that consists of the subspaces $\bar{U} \cong U$ and $\bar{W} \cong W$, and apply the induction hypothesis to this configuration. Let us choose ordered bases $\{\bar{u}_1, \dots, \bar{u}_k\}$, $\{\bar{w}_1, \dots, \bar{w}_l\}$, and $\{\bar{v}_1, \dots, \bar{v}_{n-1}\}$ in $\bar{U}$, $\bar{W}$, and $\bar{V}$. Then we choose a lift $\imath\colon \bar{V} \hookrightarrow V$. Now take the pre-images of these basis vectors in $V$ in the following way: $u_i = \imath(\bar{u}_i)$, $w_i = \imath(\bar{w}_i)$, $v_i = \imath(\bar{v}_{i-1})$. We get the required triple of bases.

- $v_1 \in U$, $v_1 \notin W$. Set $u_1 = v_1$ and again apply the induction hypothesis to the quotient $\bar{V} = V/\langle v_1 \rangle$ with the flag $\bar{V}_\bullet$ and the configuration $(\bar{U}, \bar{W})$. The only difference is that in this case $\dim \bar{U} = \dim U - 1$. After that we take the pre-images of the bases of $\bar{U}$, $\bar{W}$, and $\bar{V}$ in $V$ in a similar way.

- The case when $v_1 \notin U$, $v_1 \in W$, is analogous to the previous one (we set $w_1 = v_1$).

- If $v_1 \in U \cap W$, let us set $u_1 = w_1 = v_1$ and again apply the induction.

- The most interesting case is the last one: $v_1 \in U + W$, but it does not belong to any of these two subspaces. Consider then the set of vectors $S = \{v \mid v \in U, v_1 + v \in W\}$. Since $v_1$ belongs to the sum $U + W$, this set is nonempty. Now let $j$ be the minimal number such that $V_j$ contains vectors from $S$, and $v_j \in V_j \cap S$. Let us set $u_1 = v_j$, $w_1 = v_1 + v_j$. Now apply the induction hypothesis to the $(n-2)$-dimensional space

$\bar{V} = V/\langle v_1, v_j\rangle$ and to the configuration of two subspaces $\bar{U}$, $\bar{W}$, and the flag

$$\bar{V}_\bullet = V_2/V_1 \subset \cdots \subset V_{j-1}/V_1 = \\ = V_j/\langle v_1, v_j\rangle \subset V_{j+1}/\langle v_1, v_j\rangle \subset \cdots \subset V_n/\langle v_1, v_j\rangle.$$

We take the pre-images of vectors from $\bar{V}$ to $V$ as follows:

$$v_i = \imath(\bar{v}_{i-1}),\ \text{if } i \in [2, j-1]; \qquad v_i = \imath(\bar{v}_{i-2}) \text{ if } i \in [j+1, n],$$

where, as above, $\imath$ is an embedding of $\bar{V}$ into $V$. We have already defined the vectors $v_1$ and $v_j$.

(ii) Take a configuration $(U, W)$ and assume that there exist two triples of ordered bases $((u_1, \ldots, u_k), (w_1, \ldots, w_l), (v_1, \ldots, v_n))$ and $((u'_1, \ldots, u'_k), (w'_1, \ldots, w'_l), (v'_1, \ldots, v'_n))$, satisfying the conditions of (i), such that either the triples of sets $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ and $(\bar{\alpha}', \bar{\beta}', \bar{\gamma}')$, or the permutations $\sigma$ and $\sigma'$, corresponding to the first and the second triple of bases, respectively, are not equal.

The set $\bar{\alpha}$ can be described as follows. $i \in \bar{\alpha}$ iff $\dim U \cap V_i > \dim U \cap V_{i-1}$. This means that $\bar{\alpha} = \bar{\alpha}'$.

By the same argument we can prove that $\bar{\beta} \cup \bar{\gamma} = \bar{\beta}' \cup \bar{\gamma}'$.

Now let us prove that $\sigma = \sigma'$. This will complete the proof, since $\bar{\beta} = \{j \in \bar{\beta} \cup \bar{\gamma} \mid \sigma(j) = j\}$.

Let $j$ be the minimal number from $\beta \cup \gamma$, such that $\sigma(j) \neq \sigma'(j)$. Suppose that $\sigma(j) < \sigma'(j)$. Two cases may occur:

a) $i := \sigma'(j) \neq j$. First observe that $i \notin \bar{\alpha}$. Consider the subspace

$$\tilde{V} = (U \cap V_j) + V_{i-1} = \langle v_s, v_{\alpha_i} \mid s \leq i-1, \alpha_i \in \bar{\alpha} \cap [i, j]\rangle = \\ = \langle v'_s, v'_{\alpha_i} \mid s \leq i-1, \alpha_i \in \bar{\alpha}' \cap [i, j]\rangle.$$

Let $R$ and $R'$ denote respectively the sets $\{r \in \bar{\beta} \cup \bar{\gamma} \mid r, \sigma(r) \in [1, i-1] \cup (\bar{\alpha} \cap [i, j])\}$ and $\{r \in \bar{\beta} \cup \bar{\gamma} \mid r, \sigma'(r) \in [1, i-1] \cup (\bar{\alpha} \cap [i, j])\}$. One can easily see that

$$\dim \tilde{V} \cap W = \#R = \#R'.$$

But $\sigma(r) = \sigma'(r)$ for all $r \in [1, j-1]$, and $j$ belongs to $R$ and does not belong to $R'$. That means that the cardinalities of these two sets are different, that gives us the desired contradiction.

b) If $\sigma'(j) = j$, set $i = \sigma(j)$, and proceed as in a). $\square$

**Remark.** It is easy to see that, with such a choice of basis $\{v_1, \ldots, v_n\}$, the subspaces $U$ and $W$ are exactly the "standard" representatives in their normal form, as described in 2.5.1.

Let us now introduce a combinatorial construction that parametrizes configuration types. Namely, having a configuration, we will construct a pair of Young diagrams with some boxes distinguished.

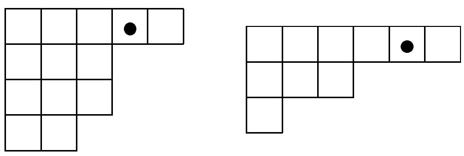
Suppose we have a configuration $(U, W)$ with bases $(u_1, \ldots, u_k)$, $(w_1, \ldots, w_l)$, and $(v_1, \ldots, v_n)$, chosen as in Prop. 3.1, the sets $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, and the involution $\sigma$ corresponding to this configuration. Consider a rectangle of size $k \times (n-k)$ and construct a path from its bottom-left to upper-right corner, such that its $j$-th step is vertical if $j$ belongs to $\bar{\alpha}$ (that is, $v_j$ is equal to some $u_i$), and horizontal otherwise. This path bounds (from below) the first Young diagram.

The second diagram will be contained in the rectangle of size $l \times (n-l)$. Again, we will construct a path bounding it. Let the $j$-th step of this path be vertical if $j \in \bar{\beta} \cup \bar{\gamma}$, and horizontal otherwise.

If $j \in \bar{\gamma}$, then the $\sigma(j)$-th step of this path is horizontal. This also means that the $j$-th and $\sigma(j)$-th steps of the path bounding the first diagram are also vertical and horizontal, respectively. In each diagram, take the box located above the $\sigma(j)$-th step and to the left of the $j$-th step, and put a dot into this box.

Let us call this pair of diagrams with dots a *marked pair.*

**Example.** Let $n = 9$, $k = 4$, $l = 3$. Suppose that $\bar{\alpha} = \{3, 5, 6, 9\}$, $\bar{\beta} = \{2, 5\}$, $\bar{\gamma} = \{9\}$, $\sigma = (7, 9)$. Then the corresponding marked pair of diagrams is the following:



**Remark.** Note that the constructed diagrams (without dots) are the same as the diagrams that correspond to the Schubert cells containing the points $U \in \mathrm{Gr}(k, V)$ and $W \in \mathrm{Gr}(l, V)$. (The correspondence between Schubert cells and Young diagrams is described, for example, in [Ful], [Man1], or any other textbook on this subject).

So, we have a combinatorial parametrization of $B$-orbits in $\mathrm{Gr}(k, V) \times \mathrm{Gr}(l, V)$. A natural question which arises is as follows: how to relate it with the paramezation described in Section 2.2, where orbits are indexed by the isomorphism classes of objects with the given dimension vector?

From Prop. 3.1 and the subsequent Remark, we can recover the decomposition of the object corresponding to an orbit. So, if by an orbit $\mathcal{O}$ we obtain the subsets $\bar{\alpha}$, $\bar{\beta}$, and $\bar{\gamma} \subset [1, n]$, and the permutation $\sigma \in S_n$, then the corresponding object $F_{\mathcal{O}}$ equals

$$F_{\mathcal{O}} = \bigoplus_{i\in\bar{\alpha}\cap\bar{\beta}} I_{i\infty} \oplus \bigoplus_{i\in\bar{\alpha}\setminus(\bar{\beta}\cup\bar{\gamma})} I_i^+ \oplus \bigoplus_{i\in\bar{\beta}\setminus\bar{\alpha}} I_i^- \oplus \bigoplus_{i\in\bar{\gamma}} I_{\sigma(i)i} \oplus \bigoplus_{i\notin\bar{\alpha}\cup\bar{\beta}\cup\sigma(\bar{\gamma})} I_{0i}.$$

This description can be easily recovered from the marked pair of diagrams corresponding to an orbit. To do this, consider the paths bounding both diagrams from below as sequences of horizontal and vertical steps, numbered from 1 to $n$, starting from the bottom-left to the upper-right corner. Now the object $F$ corresponding to this marked pair is defined as follows:

- For each dot situated above the $i$-th step and to the left of the $j$-th step, $F$ contains an indecomposable subobject $I_{ij}$;
- For each $i$, such that the $i$-steps in both diagrams are vertical and there is no dot to the left of these steps, $F$ contains $I_{i\infty}$;
- For each $i$, such that the $i$-steps in both diagrams are horizontal and there is no dot above these steps, $F$ contains $I_{0i}$;
- For each $i$, such that the $i$-step in the first diagram is vertical, and the $i$-th step in the second diagram is horizontal, $F$ contains $I_i^+$;
- And vice versa, for each $i$, such that the $i$-step is horizontal in the first diagram and vertical in the second one, $F$ contains $I_i^-$.

**Example.** According to this rule, let us recover the decomposition of the object $F$ which corresponds to the marked pair of diagrams, given in Example on Page 45:

$$F = I_{01} \oplus I_2^- \oplus \ I_3^+ \oplus I_{04} \oplus I_{5\infty} \oplus I_6^+ \oplus I_{79} \oplus I_{08}.$$

### 3.2.3 Decomposition of $Y$ into the union of $\mathrm{GL}(V)$-orbits

$\mathrm{GL}(V)$-orbits in $Y$ have a much simpler description: the $\mathrm{GL}(V)$-orbit is given only by one natural number, namely, the dimension of the intersection of a $k$-plane and an $l$-plane. For this number (denote it by $i$) we have the inequality

$$\max\{0, k+l-n\} \leq i \leq \min\{k, l\}.$$

Denote the corresponding GL($V$)-orbit by $Y_i$:

$$Y = \bigsqcup_{i\in[\max(0,k+l-n),\min(k,l)]} Y_i.$$

For each $B$-orbit the dimension of the intersection of the corresponding subspaces is equal to $\#(\bar\alpha \cap \bar\beta)$. This follows from our construction of the combinatorial data corresponding to an orbit.

### 3.2.4 Counting orbits

In this subsection we give explicit formulas for the total number $\#\mathcal{B}(Y)$ of $B$-orbits in $Y$ and for the number of $B$-orbits $\#\mathcal{B}(Y_i)$ in each $G$-orbit.

To do this, let us count the number of orbits, such that the support of the corresponding involution $\sigma \in S_n$ has exactly $2r$ elements.

First choose a $2r$-element subset Supp $\sigma$ in $\{1,\dots,n\}$. This can be done in $\binom{n}{2r}$ ways. Then let us count the number of involutions with this support. There are $2r-1$ possibilities for the first element of Supp $\sigma$, $2r-3$ for the second one, etc., so the total number equals $(2r-1)!! = 1\cdot 3\cdot\dots\cdot(2r-1)$. Let us also set by definition $(-1)!! := 1$.

After the permutation $\sigma$ is fixed, to determine the orbit we have to choose the sets $\bar\alpha\setminus\bar\gamma$ and $\bar\beta$, such that $\#(\bar\alpha\setminus\bar\gamma)$ equals to $k-r$, and $\#\bar\beta$ equals $l-r$. These subsets are chosen from $\{1,\dots,n\}\setminus$ Supp $\sigma$, so the total number of possibilities equals $\binom{n-2r}{k-r}\binom{n-2r}{l-r}$. Finally, we get the following formula:

$$\#\mathcal{B}(Y) = \sum_{r=0}^{[n/2]} \binom{n-2r}{k-r}\binom{n-2r}{l-r}\binom{n}{2r}(2r-1)!!.$$

If we want to count those possibilities where the dimension of intersection of our two subspaces $\#(\bar\alpha\cap\bar\beta)$ equals some given $i$, then after fixing the permutation choose the non-intersecting subsets $\bar\alpha\cap\bar\beta$, $\bar\alpha\setminus(\bar\beta\cup\bar\gamma)$ and $\bar\beta\setminus\bar\alpha$ from $\{1,\dots,n\}\setminus$ Supp $\sigma$, consisting of $i$, $k-r-i$ and $l-r-i$ respectively. The resulting formula is as follows:

$$\#\mathcal{B}(Y_i) = \sum_{r=0}^{[n/2]} \binom{n-2r}{i}\binom{n-2r-i}{k-r-i}\binom{n-k-r}{l-r-i}\binom{n}{2r}(2r-1)!!.$$

### 3.2.5 Stabilizers and dimensions of orbits

Now let us find the stabilizer $\mathrm{GL}(V)_{(U,W)}$ for a given configuration.

**Proposition 3.2.** *With the notation of Prop. 3.1, the stabilizer of a configuration $(U, W)$ written w.r.t. basis $(v_1, \dots, v_n)$, consists of the upper-triangular matrices $A = (a_{ij}) \in \mathrm{GL}(n)$ satisfying the following conditions:*

1. $a_{\gamma\gamma} = a_{\sigma(\gamma)\sigma(\gamma)}$ *for each* $\gamma \in \bar{\gamma}$;
2. $a_{i\alpha} = 0$ *for each* $\alpha \in \bar{\alpha}$, $i \notin \bar{\alpha}$;
3. $a_{j\beta} = 0$ *for each* $\beta \in \bar{\beta}$ *and* $j \notin \bar{\beta} \cup \bar{\gamma} \cup \sigma(\bar{\gamma})$;
4. $a_{\gamma\beta} = a_{\sigma(\gamma)\beta}$ *for each* $\beta \in \bar{\beta}$ *and* $\gamma \in \bar{\gamma}$, $\gamma < \beta$;
5. $a_{j\gamma} = -a_{j\sigma(\gamma)}$ *for each* $j \notin \bar{\beta} \cup \bar{\gamma} \cup \sigma(\bar{\gamma})$ *and* $\gamma \in \bar{\gamma}$;
6. *for each* $\gamma_1, \gamma_2 \in \bar{\gamma}$, $\gamma_1 < \gamma_2$, *one of the following cases occurs:*

   - $\sigma(\gamma_2) < \sigma(\gamma_1) < \gamma_1 < \gamma_2$*: then* $a_{\gamma_1\gamma_2} = a_{\sigma(\gamma_1)\gamma_2} = a_{\sigma(\gamma_2)\gamma_1} = a_{\sigma(\gamma_1)\sigma(\gamma_2)} = 0$;
   - $\sigma(\gamma_1) < \sigma(\gamma_2) < \gamma_1 < \gamma_2$*: then* $a_{\sigma(\gamma_2)\gamma_1} = a_{\sigma(\gamma_1)\gamma_2} = 0$, $a_{\gamma_1\gamma_2} = a_{\sigma(\gamma_1)\sigma(\gamma_2)}$;
   - $\sigma(\gamma_1) < \gamma_1 < \sigma(\gamma_2) < \gamma_2$*: then* $a_{\sigma(\gamma_1)\gamma_2} = 0$, $a_{\gamma_1\gamma_2} + a_{\gamma_1\sigma(\gamma_2)} = a_{\sigma(\gamma_1)\sigma(\gamma_2)}$.

**Corollary 3.3.** *The stabilizer of a configuration $(U, W)$ is a semidirect product of a toric and a unipotent part:*

$$\mathrm{GL}(V)_{(U,W)} = T_{(U,W)} \ltimes U_{(U,W)},$$

*where $T_{(U,W)}$ is the subgroup in the group of diagonal matrices defined by the equations* 1.*, so that* $\dim T_{(U,W)} = n - \#\bar{\gamma}$*, and $U_{(U,W)}$ is the subgroup in the group of unitriangular matrices, defined by the equations* 2.–6.

**Definition.** The codimension of the toric part of the stabilizer is said to be the *rank* of a configuration (or its corresponding orbit):

$$\mathrm{rk}(U, W) := n - \dim T_{(U,W)} = \#\bar{\gamma}.$$

*Proof of the proposition.* First of all, the stabilizer $B_{(U,W)}$ is formed by upper-triangular matrices, as a subgroup of $B$.

Next, it preserves the subspace $U = \langle v_{\alpha_1}, \dots, v_{\alpha_k} \rangle$. This means that a transformation $A \in B_{(U,W)}$ maps each $v_{\alpha_i}$ into a linear combination of $v_{\alpha_j}$, so all the elements $a_{i\alpha}$, where $\alpha \in \bar{\alpha}$, $i \notin \bar{\alpha}$, vanish. (Note that the zeros in $A$ obtained in this way also form a Young diagram corresponding to the subspace $U$, rotated 90° clockwise. This proves, in particular, that the dimension of a Schubert cell in a Grassmannian is equal to the number of boxes in the corresponding diagram.)

So, the boxes of the first Young diagram are in a one-to-one correspondence with the linear equations defining $B_U$ as a subgroup of the group of upper-triangular matrices: the box located above the $i$-th (horizontal) step and to the left of the $j$-th (vertical) step of the corresponding path (denote this box by $(i, j)$) corresponds to the equation $a_{ij} = 0$.

Similarly, the stabilizer of our configuration preserves the subspace $W$. This gives us a set of linear equations on the elements $a_{ij}$, and the number of them is equal to the number of boxes in the second diagram of the corresponding marked pair. Again, we establish a one-to-one correspondence between the boxes of this diagram and these equations, denoting boxes as in the previous paragraph. Here they are:

- $a_{j\beta} = 0$ for each $\beta \in \bar{\beta}$ and $j \notin \bar{\beta} \cap \bar{\gamma} \cap \sigma(\bar{\gamma})$, $j < \beta$. The corresponding box is $(j, \beta)$;

- $a_{j\gamma} = -a_{j\sigma(\gamma)}$ for each $j \notin \bar{\beta} \cup \bar{\gamma} \cup \sigma(\bar{\gamma})$ and $\gamma \in \bar{\gamma}$, $j < \gamma$. The corresponding box is $(j, \gamma)$;

- $a_{\sigma(\gamma)\gamma} + a_{\gamma\gamma} - a_{\sigma(\gamma)\sigma(\gamma)} = 0$ for each $\gamma \in \bar{\gamma}$. The corresponding box is $(\sigma(\gamma), \gamma)$;

- $a_{\gamma\beta} = a_{\sigma(\gamma)\beta}$ for each $\beta \in \bar{\beta}$ and $\gamma \in \bar{\gamma}$, $\gamma < \beta$. The corresponding box is $(\sigma(\gamma), \beta)$;

- $a_{\sigma(\gamma_1)\sigma(\gamma_2)} + a_{\sigma(\gamma_1)\gamma_2} = a_{\gamma_1\sigma(\gamma_2)} + a_{\gamma_1\gamma_2}$ for each $\gamma_1 < \gamma_2$. This equation corresponds to the box $(\sigma(\gamma_1), \gamma_2)$.

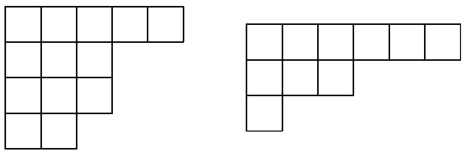
Bringing all these equations together completes the proof of the proposition. □

Once we know the stabilizer of a configuration, we can calculate its dimension (and hence the dimension of the orbit $B(U, W) \subset Y$). Analyzing the equations above, one can deduce a combinatorial interpretation of dimension in terms of Young diagrams with dots.

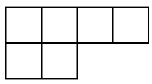
To do this, we have to introduce one more combinatorial notion. Suppose we have two rectangles of size $k \times (n-k)$ and $l \times (n-l)$, respectively, and a path in each of these rectangles bounding a Young diagram (so both paths are of the length $n$). Consider the set of all numbers $i$, such that the $i$-th steps in the paths bounding both diagrams are horizontal, and take the columns in the diagrams lying above these steps. After that let us do the same for those pairs of steps that are "simultaneously vertical", and take the rows to the left of these steps.

The intersection of columns and rows we have taken also forms a Young diagram. Let us call it a *common diagram* corresponding to the given pair of diagrams.

**Example.** The pair of Young diagrams



has the following common diagram:



By our construction of marked pairs, dots can only be situated in the boxes of the common diagram of a marked pair.

**Corollary 3.4.** *Let $(U, W)$ be a configuration of subspaces, and let $(\mathcal{Y}_1, \mathcal{Y}_2)$ be the corresponding marked pair of Young diagrams, with dots in some boxes of its common diagram $\mathcal{Y}_{com}$.*

*Now take the diagram $\mathcal{Y}_{com}$. Take all its boxes with dots and consider all the hooks with spikes in these boxes. Let $H$ be the set of boxes that belong to at least one of these hooks. Then the dimension of the $B$-orbit of $(U, W)$ equals*

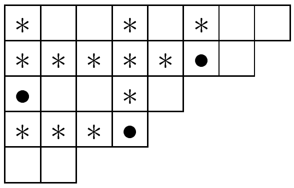
$$\dim B(U, W) = \#\mathcal{Y}_1 + \#\mathcal{Y}_2 - \#\mathcal{Y}_{com} + \#H,$$

*where $\#Y$ denotes the number of boxes in $\mathcal{Y}$.*

**Remark.** $\#H$ equals the total number of boxes contained in all the hooks, not the sum of all the hooks' lengths. That means that a box included into two hooks must be counted once, not twice!

*Proof.* In the proof of Prop. 3.2 we deal with two systems of linear equations on the matrix entries $(a_{ij})$, that correspond to stabilizing the subspaces $U$ and $W$ and consist of $\#\mathcal{Y}_1$ and $\#\mathcal{Y}_2$ equations, respectively. One can easily see that the equations corresponding to the box $(i, j)$ coincide in both systems iff the box $(i, j)$ of the common diagram does not belong to any hook, and also that the system obtained by eliminating these "double" equations is linearly independent. So, the codimension of $B_{(U,W)}$ in $B$ (that is, the dimension of $B(U, W)$) equals $\#\mathcal{Y}_1 + \#\mathcal{Y}_2 - \#\mathcal{Y}_{com} + \#H$. $\square$

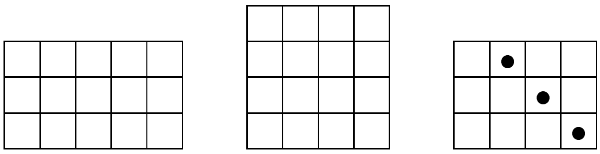
**Example.** Let the common diagram for a marked pair be as follows:



Then $\#\mathcal{Y}_{com} = 26$, $\#H = 15$ (boxes belonging to $H$ are the non-empty ones).

In particular, the dimension formula allows us to describe the minimal, or the most special, and the maximal (open) orbit. The most special orbit is zero-dimensional and corresponds to $\mathcal{Y}_1 = \mathcal{Y}_2 = \varnothing$. It is the point $(\langle v_1, \ldots, v_k\rangle, \langle v_1, \ldots, v_l\rangle) \in Y$. Both Young diagrams corresponding to the most generic orbit are rectangular, of size $k \times (n-k)$ and $l \times (n-l)$, respectively. So, their common diagram is also a rectangle of size $\min\{k,l\} \times (n - \max\{k,l\})$, with dots situated on a diagonal starting from the bottom-right corner.

**Example.** For $n = 8$, $k = 3$, and $l = 4$, the combinatorial data corresponding to the maximal orbit are as follows:



$\mathcal{Y}_1$ $\quad$ $\mathcal{Y}_2$ $\quad$ $\mathcal{Y}_{com}$

## 3.3 The weak order on the set of orbits

Starting from this point, we work over an algebraically closed ground field $\mathbb{K}$.

In the previous section we described the set of $B$-orbits in $\mathrm{Gr}(k,V) \times \mathrm{Gr}(l,V)$. There exist several partial order structures on this set. The first, and the most natural one, is defined as follows:

**Definition.** Let $\mathcal{O}$ and $\mathcal{O}'$ be two $B$-orbits in $\mathrm{Gr}(k,V) \times \mathrm{Gr}(l,V)$. We say that $\mathcal{O}$ is less or equal than $\mathcal{O}'$ w.r.t. the *strong* (or *topological*) *order*, iff $\mathcal{O} \subset \bar{\mathcal{O}}'$. (Saying "topological", we speak about the Zariski topology). Notation: $\mathcal{O} \le \mathcal{O}'$.

There exists another order on this set, usually called the weak order. Here notation and terminology are taken from [Br1].

Let $W$ be the Weyl group for $\mathrm{GL}(n)$, and let $\Delta$ be the corresponding root system. Denote the simple reflections by $s_1, \ldots, s_{n-1}$, and the corresponding simple roots by $\alpha_1, \dots, \alpha_{n-1}$. Let $P_i = B \cup Bs_iB$ be the minimal parabolic subgroup in $\mathrm{GL}(V)$ corresponding to the simple root $\alpha_i$.

We say that $\alpha_i$ *raises* an orbit $\mathcal{O}$ to $\mathcal{O}'$, if $\bar{\mathcal{O}}' = P_i\bar{\mathcal{O}} \neq \bar{\mathcal{O}}$. In this case, $\dim \mathcal{O}' = \dim \mathcal{O} + 1$. This notion allows us to define the weak order.

**Definition.** An orbit $\mathcal{O}$ is said to be less or equal than $\mathcal{O}'$ w.r.t. the *weak order* (notation: $\mathcal{O} \preceq \mathcal{O}'$), if $\bar{\mathcal{O}}'$ can be obtained as the result of several consecutive raisings of $\bar{\mathcal{O}}$ by minimal parabolic subgroups:

$$\mathcal{O} \preceq \mathcal{O}' \quad \Leftrightarrow \quad \exists (i_1, \dots, i_r) \colon \bar{\mathcal{O}}' = P_{i_r} \dots P_{i_1} \bar{\mathcal{O}}.$$

Let us represent this relation of order by an oriented graph. Consider a graph $\Gamma(Y)$ with vertices indexed by $B$-orbits in $Y$. Join $\mathcal{O}$ and $\mathcal{O}'$ with an edge of label $i$, leading to $\mathcal{O}'$, if $P_i$ raises $\mathcal{O}$ to $\mathcal{O}'$.

It is clear that the connected components of $\Gamma(Y)$ consist of the $B$-orbits contained in the same $\mathrm{GL}(V)$-orbit $Y_i$, and that every connected component has a unique maximal element (the $B$-orbit that is open in $Y_d$).

Our next aim will be to describe minimal elements w.r.t. the weak order in each connected component.

### 3.3.1 Combinatorial description of minimal parabolic subgroup action

Consider an orbit $\mathcal{O}$ and the corresponding combinatorial data: the sets $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, and the involution $\sigma \in S_n$. Let the minimal parabolic subgroup $P_i = B \cup Bs_iB$ raise the orbit $\mathcal{O}$ to the orbit $\mathcal{O}' \neq \mathcal{O}$. Now we will describe the combinatorial data $(\bar{\alpha}', \bar{\beta}', \bar{\gamma}', \sigma')$ of $\mathcal{O}'$.

Denote the transposition $(i, i+1) \in S_n$ by $\tau_i$.

The following cases may occur:

1. Suppose that

$$i \in \bar{\alpha}, \quad i \notin \bar{\beta}, \quad i+1 \notin \bar{\alpha}, \quad i+1 \in \bar{\beta},$$

or, vice versa,

$$i \notin \bar{\alpha}, \quad i \in \bar{\beta}, \quad i+1 \in \bar{\alpha}, \quad i+1 \notin \bar{\beta}.$$

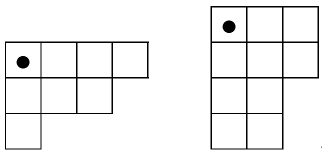
These two variants correspond to two orbits that could be risen to $\mathcal{O}'$. In this case, the new combinatorial data is given as follows:

$$\begin{aligned}
\bar{\alpha}' &= \bar{\alpha} \cup \{i+1\} \setminus \{i\}; \\
\bar{\beta}' &= \bar{\beta} \setminus \{i, i+1\}; \\
\bar{\gamma}' &= \bar{\gamma} \cup \{i+1\} \\
\sigma' &= \sigma \cdot \tau_i.
\end{aligned}$$

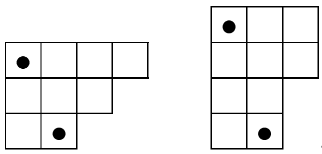
Note that $\operatorname{rk}\mathcal{O}' = \operatorname{rk}\mathcal{O} + 1$, $\dim\mathcal{O}' = \dim\mathcal{O} + 1$.

In the language of marked pairs of diagrams, this is represented as follows. If the $i$-th and the $i+1$-th steps of the path bounding the first diagram form a ravine (that is, the $i$-th step is horizontal, and the $i+1$-th step is vertical), and the corresponding intervals of the second diagram form a spike (vertical step is followed by a horizontal one) or, vice versa, we have a spike in the first diagram and a ravine in the second, both these pairs of steps can be replaced by spikes bounding a marked box.

**Example.** Apply the minimal parabolic subgroup $P_2$ to the orbit $\mathcal{O} \subset \operatorname{Gr}(3,7) \times \operatorname{Gr}(4,7)$ defined by the following marked pair:



The orbit $\mathcal{O}'$ obtained as the result of this raising is defined by the marked pair



2. In all the other cases $\bar{\alpha}' = \tau_i(\bar{\alpha})$, $\bar{\beta}' = \tau_i(\bar{\beta})$, $\bar{\gamma}' = \tau_i(\bar{\gamma})$, and the permutation $\tilde{\sigma}$ is the result of the conjugation of $\sigma$ by $\tau_i$:

$$\tilde{\sigma} = \tau_i \sigma \tau_i.$$

The ranks of these orbits are equal: $\operatorname{rk}\mathcal{O}' = \operatorname{rk}\mathcal{O}$.

### 3.3.2 The weak order from the quiver point of view

In this subsection we describe the weak order by means of the language of Auslander–Reiten quivers, developed in Chapter 2.

Let $\mathcal{O}$ and $\mathcal{O}'$ be two $B$-orbits in $Y$, such that $\mathcal{O}'$ is obtained from $\mathcal{O}$ by the action of a minimal parabolic subgroup $P_i$:

$$\overline{\mathcal{O}'} = P_i \cdot \overline{\mathcal{O}}. \tag{3.1}$$

As we showed in Chapter 2, this means that for the corresponding $\mathrm{GL}(V)$-orbits $\mathcal{O}_X$ and $\mathcal{O}'_X$ in $X = Y \times \mathrm{Fl}\,(V)$, the latter can be obtained from the former by an elementary move. But the weak order and the strong order in general do not coincide, so the converse is false. So, one would like to know which elementary moves correspond to the action of a minimal parabolic subgroup. The answer is given by the following proposition.

**Proposition 3.5.** *The equality (3.1) holds iff for the objects $F$ and $F'$, corresponding to $\mathcal{O}$ and $\mathcal{O}'$,*

$$F \lessdot F',$$

*and, moreover, the corresponding elementary move is of type I.a), I.c), I.d), I.e), or II, and the source and the sink of the corresponding admissible region belong to neighbor roads.*

*Proof.* Each minimal parabolic subgroup may be presented as the closure of the product

$$P_i = \overline{U_i^- \cdot B},$$

where $U_i^- = \{E + \tau E_{i+1,i} \mid \tau \in \mathbb{K}\}$ is a one-dimensional unipotent subgroup consisting of the matrices whose diagonal entries equal 1, and the only nonzero non-diagonal entry, situated in the $i+1$-th line and $i$-th column, equals $\tau$.

For a pair of orbits $\mathcal{O}$ and $\mathcal{O}'$ in $Y$, such that $\overline{\mathcal{O}'} = P_i\overline{\mathcal{O}}$, and a representative $(U, W) \in \mathcal{O}$, the action of $U_i^-$ gives us the curve $U_i^-(U, W) = \{(U(\tau), W(\tau)\} \subset \overline{\mathcal{O}'}$. For a general $\tau$, the point $(U(\tau), W(\tau))$ belongs to the orbit $\mathcal{O}'$.

We see that, for the canonical representative $(U, W, V_\bullet) \in \mathcal{O}_X \subset X$ corresponding to $\mathcal{O} \subset Y$, the curve $(U(\tau), W(\tau), V_\bullet) \subset \mathcal{O}'_X$ is exactly the one that was constructed in the proof of Lemma 2.6. The corresponding region has its source and sink on the roads beginning at $I_{i+1,\infty}$ and $I_{i\infty}$ and is *not* of type I.b).

Conversely, let $F \lessdot F'$. Suppose that the elementary move transferring $F$ to $F'$ is not of type I.b), and that the source and the sink of the corresponding minimal admissible region belong to the roads beginning in $I_{r\infty}$ and $I_{s\infty}$ respectively, $s < r$. Then the curve constructed in the proof of Lemma 2.6 is of the form

$$\begin{aligned} U(\tau) &= A_{rs}(\tau)U; \\ W(\tau) &= A_{rs}(\tau)W; \\ V_\bullet(\tau) &= V_\bullet, \end{aligned}$$

where $A_{rs}(\tau) = E + \tau E_{rs}$ is again a matrix with one nonzero nondiagonal entry. So, this action is given by the minimal parabolic subgroup $P_i$ iff $s = i$ and $r = i + 1$.

□

### 3.3.3 Minimal orbits

**Lemma 3.6.** *All minimal $B$-orbits w.r.t. the weak order in a given* $\mathrm{GL}(V)$-*orbit have rank 0.*

*Proof.* Assume the converse. Let $\mathcal{O}$ be a minimal orbit with a nonzero rank, and let $(\bar\alpha, \bar\beta, \bar\gamma, \sigma)$ the corresponding combinatorial data, such that $\sigma \neq Id$. Let $p \in \bar\gamma$, $p' = \sigma(p)$. Without loss of generality we can suppose that there is no other $q \in \bar\gamma$, such that $p' < \sigma(q) < q < p$.

Let $R_1$ denote the set of ravines in the first diagram, situated between $p$ and $p'$ — that is, the set of indices $i$, such that the $i$-th step in the first diagram is horizontal, and the $i+1$-th is vertical. Similarly, let $S_1$ denote the set of spikes, — that is, the set of $i$, such that the $i$-th step is vertical, and the $i+1$-st is horizontal. Denote the same sets for the second diagram by $R_2$ and $S_2$. Note that $\#R_1 = \#S_1 + 1$, and $\#R_2 = \#S_2 + 1$ — since $p'$-th steps in both diagrams are horizontal, and $p$-th steps are vertical.

Now take a $j$, such that $j \in (R_1 \setminus S_2) \cup (R_2 \setminus S_1)$. Let us show that there exists an orbit $\mathcal{O}'$, such that $\bar{\mathcal{O}} = P_j \bar{\mathcal{O}}'$. We describe the combinatorial data for this orbit.

If the permutation $\sigma$ contains the transposition $(j, j+1)$, then the combinatorial data for $\mathcal{O}'$ is as follows:

$$\begin{aligned} \bar\alpha' &= \bar\alpha \cup \{j\} \setminus \{j+1\}; \\ \bar\beta' &= \bar\beta \cup \{j\}; \\ \bar\gamma' &= \bar\gamma \setminus \{j+1\} \\ \sigma' &= \sigma \cdot \tau_j. \end{aligned}$$

Otherwise $\bar\alpha' = \tau_j(\bar\alpha)$, $\bar\beta' = \tau_j(\bar\beta)$, $\bar\gamma' = \tau_j(\bar\gamma)$, $\sigma' = \tau_j \sigma \tau_j$.

The calculation of the dimensions shows that $\dim \mathcal{O}' = \dim \mathcal{O} - 1$.

To complete the proof, we have to show that the set $(R_1 \setminus S_2) \cup (R_2 \setminus S_1)$ is nonempty:

$$\begin{aligned} \#((R_1 \setminus S_2) \cup (R_2 \setminus S_1)) &\geq \max(\#(R_1 \setminus S_2), \#(R_2 \setminus S_1)) \geq \\ &\geq \max(\#R_1 - \#R_2 + 1, \#R_2 - \#R_1 + 1) \geq 1. \end{aligned}$$

□

After that we can find all the minimal orbits in $Y_d$. One can easily see that each minimal orbit has the following combinatorial data:

$$\begin{aligned} \bar\alpha \cup \bar\beta &= \{1, \ldots, k+l-d\}; \\ \bar\alpha \cap \bar\beta &= \{1, \ldots, d\}; \\ \bar\gamma &= \varnothing; \\ \sigma &= Id. \end{aligned}$$

The dimension of all minimal orbits in $Y_d$ equals $(k-d)(l-d)$. In particular, that means that they all are closed in $Y_d$. They correspond to decompositions of the set $\{d+1, \ldots, k+l-d\}$ into two parts, $\bar{\alpha} \setminus \bar{\beta}$ and $\bar{\beta} \setminus \bar{\alpha}$, so their number is equal to $\binom{k+l-2d}{k-d}$.

Also note that the pair of Young diagrams that corresponds to a minimal orbit is complementary: one can put these two diagrams together so that they will fill a rectangle of size $(k-d) \times (l-d)$.

It is also clear that no other $B$-orbit corresponds to such pair of Young diagrams. That means that all the minimal orbits are stable under the $(B \times B)$-action, that is, they are direct products of two Schubert cells in two Grassmannians.

These results can be summarized as the following theorem.

**Theorem 3.7.** *Each $Y_d$, where $d \in \{\max(k+l-n, 0), \ldots, \min(k, l)\}$, contains $\binom{k+l-2d}{k-d}$ minimal orbits. All these orbits are closed in $Y_d$ and have dimension $(k-d)(l-d)$. They are direct products of Schubert cells.*

## 3.4 Desingularizations of the orbit closures

In this section we construct desingularizations for the $B$-orbit closures in $Y$. Given a minimal parabolic subgroup $P_i$ and an orbit closure $\bar{\mathcal{O}}$, consider the morphism

$$F_i \colon P_i \times^B \bar{\mathcal{O}} \to P_i \bar{\mathcal{O}},$$

$$(p, x) \mapsto px.$$

Knop [Kn] and Richardson–Springer [RS] showed that the following three cases may occur:

- Type U: $P_i\mathcal{O} = \mathcal{O}' \sqcup \mathcal{O}$, and $F_i$ is birational;
- Type N: $P_i\mathcal{O} = \mathcal{O}' \sqcup \mathcal{O}$, and $F_i$ is of degree 2;
- Type T: $P_i\mathcal{O} = \mathcal{O}' \sqcup \mathcal{O} \sqcup \mathcal{O}''$, and $F_i$ is birational. In this case $\dim \mathcal{O}'' = \dim \mathcal{O}$.

It turns out that in our situation the case N never occurs.

**Proposition 3.8.** *Let $\mathcal{O}$ be a $B$-orbit in $Y$ and let $P_i$ be a minimal parabolic subgroup raising this orbit. Then the map $F_i \colon P_i \times^B \mathcal{O} \to P_i\mathcal{O}$ is birational.*

*Proof.* Choose the canonical representative $x \in \mathcal{O}$ as in Prop. 3.1. A direct calculation shows that the isotropy group of $Y$ in $P_i$ equals the isotropy group of $Y$ in $B$, described in Prop. 3.2. This implies the birationality of $F_i$. □

**Remark.** The two remaining cases correspond to the two possible "raisings" described in the subsection 3.3.1: (T) corresponds to (1), and (U) corresponds to (2). In the first case, the rank of the orbit is increased by one, and in the second case, it does not change. So, the weak order is compatible with the rank function: if $\mathcal{O} \preceq \mathcal{O}'$, then $\operatorname{rk}\mathcal{O} \leq \operatorname{rk}\mathcal{O}'$. This is true in general for spherical varieties (cf., for instance, [Br1]). Note that the strong order is *not* compatible with the rank function.

Proposition 3.8 together with Theorem 3.7 allows us to construct desingularizations for $\bar{\mathcal{O}}$'s similar to Bott–Samelson desingularizations of Schubert varieties in Grassmannians.

Given an orbit $\mathcal{O}$, consider a minimal orbit $\mathcal{O}_{min}$ that is less that $\mathcal{O}$ w.r.t. the weak order. That means that there exists a sequence of minimal parabolic subgroups $(P_{i_1}, \dots, P_{i_r})$, such that

$$\bar{\mathcal{O}} = P_{i_r} \dots P_{i_1} \bar{\mathcal{O}}_{min}.$$

So, we can consider the map

$$F\colon P_{i_r} \times^B \dots \times^B P_{i_1} \times^B \bar{\mathcal{O}}_{min} \to \bar{\mathcal{O}},$$

$$F\colon (p_{i_r}, \dots, p_{i_1}, x) \mapsto p_{i_r} \dots p_{i_1} x.$$

According to Proposition 3.8, it is birational. But this is not yet a desingularization, because $\bar{\mathcal{O}}_{min}$ can be singular.

The second step of the desingularization consists in constructing a $B$-equivariant desingularization for $\bar{\mathcal{O}}_{min}$. We have already proved in Theorem 3.7 that $\bar{\mathcal{O}}_{min}$ can be presented as the direct product

$$\bar{\mathcal{O}}_{min} = X_w \times X_v$$

for some Schubert varieties $X_w \subset \operatorname{Gr}(k, V)$ and $X_v \subset \operatorname{Gr}(l, V)$.

For $X_w$ and $X_v$ one can take Bott–Samelson desingularizations

$$F_w\colon Z_w \to X_w \quad \text{and} \quad F_v\colon Z_v \to X_v.$$

(Details can be found, for instance, in [Br2]). So, we get a desingularization

$$F_w \times F_v\colon \quad Z_w \times Z_v \to X_w \times X_v = \bar{\mathcal{O}}_{min}.$$

Having this, we can combine this map with the map $F$ and get the main result of this chapter:

**Theorem 3.9.** *The map*

$$\tilde{F} = F \circ (F_w \times F_v)\colon \quad P_{i_r} \times^B \dots \times^B P_{i_1} \times^B (Z_w \times Z_v) \to \bar{\mathcal{O}}$$

*is a desingularization of* $\bar{\mathcal{O}}$.

*Proof.* We have already seen that both maps $F$ and $F_w \times F_v$ are birational morphisms. Since all the considered varieties are projective, these morphisms are proper. The space $P_{i_r} \times^B \cdots \times^B P_{i_1} \times^B (Z_w \times Z_v)$ is a homogeneous $B$-bundle over a nonsingular variety, hence it is nonsingular itself. ☐

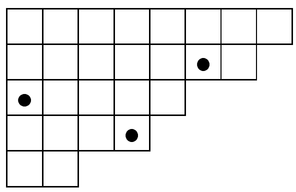
## 3.5 Bruhat order in a $B \times B$-orbit

Let $C_w \times C_v \subset \mathrm{Gr}(k, V) \times \mathrm{Gr}(l, V)$ be a $B \times B$-orbit, that is, the direct product of two Schubert cells in the Grassmannians. Let $(\mathcal{Y}_1, \mathcal{Y}_2)$ be the corresponding pair of Young diagrams. Take their common diagram $\mathcal{Y}_{com}$, as described in Section 3.2.2; as we showed above, $B$-orbits in $C_w \times C_v$ correspond to rook placements in $\mathcal{Y}_{com}$, such that the rooks do not attack each other. In this section, we describe the inclusion order on the closures of these $B$-orbits.

Throughout this section, we fix $C_w \times C_v$ and, thus, the Young diagrams $\mathcal{Y}_1$, $\mathcal{Y}_2$, and their common diagram $\mathcal{Y}_{com}$.

**Definition.** For a rook placement in the diagram $\mathcal{Y}_{com}$, define an *associated Young tableau* $T$ of shape $\mathcal{Y}_{com}$ as follows: to a box we assign the number of rooks located (non-strictly) to the South-East of it. Obviously, in such a way we obtain a tableau with the entries weakly decreasing by rows and columns.

**Example.** With a rook placement



we associate the following tableau:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 3 | 2 | 2 | 2 | 1 | 1 | 0 | 0 |
| 3 | 2 | 2 | 2 | 1 | 1 | 0 | |
| 2 | 1 | 1 | 1 | 0 | | | |
| 1 | 1 | 1 | 1 | | | | |
| 0 | 0 | | | | | | |

.

Thus, we obtain a map from the set of $B$-orbits in $C_w \times C_v$ into Young tableaux of shape $\mathcal{Y}_{com}$: having an orbit, we construct a rook placement in $\mathcal{Y}_{com}$, and then we associate to the latter a tableau of shape $\mathcal{Y}_{com}$.

**Definition.** Define a partial order on tableaux of given shape $\mathcal{Y}_{com}$ as follows: we shall say that $T_1 \leq T_2$, if for each box in $\mathcal{Y}_{com}$ the entry from $T_1$ in this box is less or equal than the corresponding entry from $T_2$.

We claim that this order is equal to the topological order on the set of $B$-orbits:

**Theorem 3.10.** *Let $\mathcal{O}_1, \mathcal{O}_2 \subset C_w \times C_v$ be two $B$-orbits, and let $T_1, T_2$ be the associated tableaux. Then $\mathcal{O}_1 \subset \overline{\mathcal{O}}_2 \Leftrightarrow T_1 \leq T_2$.*

To prove the theorem, begin with a combinatorial lemma.

**Lemma 3.11.** *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two rook placements in $\mathcal{Y}_{com}$, with the corresponding tableaux $T_1$, $T_2$. Then the inequality $T_1 \leq T_2$ holds if and only if $\mathcal{R}_2$ can be obtained from $\mathcal{R}_1$ by a sequence of the following operations:*

1. *Adding a rook;*
2. *Moving a rook to SE;*
3. *For a rectangle with two rooks situated in its NE and SW corners, replacing them with the rooks in its NW and SE corners.*

*Proof.* The "if" assertion is evident. Let us prove the converse.

We will proceed by induction on the number of rooks $R$ such that $R \in \mathcal{R}_1 \setminus \mathcal{R}_2$. If $\#(\mathcal{R}_1 \setminus \mathcal{R}_2) = 0$, there is nothing to prove: $\mathcal{R}_2$ is obtained by $\mathcal{R}_1$ by adding rooks.

If $\#(\mathcal{R}_1 \setminus \mathcal{R}_2) \neq 0$, let us show how to find a rook placement $\mathcal{R}'$, obtained from $\mathcal{R}_1$ by an operation of type (ii) or (iii), and such that $T' \leq T_2$. To do this, take a rightmost rook $R_1$ from $\mathcal{R}_1 \setminus \mathcal{R}_2$, and a rightmost rook $R_2$ from $\mathcal{R}_2 \setminus \mathcal{R}_1$. Since $T_1 \leq T_2$, the rook $R_2$ is located to SE from $R_1$. Three cases can occur:

(1) $R_1$ and $R_2$ are in the same row. Then $\mathcal{R}'$ is obtained from $\mathcal{R}_1$ by moving the rook $R_1$ into the position $R_2$. Obviously, for the tableaux $T'$ and $T_2$ corresponding to $\mathcal{R}'$ and $\mathcal{R}_2$, the inequality $T' \leq T_2$ holds, and $\#(\mathcal{R}' \setminus \mathcal{R}_2) = \#(\mathcal{R}_1 \setminus \mathcal{R}_2) - 1$.

(2) There is no rook from $\mathcal{R}_1$ in the row occupied by $R_2$. This case is completely analogous to the prevoius one: again we move $R_1$ into the position $R_2$.

(3) There is a rook $R' \in \mathcal{R}_1$, $R' \neq R_1$, situated in the row occupied by $R_2$. Since $R_1$ is the rightmost rook from $\mathcal{R}_1$, $R'$ is situated to the SW from $R_1$. Then we can replace $R_1$ and $R'$ by the two other corners of the corresponding rectangle, as described in (iii). We obtain the rook placement $\mathcal{R}'$, such that its rightmost rook is in the same row with the rightmost rook $R_2 \in \mathcal{R}_2$. The situation is thus reduced to the case (1). $\square$

*Proof of Theorem.* In Section 2.4.3, we classify the minimal degenerations of $B$-orbit closures. It can be easily seen that the operations (i)–(iii) on rook placements correspond to the degenerations of type I.a). The degenerations of this type take a $B$-orbit into another one, such that the both $B$-orbits are situated in the same $B \times B$-orbit.

So, if $T_1 \leq T_2$, then, due to Lemma, the rook placement $\mathcal{R}_2$ can be obtained from $\mathcal{R}_2$ by a sequence of operations (i)–(iii), and each of these operations corresponds to a minimal degeneration of type I.a). Hence, $\mathcal{O}_1 \subset \overline{\mathcal{O}}_2$. □

**Remark.** As described in Section 3.2, instead of considering rook placements one can consider involutive permutations from $S_n$, where $n = \dim V$. So, the rook placements in a given Young diagram $\mathcal{Y}_{com}$ give us a certain subset $S_n^2(\mathcal{Y}) \subset S_n^2$ in the set of involutive permutations (or *link patterns*, see [Mel1] for details). In [Mel1], [Mel2], Anna Melnikov studies $B$-orbits on strictly upper-triangular matrices of the nilpotent order 2. They are indexed by the set $S_n^2$ of involutive permutations, so one gets a partial order on $S_n^2$, arising from the topological order on the $B$-orbits.

As one can see from [Mel1, 1.5], the restriction of this partial order onto each set $S_n^2(\mathcal{Y})$ coincides with the ours. It would be interesting to investigate any deeper relations between the situation considered by Melnikov and our situation.

# Appendix A

# Some examples

In the Appendix, we consider $\mathrm{Gr}(k,n) \times \mathrm{Gr}(l,n)$ for small $n$ and give the inclusion diagrams of $B$-orbit closures in these varieties.

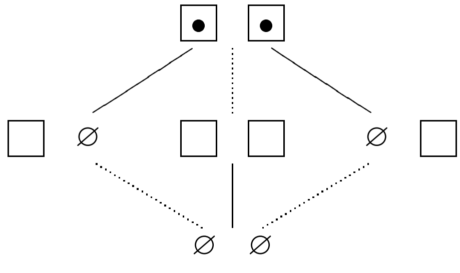
Here we follow the notation from Section 3.2.2: orbits are parametrized by marked pairs of Young diagrams.

The weak order is represented by solid edges. Two marked pairs are connected with a solid edge labeled with an integer $i$ if and only if the orbit corresponding to the upper marked pair can be obtained from the one corresponding to the lower marked pair by the action of the minimal parabolic subgroup $P_i$.

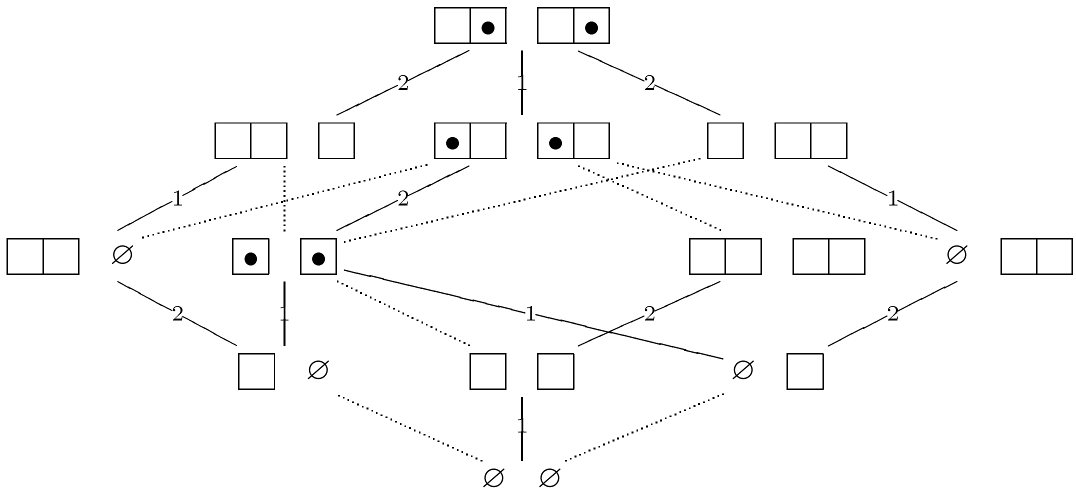
Dotted edges correspond to the pairs of orbits $(\mathcal{O}, \mathcal{O}')$ of codimension 1, such that $\mathcal{O} \subset \mathcal{O}'$, but $\mathcal{O}'$ cannot be obtained from $\mathcal{O}$ by action of any parabolic.

Each horizontal level of the inclusion diagram contains the orbits of a given dimension; the orbit of the maximal dimension is at the top of the diagram, and the zero-dimensional orbit is at the bottom.

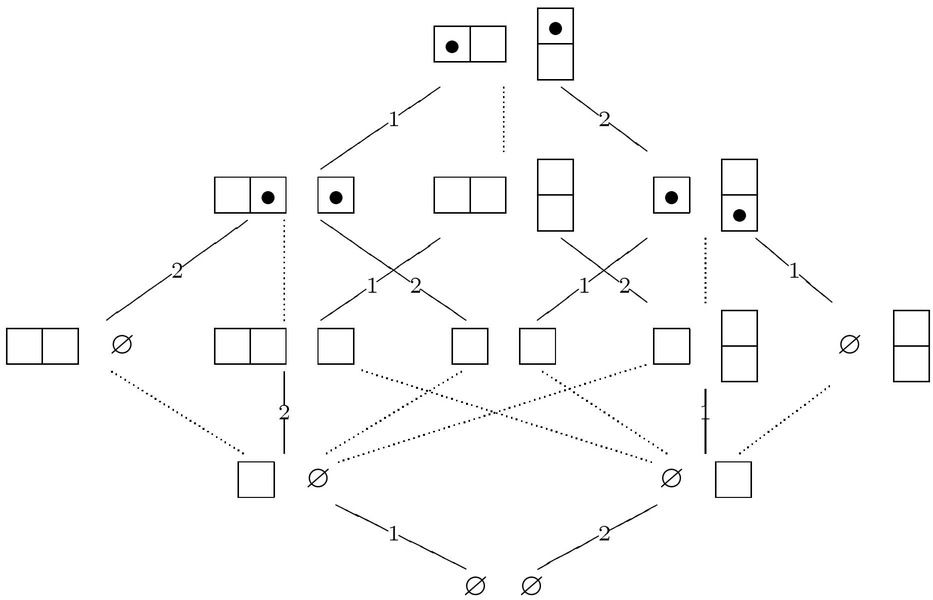
## A.1 $\mathbb{P}^1 \times \mathbb{P}^1 = \mathrm{Gr}(1,2) \times \mathrm{Gr}(1,2)$



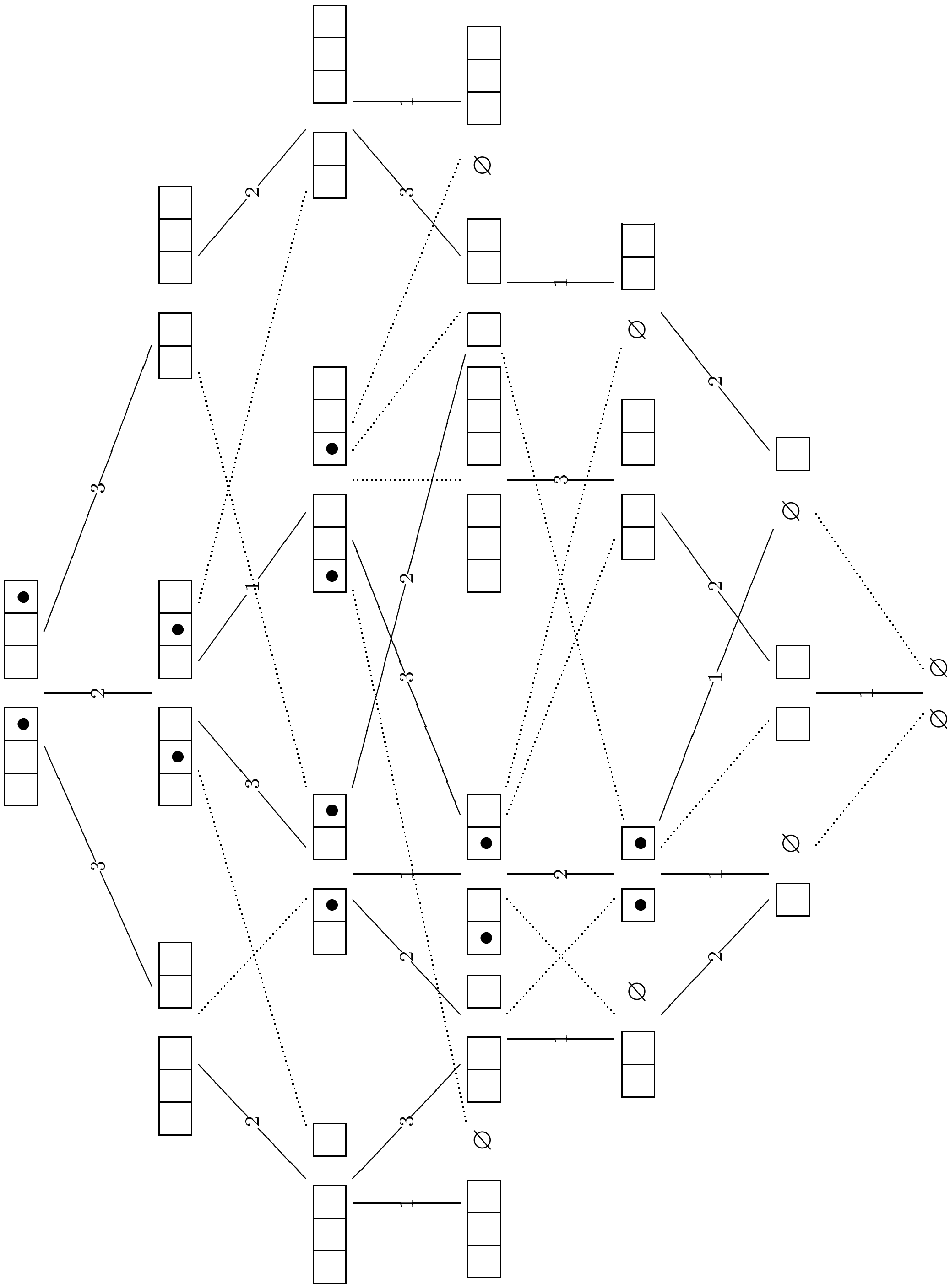
## A.2 $\mathbb{P}^2 \times \mathbb{P}^2 = \mathrm{Gr}(1,3) \times \mathrm{Gr}(1,3)$



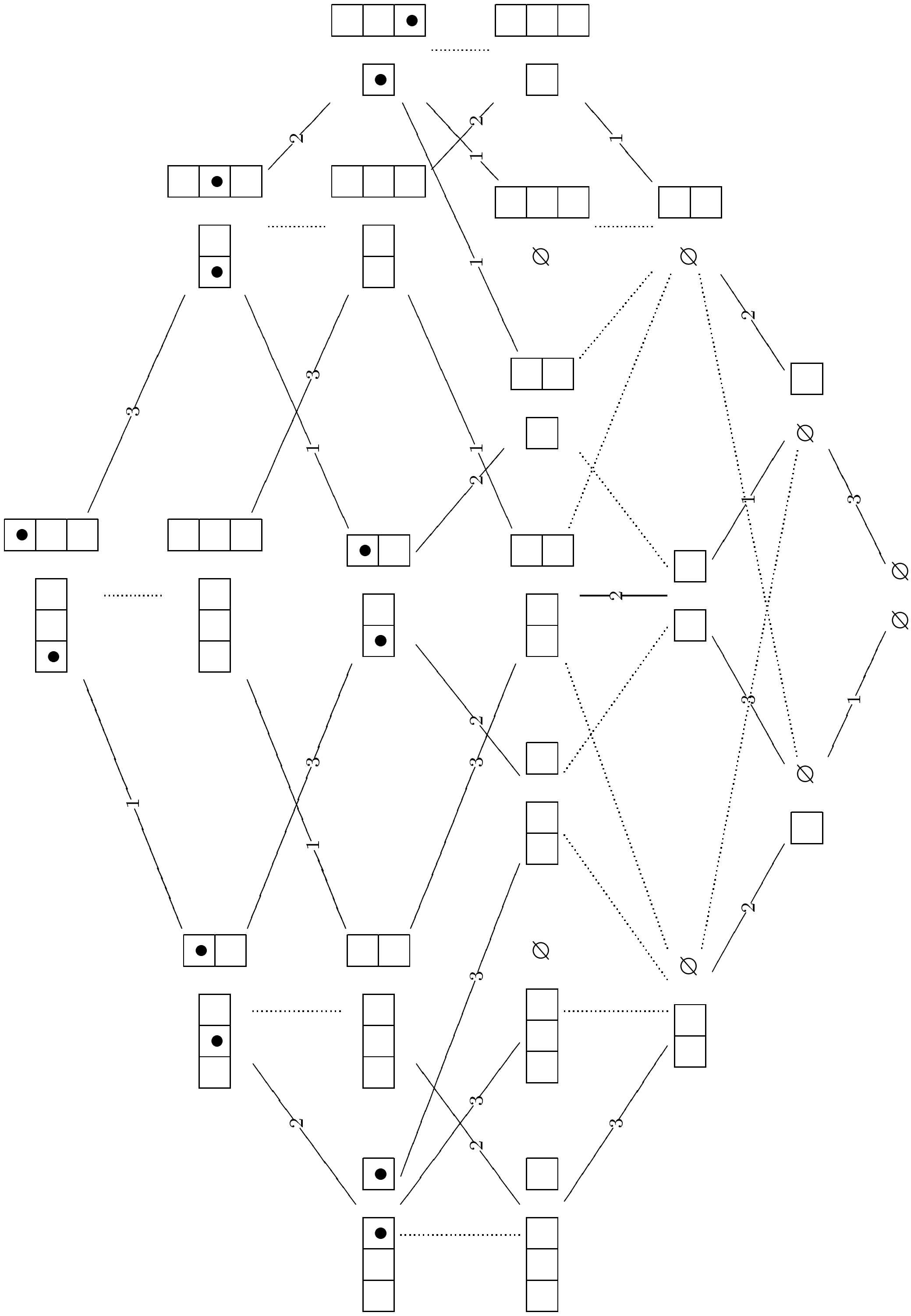

## A.3 $\mathbb{P}^2 \times \mathbb{P}^{2*} = \mathrm{Gr}(1,3) \times \mathrm{Gr}(2,3)$



## A.4 $\mathbb{P}^3 \times \mathbb{P}^3 = \mathrm{Gr}(1,4) \times \mathrm{Gr}(1,4)$



## A.5 $\mathbb{P}^3 \times \mathbb{P}^{3*} = \mathrm{Gr}(1,4) \times \mathrm{Gr}(3,4)$



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## RÉSUMÉ

Soit $X$ le produit direct de deux grassmanniennes des sous-espaces de dimensions $k$, $l$ d'un espace vectoriel $V$. Nous étudions les orbites d'un sous-groupe de Borel $B$ de $\mathrm{GL}(V)$ opérant diagonalement dans $X$, et les adhérences de Zariski de ces orbites, en anagogie avec les cellules et les variétés de Schubert dans les grassmanniennes. On vérifie sans peine que ces orbites sont en nombre fini. Elles ont été décrites de façon combinatoire par P. Magyar, J. Weyman et A. Zelevinsky. Nous obtenons un critère pour l'inclusion d'une orbite dans l'adhérence d'une autre orbite, et nous construisons une résolution de ces adhérences d'orbites, analogue aux désingularisations de Bott–Samelson de variétés de Schubert.



## ABSTRACT

Let $X$ be the direct product of two Grassmannians of $k$- and $l$-planes in a finite-dimensional vector space $V$. We study the orbits of a Borel subgroup $B \subset \mathrm{GL}(V)$ acting diagonally on $X$, as well as their Zariski closures, in analogy with Schibert cells and Schubert varieties in Grassmannians. One easily shows that the number of these orbits is finite. Their combinatorial description was obtained by P. Magyar, J. Weyman, and A. Zelevinsky. We obtain a criterion to check whether one orbit lies in the closure of another one. We also construct a resolution of singularities for the closures of these orbits, which is analogous to the Bott–Samelson desingularization of Schubert varieties.



## MOTS-CLÉS

Variétés sphériques, grassmanniennes, variétés de drapeaux multiples, représentations de carquois, décomposition de Schubert, désingularisation de Bott–Samelson, carquois d'Auslander–Reiten

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