

*To Askold Khovanskii on his 65th birthday  
with gratitude and admiration*

## Schubert calculus and Gelfand–Zetlin polytopes

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**Abstract.** A new approach is described to the Schubert calculus on complete flag varieties, using the volume polynomial associated with Gelfand–Zetlin polytopes. This approach makes it possible to compute the intersection products of Schubert cycles by intersecting faces of a polytope.

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**Keywords:** Flag variety, Schubert calculus, Gelfand–Zetlin polytope, volume polynomial.

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## 1. Introduction

**1.1. Main results.** In this paper we explore the connection between the Schubert calculus and the volume polynomial on spaces of convex polytopes. We give various representations of Schubert cycles in a complete flag variety by sums of faces of the Gelfand–Zetlin polytope. Our work is motivated by the rich interplay between algebraic geometry and convex polytopes, originally explored for toric varieties and recently extended to a more general setting in [10].

One of our main tools is a construction in [22] which associates with each convex polytope  $P \subset \mathbb{R}^d$  a graded commutative ring  $R_P$  (called the *polytope ring*) satisfying Poincaré duality (see [23] or §2). For an *integrally simple* polytope  $P$  (simple means that there are exactly  $d = \dim(P)$  edges meeting at each vertex, and integrally simple means that primitive integer vectors parallel to the edges generate the lattice  $\mathbb{Z}^d$ ), the ring  $R_P$  is isomorphic to the Chow ring of the corresponding smooth toric variety  $X_P$  [22]. Faces of  $P$  give rise to certain elements of  $R_P$ , which generate  $R_P$  as an additive group. If  $[F]$  is the element of  $R_P$  corresponding to a face  $F$ , then  $[F] \cdot [G] = [F \cap G]$  in  $R_P$ , provided that  $F$  and  $G$  are transverse. Individual faces of  $P$  represent cycles given by the closures of the torus orbits in  $X_P$ . In this paper we are primarily interested in the case when  $P$  is not simple. Kiumars Kaveh has related the polytope rings of some non-simple polytopes to the Chow rings of smooth non-toric spherical varieties [9]. In particular, he observed that the ring  $R_P$  for the *Gelfand–Zetlin polytope*  $P = P_\lambda$  (which is not simple) associated with a strictly dominant weight  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$  of the group  $GL_n(\mathbb{C})$  is isomorphic to the Chow ring of the variety  $X$  of complete flags in  $\mathbb{C}^n$ . Recall that the Gelfand–Zetlin polytope  $P_\lambda \subset \mathbb{R}^d$ , where  $d := n(n-1)/2$  denotes the dimension of the flag variety, is given by  $2d$  inequalities depending on  $\lambda$  (see §3.1).

When  $P$  is not simple, there is no obvious correspondence between faces of  $P$  and elements of  $R_P$ . One of the results in the present paper is a general construction that associates with each element of  $R_P$  a linear combination of faces of  $P$  (though not every face of  $P$  corresponds to an element of  $R_P$ ). Namely, we embed the ring  $R_P$  in a  $\mathbb{Z}$ -module  $M_P$  whose elements can be regarded as linear combinations of arbitrary faces of  $P$  modulo certain relations (see §2). The module  $M_P$  depends on the choice of a *resolution* of  $P$ . On the algebro-geometric level,  $R_P$  can be regarded as the subring of the Chow ring of the singular toric variety  $X_P$  generated by the Picard group, and  $M_P$  can be constructed using a resolution of singularities for  $X_P$ . However, we describe  $M_P$  in elementary terms using convex geometry. A crucial

feature of such representations by sums of faces is that we can still multiply elements of  $R_P$  by intersecting faces (assuming that the faces we intersect are transverse).

While our construction applies to any convex polytope  $P$ , it is especially interesting to study the case when  $P = P_\lambda$  is a Gelfand–Zetlin polytope, due to the isomorphism  $R_P \simeq CH^*(X)$  for the flag variety  $X$ . We recall that  $CH^*(X)$  (as a group) is a free Abelian group with a basis of Schubert cycles  $[X^w]$ , where  $w$  runs through all the permutations in  $S_n$  (see the definition of Schubert cycles in § 4.1). In particular, our construction lets us represent Schubert cycles as linear combinations of faces of the Gelfand–Zetlin polytope in many different ways (see Theorem 4.3, Proposition 3.2, and Corollary 4.5), and this has applications to the Schubert calculus.

Though the relation between Schubert varieties and certain faces of the Gelfand–Zetlin polytope was first investigated in [15], and then by different methods also in [16] and [12], only our approach develops this relation to such an extent that the Schubert calculus can be modelled by the Gelfand–Zetlin polytope. The results in [15] and [16] cannot be applied to the Schubert calculus, since only one representation by a sum of faces is constructed for each Schubert variety. The polytope ring  $R_P$  and the  $\mathbb{Z}$ -module  $M_P$  enable us to obtain new representations for Schubert cycles. In particular, given two Schubert cycles  $[X^w]$  and  $[X^{w'}]$ , we can represent  $[X^w]$  and  $[X^{w'}]$  as sums of faces so that every face appearing in the decomposition of  $[X^w]$  is transverse to every face appearing in the decomposition of  $[X^{w'}]$ . Hence, the intersection of any two Schubert cycles can be represented by a linear combination of faces with non-negative coefficients, which is closely related to a central problem of the Schubert calculus — a combinatorial interpretation of the positivity of structure constants (see § 1.2). More precisely, we get the following result (see also Corollary 4.6).

**Theorem 1.1.** *The product of any two Schubert cycles  $[X^w]$  and  $[X^u]$  can be represented as the sum of faces*

$$[X^w] \cdot [X^u] = \sum_{\substack{w(F)=w \\ w(F^*)=w_0 u w_0^{-1}}} [F \cap F^*],$$

where  $F$  and  $F^*$  run over the reduced Kogan faces and the dual Kogan faces, respectively, of the polytope  $P_\lambda$ .

Here  $w_0 \in S_n$  denotes the longest permutation, which takes  $i$  to  $(n - i + 1)$ . The *rc-faces* in [5] we will call *reduced dual Kogan faces* (see § 4.3). *Kogan faces* are defined in § 3.3 and are characterized by the property that they contain the simple vertex  $v$  of the Gelfand–Zetlin polytope with minimal sum of coordinates (dual Kogan faces, correspondingly, contain the vertex  $v^*$  with maximal sum of coordinates). Each Kogan face intersects each dual Kogan face transversely (since no facet contains both  $v$  and  $v^*$ ). For each Kogan or dual Kogan face  $F$ , one can define the permutation  $w(F) \in S_n$  by assigning an elementary transposition to every facet containing  $v$  or  $v^*$  and then multiplying them in a certain order (see details in § 3.3).

The connection between the Schubert calculus and Gelfand–Zetlin polytopes stems from the representation theory for the group  $GL_n(\mathbb{C})$ . Recall that by the

definition of Gelfand–Zetlin polytopes the integer points inside and on the boundary of  $P_\lambda$  parametrize a natural basis (a *Gelfand–Zetlin basis*) in the irreducible highest-weight  $GL_n$ -module  $V_\lambda$  with highest weight  $\lambda$ . In particular, with every integer point  $z \in P_\lambda$  we can associate its weight  $p(z)$  in the weight lattice of  $GL_n$  (that is, in the character lattice of a maximal torus in  $GL_n(\mathbb{C})$ ). On the other hand, the Borel–Weil–Bott theorem describes the module  $V_\lambda$  geometrically as the dual space to the space of global sections of some line bundle  $\mathcal{L}_\lambda$  on the flag variety  $X$  (see §5). Thus, a basis in the space of global sections of a line bundle on  $X$  is parametrized by integer points in the corresponding Gelfand–Zetlin polytope. This lets us use methods from the theory of Newton polytopes. Similarly, the space of global sections of the line bundle  $\mathcal{L}_\lambda$  restricted to the Schubert variety  $X^w$  is dual to a  $B^-$ -submodule of the module  $V_\lambda$ , namely, to the so-called *Demazure submodule*  $V_{\lambda,w}^-$  (here  $B^- \subset GL_n$  denotes the subgroup of lower-triangular matrices). It is natural to ask whether a basis in  $V_{\lambda,w}^-$  can be parametrized by integer points in faces of the Gelfand–Zetlin polytope. The answer is given by the following theorem. For each Schubert variety  $X^w$  and a strictly dominant weight  $\lambda$ , we realize the corresponding Demazure character  $\chi_w(\lambda)$  of the Demazure module  $V_{\lambda,w}^-$  as the exponential sum over integer points in a union of reduced Kogan faces (see also Theorem 5.1).

**Theorem 1.2.** *For each permutation  $w \in S_n$  the Demazure character  $\chi^w(\lambda)$  has the form*

$$\chi^w(\lambda) = \sum_{z \in A_{\lambda,w} \cap \mathbb{Z}^d} e^{p(z)},$$

where  $A_{\lambda,w} := \bigcup_{w(F_\lambda) = w} F_\lambda$  is the union of all reduced Kogan faces with the permutation  $w$  in the Gelfand–Zetlin polytope  $P_\lambda$ .

This generalizes the identity in Corollary 15.2 of [21] for the Demazure character of a 132-avoiding, or in other terms *Kempf*, permutation  $w$  (such permutations are also said to be *dominant*, but we will use the term ‘Kempf’ instead). We note that a permutation is Kempf if and only if there is a unique reduced Kogan face with this permutation (see [15], Proposition 2.3.2), and this is exactly the face considered in [21]. Theorem 1.2 enables us to study the geometry of Schubert varieties by methods of the theory of Newton polytopes (see §5).

To prove our formula for the Demazure character we use elementary convex geometry together with a simple combinatorial procedure introduced in [14] and called *mitosis* (see also [19] for an elementary exposition) for dealing with divided difference operators. In particular, our proof yields a geometric realization of mitosis (see §6.2). As a by-product, we construct a minimal realization of a simplex as a cubic complex different from previously known realizations (see Proposition 6.6).

This paper is organized as follows. In §2 we recall the definition of the polytope ring  $R_P$ , discuss its properties, and construct the module  $M_P$  for a non-simple  $P$ . In §3 we study the polytope rings of Gelfand–Zetlin polytopes. In §4 we represent Schubert cycles by faces. In §5 we give formulae for Demazure characters, Hilbert functions, and degrees of Schubert varieties in terms of faces, and we deduce some of the results in §4 from these formulae. In §6 we introduce a simple geometric version of mitosis (*paramitosis*) and use it to prove formulae for Demazure characters in §5.

**1.2. History of the Schubert calculus.** The groundwork for the Schubert calculus was laid by the German mathematician Hermann Schubert in the 19th century. He developed a general method for solving enumerative geometry problems. For instance, a classical problem on the number of lines meeting four given lines in 3-dimensional space reduces to the computation of intersection products of Schubert cycles in the Grassmannian  $G(2, 4)$ . For an arbitrary Grassmannian there exists an algorithm called the *Littlewood–Richardson rule* for computing the structure constants of the Chow ring in the basis of Schubert cycles. This algorithm gives a combinatorial proof of the non-negativity of the structure constants—each of them turns out to be equal to the number of Young diagrams with certain properties (see [8] or [18], Chap. 1).

For the variety of complete flags there is a simple algorithm for multiplying Schubert cycles, but it does not yield a combinatorial proof of the non-negativity of the structure constants. However, the non-negativity follows easily from geometric arguments as in the case of Grassmannians: since any variety of (partial) flags is a homogeneous space under the action of  $GL_n$ , any two subvarieties can be made transverse by the group action in view of the Kleiman transversality theorem [13]. It is not hard to deduce from this that the intersection product of any two Schubert cycles is a linear combination of Schubert cycles with non-negative coefficients. A combinatorial interpretation of positivity of the structure constants was recently obtained for two-step flag varieties [5], but the related combinatorics is much more complicated than the classical Littlewood–Richardson rule for Grassmannians. The case of a complete flag variety is still open.

Since the Chow ring of the variety of complete flags in  $\mathbb{C}^n$  is generated by the Picard group, this ring can be represented as a quotient ring of the ring  $\mathbb{Z}[x_1, \dots, x_n]$  of polynomials in  $n$  variables (see [18], Theorem 3.6.15). Such a description is called the *Borel presentation*. A realization of Schubert cycles by polynomials in the Borel presentation was obtained in [3] and [6] using *divided difference operators* or *Demazure operators* (see Theorem 4.2). Thus, Schubert cycles can be multiplied as polynomials in the ring  $\mathbb{Z}[x_1, \dots, x_n]$ .

It is interesting that Demazure operators (more precisely, their  $K$ -theoretic versions) also play an important role in representation theory; namely, they enable one to compute Demazure characters. This was mentioned already in [6], but a rigorous proof appeared later in [1] (see Theorem 5.6). This is a manifestation of the connections between the Schubert calculus and representation theory. Another manifestation is that the Littlewood–Richardson rule for Grassmannians also gives a rule for decomposing the tensor product of two irreducible  $GL_n$ -modules into irreducible modules.

In [17] Schubert polynomials were defined. These are the most natural representatives of Schubert cycles in the ring  $\mathbb{Z}[x_1, \dots, x_n]$ . Schubert polynomials became a popular theme in algebraic combinatorics. A striking result in this area is the Fomin–Kirillov theorem (see [7] or §4.2), which implies that every Schubert polynomial is a linear combination of monomials with non-negative integer coefficients (the non-negativity of the coefficients is not obvious from the definition of Schubert polynomials).

We note that the approach to the Schubert calculus developed in the present paper gives a new combinatorial model of the Schubert calculus different from the model based on Schubert polynomials (see Remark 2.5).

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## 2. Polytope ring

**2.1. Rings associated with polynomials.** Following [22], we associate a graded commutative ring with any homogeneous polynomial. We will later specialize to the case of the volume polynomial on a space of polytopes with a given normal fan. Let  $\Lambda_f$  be a lattice, that is, a free  $\mathbb{Z}$ -module, and let  $f$  be a homogeneous polynomial on the real vector space  $V_f = \Lambda_f \otimes \mathbb{R}$  containing the lattice  $\Lambda_f$ . The symmetric algebra  $\text{Sym}(\Lambda_f)$  of  $\Lambda_f$  can be thought of as the ring of differential operators with constant integer coefficients acting on  $\mathbb{R}[V_f]$ , the space of all polynomials on  $V_f$ . If  $D \in \text{Sym}(\Lambda_f)$  and  $\phi \in \mathbb{R}[V_f]$ , then we write  $D\phi \in \mathbb{R}[V_f]$  for the result of this action. Define  $A_f$  as the homogeneous ideal in  $\text{Sym}(\Lambda_f)$  consisting of all differential operators  $D$  such that  $Df = 0$ . Let  $R_f = \text{Sym}(\Lambda_f)/A_f$ . We call this ring *the ring associated with the polynomial  $f$* .

Let  $\Lambda_g$  be another lattice and let  $\sigma: \Lambda_g \rightarrow \Lambda_f$  be a homomorphism of lattices. Define the polynomial  $g \in \mathbb{R}[V_g]$  as  $\sigma^*(f) = f \circ \sigma$ . We want to describe a relation between the rings  $R_f$  and  $R_g$  associated with these polynomials. Unfortunately, there is no natural homomorphism between these rings. However, we do have the following result.

**Proposition 2.1.** *There exist a natural Abelian group  $M_{f,g}$ , a natural epimorphism  $\pi: R_f \rightarrow M_{f,g}$ , and a natural monomorphism  $\iota: R_g \rightarrow M_{f,g}$  such that*

$$\pi(\tilde{\alpha}\tilde{\beta}) = \iota(\alpha\beta)$$

whenever  $\pi(\tilde{\alpha}) = \iota(\alpha)$  and  $\pi(\tilde{\beta}) = \iota(\beta)$ .

This proposition can be used in the following way. The elements of  $R_g$  can be embedded naturally in  $M_{f,g}$ . Although elements of  $M_{f,g}$  cannot be multiplied in general, we can consider the lifts to  $R_f$  of two elements coming from  $R_g$ , multiply them in  $R_f$ , and project the product back to  $M_{f,g}$ . In many cases this is easier than multiplying two elements of  $R_g$  directly.

*Proof.* Consider the  $\mathbb{Z}$ -submodule  $A_{f,g}$  of  $\text{Sym}(\Lambda_f)$  consisting of all operators  $D$  such that  $\sigma^*(Df) = 0$ . Let  $M_{f,g} = \text{Sym}(\Lambda_f)/A_{f,g}$ . Clearly,  $A_f \subset A_{f,g}$ ; thus, we obtain a natural projection  $\pi: R_f \rightarrow M_{f,g}$ . Let  $\sigma_*: \text{Sym}(\Lambda_g) \rightarrow \text{Sym}(\Lambda_f)$  be the homomorphism induced by the map  $\sigma$ . For a differential operator  $D \in \text{Sym}(\Lambda_g)$

let  $[D]$  denote the class of  $D$  in the ring  $R_g$ . We define  $\iota([D])$  as the class in  $M_{f,g}$  of the operator  $\sigma_*(D)$ .

To verify that  $\iota([D])$  is well defined, we need the formula

$$\sigma^*(\sigma_*(D)\phi) = D\sigma^*\phi$$

for every  $\phi \in \mathbb{R}[V_f]$ . Indeed, this formula is obviously true if  $D \in \Lambda_g$ , and both parts of the formula depend multiplicatively on  $D$ . In particular, we have

$$\sigma^*(\sigma_*(D)f) = D\sigma^*f = Dg,$$

which is equal to 0 whenever  $D$  is in  $A_g$ . It follows that the element  $\iota([D])$  is well defined: if  $D \in A_g$ , then  $\sigma_*(D) \in A_{f,g}$ . It also follows from the same formula that  $\iota$  is injective: if  $\iota([D]) = 0$ , that is,  $\sigma_*(D) \in A_{f,g}$ , then  $D \in A_g$ .

It remains to prove that  $\pi(\tilde{\alpha}\tilde{\beta}) = \iota(\alpha\beta)$  whenever  $\pi(\tilde{\alpha}) = \iota(\alpha)$  and  $\pi(\tilde{\beta}) = \iota(\beta)$ . But this is an immediate consequence of the formula  $\sigma_*(DE) = \sigma_*(D)\sigma_*(E)$ .  $\square$

**Example 2.2.** Here is an example illustrating Proposition 2.1. Consider the polynomial  $f(x, y, z) = (x + y)^2 + xz$  defined on the space  $\mathbb{R}^3$ . Assume that the lattice  $\Lambda_f$  coincides with the standard integer lattice  $\mathbb{Z}^3$ . Then the ring  $R_f$  is generated over the integers by the classes  $[\partial_x]$ ,  $[\partial_y]$ , and  $[\partial_z]$  of the differential operators  $\partial_x$ ,  $\partial_y$ , and  $\partial_z$ , respectively. These classes satisfy the relations

$$[\partial_x]^2 = [\partial_y]^2 = [\partial_x][\partial_y] = 2[\partial_x][\partial_z], \quad [\partial_y][\partial_z] = [\partial_z]^2 = 0,$$

as well as the relations implied by the fact that the class of any differential operator of order 3 or higher is equal to 0. The elements  $1, [\partial_x], [\partial_y], [\partial_z], [\partial_x]^2$  form an additive basis in  $R_f$  (that is, they freely generate  $R_f$  as a  $\mathbb{Z}$ -module). Hence, the additive group of the ring  $R_f$  has rank 5. Consider the  $\mathbb{Z}$ -module homomorphism  $\phi: \mathbb{Z}^2 \rightarrow \mathbb{Z}^3$  given by the formula  $\phi(\xi, \eta) = (\xi, \eta, 0)$ . Then the polynomial  $g = \phi^*f$  has the form  $(\xi + \eta)^2$ . The corresponding ring  $R_g$  is generated by the class  $[\partial_\xi] = [\partial_\eta]$  with the relation  $[\partial_\xi]^3 = 0$ . Therefore, the additive group of the ring  $R_g$  has rank 3 and is freely generated by the elements  $1, [\partial_\xi]$ , and  $[\partial_\xi]^2$ . Now consider the  $\mathbb{Z}$ -module  $M_{f,g}$ . Its elements are in a bijective correspondence with the restrictions of the polynomials  $Df$  to the subspace  $z = 0$ , where  $D$  runs through all differential operators in  $\text{Sym}(\Lambda_f)$ . The corresponding space of polynomials has rank 4 and is freely generated by the polynomials  $1, 2(x + y), x$ , and  $(x + y)^2$ . We will identify the elements of the module  $M_{f,g}$  with the corresponding polynomials. The projection  $\pi: R_f \rightarrow M_{f,g}$  takes the elements  $[\partial_x]$  and  $[\partial_y]$  to the same polynomial  $2(x + y)$ ; in particular, the map  $\pi$  is not injective. The injection  $\iota: R_g \rightarrow M_{f,g}$  takes the additive generators  $1, [\partial_\xi]$ , and  $[\partial_\xi]^2$  of the ring  $R_g$  to the polynomials  $(x + y)^2, 2(x + y)$ , and  $x$ . In particular, the polynomial  $x$  does not belong to the image of the map  $\iota$ .

**2.2. The volume polynomial.** Consider the set of all convex polytopes of dimension  $d$  in  $\mathbb{R}^d$ . This set can be endowed with the structure of a commutative semi-group using the *Minkowski sum*

$$P_1 + P_2 = \{x_1 + x_2 \in \mathbb{R}^d \mid x_1 \in P_1, x_2 \in P_2\}.$$

It is not hard to check that this semigroup has the cancellation property. Polytopes can also be multiplied by non-negative real numbers, which reduces to a homothety

$$\lambda P = \{\lambda x \mid x \in P\}, \quad \lambda \geq 0.$$

Hence, we can embed the semigroup of convex polytopes in its Grothendieck group  $V$ , a real (infinite-dimensional) vector space. The elements of  $V$  are called *virtual polytopes*. We recall that two convex polytopes  $P$  and  $Q$  are said to be *analogous* if they have the same normal fan, that is, there is a one-to-one correspondence between the faces of  $P$  and the faces of  $Q$  such that any linear functional whose restriction to  $P$  attains its maximal value at a given face  $F \subseteq P$  has the property that its restriction to  $Q$  attains its maximal value on the corresponding face of  $Q$  (the set of linear functionals whose restrictions to  $P$  attain their maximal values on a face  $F \subseteq P$  form a cone  $C_F$ , and the *normal fan* of  $P$  is defined as the set of cones  $C_F$  corresponding to all faces  $F \subseteq P$ ). A virtual polytope is said to be analogous to  $P$  if it can be represented as a difference of two convex polytopes analogous to  $P$ . The set of all virtual polytopes analogous to  $P$  forms a finite-dimensional subspace  $V_P \subset V$ . On the vector space  $V$  there is defined a homogeneous polynomial  $\text{vol}$  of degree  $d$ , called the *volume polynomial*. We fix a constant (translation-invariant) volume form on  $\mathbb{R}^d$ . If an integer lattice  $\mathbb{Z}^d \subset \mathbb{R}^d$  is fixed, we will always choose this volume form to take the value 1 on the fundamental parallelepiped of  $\mathbb{Z}^d$ . The volume form on  $\mathbb{R}^d$  being fixed, the volume polynomial on the space  $V$  is uniquely characterized by the property that its value  $\text{vol}(P)$  on any convex polytope  $P$  is equal to the volume of  $P$ . We will be interested in the restriction  $\text{vol}_P$  of the volume polynomial  $\text{vol}$  to the subspace  $V_P$ .

Consider an integer convex polytope  $P$  (that is, a convex polytope with integer vertices) of dimension  $d$ , not necessarily simple. Let  $\Lambda_P$  be a lattice in  $V_P$  generated by some integer polytopes analogous to  $P$  (we do not assume that  $\Lambda_P$  contains all integer polytopes analogous to  $P$ , and thus this lattice may depend on some additional parameters and not just on  $P$ ). Suppose that  $Q$  is a convex polytope with integer vertices whose normal fan is a simplicial subdivision of the normal fan of  $P$ . In this case,  $Q$  is called a *resolution* of  $P$  (note that, since the normal fan of  $Q$  is simplicial, the polytope  $Q$  is simple). With the volume polynomial  $\text{vol}_P$  restricted to the lattice  $\Lambda_P$  we associate the *polytope ring*  $R_P := R_{\text{vol}_P}$ . Similarly, for the simple polytope  $Q$  we consider the ring  $R_Q := R_{\text{vol}_Q}$  associated with the volume polynomial  $\text{vol}_Q$  on the lattice  $\Lambda_Q$  (we always assume that this lattice is generated by *all* the integer polytopes analogous to  $Q$ ). We will use the  $\mathbb{Z}$ -module  $M_{Q,P} := M_{\text{vol}_Q, \text{vol}_P}$  introduced in Proposition 2.1 together with the homomorphisms  $\iota: R_P \rightarrow M_{Q,P}$  and  $\pi: R_Q \rightarrow M_{Q,P}$ . Since  $\iota$  is a canonical embedding, we will identify elements of  $R_P$  with their  $\iota$ -images in  $M_{Q,P}$ . With each face  $\tilde{F}$  of  $Q$  we can associate a face  $F$  of  $P$  with the property  $C_{\tilde{F}} \subset C_F$ , which we call the  *$P$ -degeneration* of  $\tilde{F}$  (or just the *degeneration* of  $\tilde{F}$  if  $P$  is fixed). A face  $F$  of  $P$  is said to be *regular (with respect to  $Q$ )* if there is only one face  $\tilde{F}$  of  $Q$  such that  $F$  is the degeneration of  $\tilde{F}$ .

**Proposition 2.3.** *Suppose that  $v$  is a simple vertex of  $P$ , that is, exactly  $d = \dim(P)$  facets of  $P$  meet at  $v$ . Moreover, suppose that no facet of  $Q$  degenerates into a face of smaller dimension. Then any face of  $P$  containing  $v$  is regular.*



*Proof.* Let  $\Gamma_1, \dots, \Gamma_d$  be all the facets of  $P$  containing the vertex  $v$  (which are clearly regular). Denote by  $\tilde{\Gamma}_1, \dots, \tilde{\Gamma}_d$  the corresponding (parallel) facets of  $Q$ . Note that the intersections of different subsets of  $\{\Gamma_1, \dots, \Gamma_d\}$  are different faces of  $P$ . Clearly, any intersection of facets  $\tilde{\Gamma}_i$  degenerates into the intersection of the corresponding facets  $\Gamma_i$  (which has the same dimension), and all other faces of  $Q$  degenerate into faces of  $P$  not containing  $v$ .  $\square$

### 2.3. Structure of polytope rings.

We now give more details on the structure of the ring  $R_Q$ . For every facet  $\Gamma$  of  $Q$  there is a differential operator  $\partial_\Gamma \in \text{Sym}(\Lambda_Q)$  such that, for every convex polytope  $Q'$  analogous to  $Q$ , the number  $\partial_\Gamma \text{vol}_Q(Q')$  is the  $(d-1)$ -dimensional volume of the facet of  $Q'$  parallel to  $\Gamma$ . The ideal  $A_Q := A_{\text{vol}_Q}$  is very easy to describe. It is generated (as an ideal) by the following two groups of differential operators [23]:

- the images of integer vectors  $a \in \mathbb{Z}^d$  under the natural inclusion of  $\mathbb{Z}^d$  in  $\Lambda_Q = \text{Sym}^1(\Lambda_Q)$  such that  $Q + a$  is the parallel translation of  $Q$  by the vector  $a$ ;
- the operators of the form  $\partial_{\Gamma_1} \cdots \partial_{\Gamma_k}$ , where  $\Gamma_1 \cap \cdots \cap \Gamma_k = \emptyset$ .

The volume polynomial on the spaces  $V_Q$  was previously used in [22] to describe the cohomology rings of smooth toric varieties. We briefly recall this description. Every integer polytope  $Q$  determines a polarized toric variety  $X_Q$ . If  $Q$  is integrally simple, then  $X_Q$  is smooth. In this case the Chow ring of  $X_Q$  (or, equivalently, the cohomology ring  $H^{2*}(X_Q, \mathbb{Z})$ ) is isomorphic to  $R_Q$  ([22], § 1.4).

This description is very useful. First, it is functorial. Second, it is clear from the definition that the non-zero homogeneous components of the ring  $R_Q$  have degrees  $\leq d$  (since the volume polynomial has degree  $d$ ) and that  $R_Q$  has a non-degenerate pairing (Poincaré duality) defined by  $(D_1, D_2) := D_1 D_2(\text{vol}_Q) \in \mathbb{Z}$  for any two homogeneous elements  $D_1, D_2 \in \text{Sym}(\Lambda_Q)$  of complementary degrees. The Poincaré duality on the ring  $R_Q$  is a key ingredient in the proof of the isomorphism between  $R_Q$  and  $H^{2*}(X_Q, \mathbb{Z})$  (see [9] for more details). We note that there is another functorial description [4] of the Chow ring of  $X_Q$  via piecewise polynomial functions on fans, but for this description the upper bound on the degrees and the Poincaré duality are harder to check directly. Also, the first known (non-functorial) description of the Chow ring (by generators and relations) follows easily from the definition of the ring  $R_Q$  (see, for example, [23]). So it seems that the polytope rings give the most convenient description of the Chow rings of smooth toric varieties.

Note that if a polytope  $P$  is not simple, then the ring  $R_P$  makes sense, has all non-zero homogeneous components in degrees  $\leq d$ , and satisfies Poincaré duality. However, its relation to the Chow ring of the (now singular) toric variety  $X_P$  is unclear, partly because the latter no longer enjoys Poincaré duality. On the other hand, the ring  $R_P$  for non-simple polytopes is sometimes related to the Chow rings of smooth non-toric varieties, as was noted by Kaveh [9].

We now discuss some important properties of the isomorphism  $R_Q \simeq CH^*(X_Q)$  for a simple polytope  $Q$ . This isomorphism lets us identify the algebraic cycles on  $X_Q$  with linear combinations of faces of  $Q$ . The dimension of the space  $V_Q$  is equal to the number  $N(Q)$  of facets of  $Q$  (since we can shift all support hyperplanes of  $Q$  independently). We note that for a non-simple polytope  $P$  the dimension of  $V_P$  is strictly less than  $N(P)$  (for example, if  $P$  is an octahedron, then  $V_P$  has dimension 4). For simple  $Q$  the space  $V_Q$  has natural coordinates called the *support*

numbers. There are as many support numbers as facets of  $Q$ . The support numbers are defined by fixing  $N = N(Q)$  linear functionals  $\xi_\Gamma$  on  $\mathbb{R}^d$  corresponding to facets  $\Gamma$  of  $Q$  such that every facet  $\Gamma$  of  $Q$  is contained in the hyperplane  $\xi_\Gamma(x) = H_\Gamma$  for some constant  $H_\Gamma$ , and the points of  $Q$  satisfy the inequalities  $\xi_\Gamma(x) \leq H_\Gamma$ . If  $\Gamma_1, \dots, \Gamma_N$  are all the facets of  $Q$ , then any collection of real numbers  $(H_{\Gamma_1}, \dots, H_{\Gamma_N})$  defines a unique (possibly virtual) polytope in  $V_Q$ . When dealing with integer polytopes, we always choose  $\xi_\Gamma$  to be a primitive integer covector orthogonal to  $\Gamma$ . In this case  $H_\Gamma$  is (up to a sign) the integer distance between the origin and the hyperplane containing  $\Gamma$ .

If we choose the volume form and the linear functionals  $\xi_\Gamma$  to be consistent with the integer lattice (in the sense explained above), then the differential operators  $\partial_\Gamma$  coincide with the partial derivatives with respect to the support numbers  $H_\Gamma$ . For a face  $F = \Gamma_1 \cap \dots \cap \Gamma_k$  of codimension  $k$  we set  $\partial_F = \partial_{\Gamma_1} \dots \partial_{\Gamma_k}$  and denote by  $[F]$  the class of  $\partial_F$  in the ring  $R_Q$ . The elements  $[F]$  corresponding to the faces of  $Q$  generate  $R_Q$  as an Abelian group. Moreover, it suffices to take certain special faces, called *separatrices* in [23], as generators. There is an explicit algorithm to represent the product  $[F_1] \cdot [F_2] \in R_Q$  as a linear combination of faces, that is, of elements of the form  $[F]$  corresponding to faces  $F$  of  $Q$ . This algorithm resembles the well-known algorithm from intersection theory: we need to replace  $[F_1]$  by a linear combination of faces that are transverse to  $F_2$ . The linear relations between facets of  $Q$  follow immediately from the description of the ideal  $A_Q$  given above. They have the form

$$\sum \xi_\Gamma(a)[\Gamma] = 0, \tag{1}$$

where  $a \in \mathbb{R}^d$  is any vector, and the sum is over all facets of  $Q$ . Indeed, the volume polynomial is invariant under parallel translations. Therefore, the  $t$ -derivative of  $\text{vol}(\cdot + ta)$  is zero (where  $\cdot$  replaces any fixed element of  $V_Q$ ). By the chain rule, this derivative is equal to  $\sum \xi_\Gamma(a)\partial_\Gamma \text{vol}(\cdot)$ . Any linear relation between the elements  $[\Gamma]$  has this form (see [23]).

If  $Q$  is a resolution of  $P$ , then we will be interested in representations of elements  $\alpha \in R_P$  by linear combinations of faces of  $Q$ , that is, in the form

$$\alpha = \pi\left(\sum [F]\right),$$

where the summation is over some set of faces of  $Q$ . Then Proposition 2.1 enables us to compute the product of two elements  $\alpha, \alpha' \in R_P$  as follows. If we find a representation

$$\alpha' = \pi\left(\sum [F']\right),$$

such that each  $F'$  is transverse to each  $F$ , then

$$\alpha \cdot \alpha' = \pi\left(\sum [F \cap F']\right).$$

In the sequel we will also use the following lemma, which is a direct corollary of the definition of  $M_{Q,P}$ .

**Lemma 2.4.** *Let  $\alpha$  and  $\beta$  be two homogeneous elements in  $R_Q$  of the same degree. Then  $\pi(\alpha) = \pi(\beta)$  in  $M_{Q,P}$  if and only if  $\pi(\alpha\gamma) = \pi(\beta\gamma)$  for all homogeneous elements  $\gamma \in R_Q$  of complementary degree such that  $\pi(\gamma) \in R_P \subset M_{Q,P}$ .*

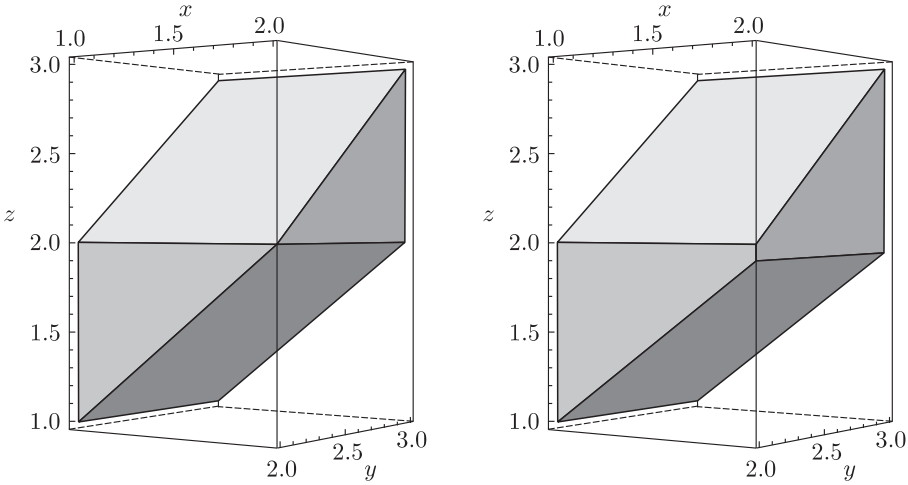


Figure 1. A Gelfand–Zetlin polytope for  $GL_3$  and its resolution.

**2.4. Example: Gelfand–Zetlin polytopes in  $\mathbb{R}^3$ .** Consider the polytope  $P$  in  $\mathbb{R}^3$  given by the following linear inequalities:

$$a \leq x \leq b, \quad b \leq y \leq c, \quad x \leq z \leq y.$$

This is a 3-dimensional Gelfand–Zetlin polytope (see Figure 1). The defining system of linear inequalities for  $P$  is usually represented schematically as follows:

$$\begin{array}{ccc} a & b & c \\ & x & y \\ & & z \end{array}$$

For instance, the fact that  $x$  lies between  $a$  and  $b$  one row below means that this coordinate satisfies the inequalities  $a \leq x \leq b$ .

The polytope  $P$  can be obtained from the parallelepiped  $[a, b] \times [b, c] \times [a, c]$  by removing the two prisms

$$\{a \leq z < x \leq b, b \leq y \leq c\}, \quad \{b \leq y < z \leq c, a \leq x \leq b\}.$$

Therefore, the volume of  $P$  is equal to

$$(b - a)(c - b)(c - a) - \frac{(b - a)^2(c - b)}{2} - \frac{(c - b)^2(b - a)}{2} = \frac{1}{2}(b - a)(c - b)(c - a)$$

(one can also see geometrically, without any computations, that the parts we are removing make up exactly half the volume of the whole parallelepiped). The ring  $R_P$  is spanned by the classes of the partial differentiations  $\partial_a, \partial_b$ , and  $\partial_c$ . Moreover, since the volume of  $P$  will not change if we shift  $a, b$ , and  $c$  simultaneously by the same real number, we have  $\partial_a + \partial_b + \partial_c = 0$  in  $R_P$ . A distinguished set of additive generators of  $R_P$  is given by Schubert polynomials in  $-\partial_a$  and  $-\partial_b$ , namely,

$$\begin{aligned} \mathfrak{S}_{s_1 s_2 s_1} &= -\partial_a^2 \partial_b, & \mathfrak{S}_{s_1 s_2} &= \partial_a \partial_b, & \mathfrak{S}_{s_2 s_1} &= \partial_a^2, \\ \mathfrak{S}_{s_2} &= -\partial_a - \partial_b, & \mathfrak{S}_{s_1} &= -\partial_a, & \mathfrak{S}_{\text{id}} &= 1. \end{aligned}$$

Let us now consider the simple polytope  $Q$  given by the inequalities

$$a \leq x \leq b, \quad b \leq y \leq c, \quad x \leq z \leq y + \varepsilon,$$

where  $\varepsilon > 0$  is a fixed small number. The polytope  $Q$  can also be obtained from the parallelepiped  $[a, b] \times [b, c] \times [a, c + \varepsilon]$  by removing the two prisms

$$\{a \leq z < x \leq b, b \leq y \leq c\}, \quad \{b \leq y < z - \varepsilon \leq c, a \leq x \leq b\}.$$

Therefore, the volume of  $Q$  is equal to

$$\begin{aligned} (b - a)(c - b)(c - a + \varepsilon) - \frac{(b - a)^2(c - b)}{2} - \frac{(c - b)^2(b - a)}{2} \\ = \frac{1}{2}(b - a)(c - b)(c - a) + \varepsilon(b - a)(c - b). \end{aligned}$$

This is a polynomial in  $a, b, c$ , and  $\varepsilon$ .

The ring  $R_Q$  is multiplicatively generated by the partial differentiations  $\tilde{\partial}_a, \tilde{\partial}_b, \tilde{\partial}_c$  (the tildes are just to distinguish these elements of  $R_Q$  from the elements  $\partial_a, \partial_b, \partial_c \in R_P$ ), and  $\partial_\varepsilon$ . We note that  $\tilde{\partial}_a + \tilde{\partial}_b + \tilde{\partial}_c = 0$  also in the ring  $R_Q$ . We have

$$\tilde{\partial}_a = -[x = a], \quad \tilde{\partial}_b = -[y = b] + [x = b], \quad \tilde{\partial}_c = [y = c], \quad \tilde{\partial}_\varepsilon = [z = y + \varepsilon].$$

The formula (1) gives three linear relations between facets of  $Q$ :

$$\begin{aligned} -[x = a] + [x = b] + [x = z] &= 0, \\ -[y = b] - [z = y + \varepsilon] + [y = c] &= 0, \\ -[x = z] + [z = y + \varepsilon] &= 0. \end{aligned}$$

We can represent the Schubert polynomials in  $\partial_a$  and  $\partial_b$  as the  $\pi$ -images of certain elements of  $R_Q$  as follows:

$$\begin{aligned} \mathfrak{S}_{s_1} &= \pi([x = a]), & \mathfrak{S}_{s_2} &= \pi([y = b] + [x = z]), \\ \mathfrak{S}_{s_2s_1} &= \pi([x = z = a]), & \mathfrak{S}_{s_1s_2} &= \pi([x = a, y = b]). \end{aligned}$$

All the faces of  $Q$  appearing on the right-hand sides of these equalities degenerate into regular faces of  $P$ . For instance, the expression for  $\mathfrak{S}_{s_2s_1}$  is obtained as follows:

$$\begin{aligned} \mathfrak{S}_{s_2s_1} &= \partial_a^2 = \pi(\tilde{\partial}_a^2) = \pi([x = a] \cdot [x = a]) \\ &= \pi([x = a] \cdot ([x = b] + [z = x])) = \pi([x = a] \cdot [x = b] + [x = z = a]). \end{aligned}$$

The first term on the right-hand side vanishes, because the faces  $\{x = a\}$  and  $\{x = b\}$  are disjoint.

In this way it is easy to justify all the heuristic calculations with faces in § 4 of [12].

*Remark 2.5.* Although the Schubert polynomial  $\mathfrak{S}_{s_2} = -\partial_a - \partial_b$  can be represented as the image of the sum of two faces,

$$\mathfrak{S}_{s_2} = \pi([y = b] + [x = z]),$$

there is no term-by-term equality between monomials and faces in its decomposition. The point is that the images  $\pi([y = b])$  and  $\pi([x = z])$  of the faces do not lie in the ring  $R_P$  (regarded as a submodule of  $M_{Q,P}$ ), though their sum does.

Indeed, using the definition of the  $\mathbb{Z}$ -module  $M_{Q,P}$  and the explicit formulae for the volume polynomials of the polytopes  $P$  and  $Q$  (see above), it is easy to deduce that the linear relations between  $\pi(\tilde{\partial}_a)$ ,  $\pi(\tilde{\partial}_b)$ ,  $\pi(\tilde{\partial}_c)$ , and  $\pi(\tilde{\partial}_\varepsilon)$  are generated by the single relation  $\pi(\tilde{\partial}_a) + \pi(\tilde{\partial}_b) + \pi(\tilde{\partial}_c) = 0$ . Therefore,  $\pi(\tilde{\partial}_\varepsilon) = \pi([z = y + \varepsilon]) = \pi([x = z])$  cannot be expressed as a linear combination of  $\tilde{\partial}_a$  and  $\tilde{\partial}_b$  in  $M_{Q,P}$ , and thus does not lie in the ring  $R_P \subset M_{Q,P}$ .

### 3. The Gelfand–Zetlin polytope and its ring

**3.1. The Gelfand–Zetlin polytope.** We now consider the ring  $R_P$  for the Gelfand–Zetlin polytope  $P = P_\lambda$  associated with a strictly dominant weight  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$  of the group  $GL_n(\mathbb{C})$ , that is, with an  $n$ -tuple of integers  $\lambda_i$  such that  $\lambda_i < \lambda_{i+1}$  for all  $i = 1, \dots, n - 1$ . We recall that the Gelfand–Zetlin polytope  $P_\lambda$  is a convex integer polytope in  $\mathbb{R}^d$ , where  $d = n(n - 1)/2$ , with the property that the integer points inside and on the boundary of  $P_\lambda$  parametrize a natural basis in the irreducible representation of  $GL_n(\mathbb{C})$  with highest weight  $\lambda$ . It can be given by the system of inequalities

$$\begin{array}{cccccccc}
 \lambda_1 & & \lambda_2 & & \lambda_3 & & \dots & & \lambda_n \\
 & \lambda_{1,1} & & \lambda_{1,2} & & \dots & & \lambda_{1,n-1} & \\
 & & \lambda_{2,1} & & \dots & & \lambda_{2,n-2} & & \\
 & & & \ddots & & \ddots & & & \\
 & & & \lambda_{n-2,1} & & \lambda_{n-2,2} & & & \\
 & & & & \lambda_{n-1,1} & & & & 
 \end{array} \tag{GZ}$$

where  $(\lambda_{1,1}, \dots, \lambda_{1,n-1}; \lambda_{2,1}, \dots, \lambda_{2,n-2}; \dots; \lambda_{n-2,1}, \lambda_{n-2,2}; \lambda_{n-1,1})$  are coordinates in  $\mathbb{R}^d$  and the notation

$$\begin{array}{cc}
 a & b \\
 & c
 \end{array}$$

means that  $a \leq c \leq b$ . See Figure 1 for a picture of the Gelfand–Zetlin polytope for  $G = GL_3$ . We note that Gelfand–Zetlin polytopes  $P_\lambda$  and  $P_\mu$  are analogous for any two strictly dominant weights  $\lambda$  and  $\mu$ . For what follows, we set  $P = P_\lambda$  for some strictly dominant weight  $\lambda$ , and define  $\Lambda_P$  as the lattice spanned by all Gelfand–Zetlin polytopes  $P_\mu$ , where  $\mu$  runs through all strictly dominant weights. The correspondence  $\mu \mapsto P_\mu$  establishes a natural isomorphism between the lattices  $\mathbb{Z}^n$  and  $\Lambda_P$ . In other words, virtual polytopes in  $\Lambda_P$  are parametrized by arbitrary  $n$ -tuples of integers, not necessarily strictly increasing. One can show that the ring  $R_P$  does not change if  $\Lambda_P$  is replaced by the lattice generated by *all* polytopes analogous to  $P_\lambda$ , but we will not need this.

We recall that with each complete flag  $W = W_1 \subset \dots \subset W^{n-1}$  in  $\mathbb{C}^n$  one associates the one-dimensional vector spaces  $L_i(W) = W_i/W_{i-1}$ . The disjoint union of the sets of the form  $\{W\} \times L_i(W)$  with the natural projection on  $X$  given by  $\{W\} \times L_i(W) \mapsto W$  has the structure of a line bundle over  $X$ . This line bundle  $\mathcal{L}_i$  is called a *tautological quotient line bundle* over  $X$ .

**Theorem 3.1** [9]. *The ring  $R_P$  is isomorphic to the Chow ring (and to the cohomology ring) of the complete flag variety  $X$  for  $GL_n(\mathbb{C})$  (note that  $\dim(X) = d$ ) in such a way that the images in  $R_P$  of the differential operators  $\frac{\partial}{\partial \lambda_1}, \dots, \frac{\partial}{\partial \lambda_n}$  are mapped to the first Chern classes of the tautological quotient line bundles  $\mathcal{L}_1, \dots, \mathcal{L}_n$  over  $X$ .*

This theorem can also be deduced directly from the Borel presentation for the cohomology ring  $H^*(X, \mathbb{Z})$  using the fact that the volume of  $P_\lambda$  (regarded as a function of  $\lambda$ ) is equal to  $\prod_{i < j} (\lambda_i - \lambda_j)$  times a constant.

Along with the Gelfand–Zetlin polytope  $P$ , we consider a resolution  $Q$  of it such that the number of facets in  $Q$  is the same as the number of facets in  $P$ , and every support hyperplane of  $Q$  intersecting  $Q$  in a facet is sufficiently close to a parallel support hyperplane of  $P$  intersecting  $P$  in a facet. This establishes a one-to-one correspondence between facets of  $Q$  and facets of  $P$  such that corresponding facets are parallel. Nothing in what follows will depend on a particular choice of  $Q$ .

**3.2. Faces and face diagrams.** It will be convenient to represent faces of  $P$  by *face diagrams*. First, we replace all the symbols  $\lambda_j$  and  $\lambda_{i,j}$  in Table (GZ) by dots. Every face of  $P$  will be given by a system of equations of the form  $a = b$ , where  $a$  and  $b$  are coordinates represented by adjacent dots in two consecutive rows. To represent such an equation, we draw a line segment connecting the corresponding dots (these line segments go from northeast to southwest or from northwest to southeast). Thus, a system of equations defining a face of  $P$  is represented by a collection of line segments called the *face diagram*.<sup>1</sup> *Rows* of a face diagram are defined as the collections of dots corresponding to the coordinates  $\lambda_{i,j}$  with a fixed  $i$ , and *columns* are by definition collections of dots with a fixed  $j$  (columns look like diagonals in our pictures).

Let  $F$  be a regular face of  $P$  and  $\tilde{F}$  the corresponding face of  $Q$ , so that  $F$  is the degeneration of  $\tilde{F}$ . We will often write  $[F]$  for the class  $[\tilde{F}]$  in the polytope ring  $R_Q$ . Note that, in general,  $\pi[F]$  does not belong to  $R_P$ .

Every facet of  $P$  is regular. For  $i = 0, \dots, n - 1$  and  $j = 1, \dots, n - i - 1$ , let  $\Gamma_{i,j}$  denote the facet of  $P$  given by the equation  $\lambda_{i,j} = \lambda_{i+1,j}$ , where we set  $\lambda_{0,j} = \lambda_j$ . Similarly, for  $i = 0, \dots, n - 1$  and  $j = 2, \dots, n - i$  we let  $\Gamma_{i,j}^-$  denote the facet given by  $\lambda_{i,j} = \lambda_{i+1,j-1}$ . Clearly, any facet of  $P$  is either one of  $\Gamma_{i,j}$  or one of  $\Gamma_{i,j}^-$ .

The next proposition describes all linear relations between facets of  $P$ .

**Proposition 3.2.** *The following linear relations hold in  $R_Q$ :*

$$[\Gamma_{i,j}] - [\Gamma_{i,j}^-] - [\Gamma_{i-1,j}] + [\Gamma_{i-1,j+1}^-] = 0,$$

where terms should be ignored if their indices are out of range. Moreover, all linear relations are generated by these.

We call this relation the *4-term relation* at  $(i, j)$ .

*Proof.* Let  $e_{i,j}$  be the standard basis in  $\mathbb{R}^d$ . The 4-term relation at  $(i, j)$  is exactly the relation

$$\sum_{\Gamma} \xi_{\Gamma}(e_{i,j})[\Gamma] = 0,$$

---

<sup>1</sup> Our face diagrams (as well as the diagrams in [12]) are reflections of the diagrams in [15] with respect to a horizontal line.

where the summation is over all facets of  $Q$ . In fact, there are at most four facets  $\Gamma$  of  $Q$  such that  $\xi_\Gamma(e_{i,j}) \neq 0$ , namely,  $\Gamma_{i,j}$ ,  $\Gamma_{i-1,j}$ ,  $\Gamma_{i,j}^-$ , and  $\Gamma_{i-1,j+1}^-$ . It is straightforward to check that the coefficients are as stated.  $\square$

**3.3. Kogan faces.** In what follows we will mostly consider faces of the Gelfand–Zetlin polytope given by equations of the type<sup>2</sup>  $\lambda_{i,j} = \lambda_{i+1,j}$  (that is, intersections of facets of the form  $\Gamma_{i,j}$ ). We will call such faces *Kogan faces*. With each Kogan face  $F$  we associate a permutation  $w(F)$  as follows. First, assign to each equation  $\lambda_{i,j} = \lambda_{i+1,j}$  the simple transposition  $s_{i+j} = (i + j, i + j + 1)$ . Now compose all simple transpositions corresponding to the equations defining  $F$  by going from left to right in each row of the diagram for  $F$  and by going from the bottom row to the top row. The word we obtain is a decomposition of some permutation  $w(F)$  (we multiply permutations from right to left, that is, a decomposition  $w = w_1 w_2$  means that  $w(i) = w_1(w_2(i))$  for all  $i = 1, \dots, n$ ). The face  $F$  is said to be *reduced* if this decomposition is reduced (in what follows we only consider permutations for reduced faces).<sup>3</sup> We recall that a decomposition  $w = s_{i_1} \cdots s_{i_l}$  is said to be *reduced* if its length is minimal, that is, the permutation  $w$  cannot be decomposed into a product of fewer than  $l$  elementary transpositions. The reduced Kogan faces of the Gelfand–Zetlin polytope are in bijective correspondence with the reduced *rc-graphs* (also called *pipe-dreams*; see [15], 2.2.1). We note that the permutations associated with a face and with the corresponding pipe-dream are the same.

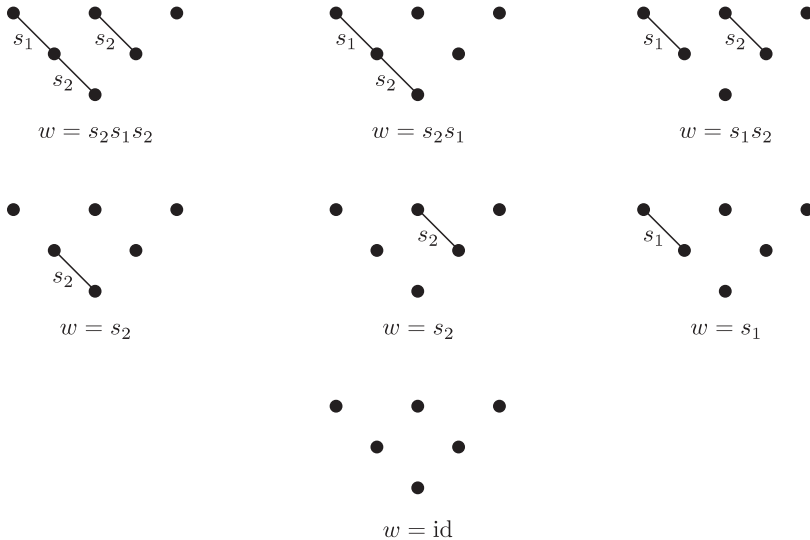


Figure 2. Reduced Kogan face diagrams for the 3-dimensional Gelfand–Zetlin polytope.

<sup>2</sup>That is, of type  $L$  in the notation of [12], which is the same as equations of type  $A$  in [15] (his  $\lambda_{i+j,i}$  is our  $\lambda_{i,j}$ ).

<sup>3</sup>Note that our definition of  $w(F)$  does not agree with [15], 2.2.1: his  $w(F)$  is our  $w(F)^{-1}$ . However, this difference does not affect the definition of reduced faces.

All reduced Kogan face diagrams for  $n = 3$  with the corresponding permutations are shown in Figure 2.

**Proposition 3.3.** *All Kogan faces are regular.*

*Proof.* There is a unique Kogan vertex. This vertex is simple and is contained in any other Kogan face. The result now follows from Proposition 2.3.  $\square$

Using the 4-term relations, we can express  $[\Gamma_{0,j+1}^-]$  in terms of Kogan facets:

$$\begin{aligned}
 [\Gamma_{0,j+1}^-] &= [\Gamma_{0,j}] - [\Gamma_{1,j}] + [\Gamma_{1,j}^-] = [\Gamma_{0,j}] - [\Gamma_{1,j}] + [\Gamma_{1,j-1}] \\
 &\quad - [\Gamma_{2,j-1}] + [\Gamma_{2,j-1}^-] = \cdots = \sum_{i=0}^{j-1} [\Gamma_{i,j-i}] - [\Gamma_{i+1,j-i}].
 \end{aligned}$$

We define the  $k$ -antidiagonal sum of facets  $AD_k$  as the sum of all elements of the form  $[\Gamma_{i,j}]$  with  $i + j = k$  fixed (including the case  $i = 0$ ). Let  $\Gamma_j = \Gamma_{0,j}$  and  $\Gamma_j^- = \Gamma_{0,j}^-$ . The computation above shows that

$$[\Gamma_{j+1}] - [\Gamma_{j+1}^-] = AD_{j+1} - AD_j.$$

**Proposition 3.4.** *The following identities hold in  $R_P$ :*

$$\frac{\partial}{\partial \lambda_1} = \pi(-[\Gamma_1]), \quad \frac{\partial}{\partial \lambda_2} = \pi([\Gamma_2^-] - [\Gamma_2]), \quad \dots, \quad \frac{\partial}{\partial \lambda_n} = \pi([\Gamma_n^-]).$$

*Proof.* Let  $\partial_j$  be the image of the vector  $\frac{\partial}{\partial \lambda_j}$  under the natural inclusion  $\Lambda_P \rightarrow \Lambda_Q$ . Denote by  $H_j$  and  $H_j^-$  the support numbers corresponding to the facets  $\Gamma_j$  and  $\Gamma_j^-$ , respectively. Thus,  $H_j$  and  $H_j^-$  are linear functionals on  $\Lambda_Q$ . By the chain rule we have

$$\partial_j = \sum_{k=1}^{n-1} H_k(\partial_j)[\Gamma_j] + \sum_{k=2}^n H_k^-(\partial_j)[\Gamma_j^-]$$

in  $R_Q$ , since  $[\Gamma_j] = \partial/\partial H_j$ , and similarly for  $[\Gamma_j^-]$ . It suffices to note that  $H_k(\partial_j) = -\delta_{kj}$  and  $H_k^-(\partial_j) = \delta_{kj}$ , where  $\delta_{kj}$  is the Kronecker delta.  $\square$

### 4. Schubert cycles and faces

**4.1. Schubert cycles.** For the rest of the paper we set  $G = GL_n(\mathbb{C})$ . Let  $B$  and  $B^-$  be the subgroups of upper-triangular and lower-triangular matrices in  $G$ , respectively. The Weyl group of  $G$  can be identified with the symmetric group  $S_n$ : a permutation  $w \in S_n$  corresponds to the element of  $G$  acting on the standard basis vectors  $e_i$  by the formula  $e_i \mapsto e_{w(i)}$ . For each  $w \in S_n$ , we define the *Schubert variety*  $X^w$  to be the closure of the  $B^-$ -orbit of  $w$  in the flag variety  $X = G/B$ . It is easy to check that the length  $l(w)$  of  $w$  is equal to the codimension of  $X^w$  in  $X$ . The class  $[X^w]$  of  $X^w$  in  $CH^{l(w)}(X)$  is called the *Schubert cycle* corresponding to the permutation  $w$ .

*Remark 4.1.* The notation in [16] is different from ours. Namely, they consider the flag variety  $B^- \setminus G$ . Under the isomorphism  $G/B \rightarrow B^- \setminus G$  sending  $gB$  to  $w_0 g^{-1} w_0^{-1} B^-$  our Schubert variety  $X^w$  is mapped to the Schubert variety  $X_{w_0^{-1} w w_0}$  in the notation of [16], § 4.



**4.2. Schubert polynomials.** We now recall the notion of a *Schubert polynomial* [3], [17]. With each elementary transposition  $s_i = (i, i + 1)$  we associate a corresponding divided difference operator (acting on polynomials in the variables  $x_1, x_2, \dots$ ) by the formula

$$A_i(f) = \frac{f - s_i(f)}{x_i - x_{i+1}},$$

where  $s_i(f)$  is the polynomial  $f$  with the variables  $x_i$  and  $x_{i+1}$  interchanged. For a permutation  $w$ , consider a reduced (that is, shortest) decomposition  $w^{-1}w_0 = s_{i_1} \cdots s_{i_k}$  of the permutation  $w^{-1}w_0$  into a product of elementary transpositions. The Schubert polynomial  $\mathfrak{S}_w$  is defined by the formula

$$\mathfrak{S}_w(x_1, x_2, \dots) = A_{i_1} \cdots A_{i_k}(x_1^{n-1}x_2^{n-2} \cdots x_{n-1}).$$

**Theorem 4.2** [3]. *The class  $[X^w]$  of the Schubert variety  $X^w$  in  $CH(X)$  is equal to  $\mathfrak{S}_w(x_1, x_2, \dots)$ , where  $x_i = -c_1(\mathcal{L}_i)$  is the negative of the first Chern class of the tautological quotient line bundle  $\mathcal{L}_i$ . Under our identification  $CH(X) = R_P$ ,*

$$[X^w] = \mathfrak{S}_w\left(-\frac{\partial}{\partial \lambda_1}, \dots, -\frac{\partial}{\partial \lambda_n}\right).$$

We now recall the Fomin–Kirillov theorem [7]. With each face  $F$  we associate a monomial  $x(F)$  in the variables  $x_1, \dots, x_{n-1}$  by assigning  $x_j$  to each equation  $\lambda_{i,j} = \lambda_{i+1,j}$  giving  $F$  and then taking the product of all these variables (here the order does not matter, of course). The Fomin–Kirillov theorem states that the Schubert polynomial  $\mathfrak{S}_w$  of the Schubert cycle  $[X^w]$  is equal to

$$\sum_{w(F)=w} x(F),$$

where the sum is taken only over the reduced Kogan faces.

**4.3. Representation of Schubert cycles by faces.** The polytope ring provides a natural setting in which one can directly identify Schubert cycles with linear combinations of faces, sidestepping the use of Schubert polynomials. The following theorem is an immediate analogue of the Fomin–Kirillov theorem: it shows that each Schubert cycle can be represented by a sum of faces in exactly the same way as the corresponding Schubert polynomial can be represented by a sum of monomials.

**Theorem 4.3.** *The Schubert cycle  $[X^w]$ , regarded as an element of the Gelfand–Zetlin polytope ring, can be represented by the linear combination of faces*

$$[X^w] = \pi\left(\sum_{w(F)=w} [F]\right),$$

where the sum is taken only over the reduced dual Kogan faces (all these faces are regular).

The proof of this theorem will be given in §5.2. It uses the combinatorics and the geometry of the Gelfand–Zetlin polytope together with the Demazure character formula.

Despite the similarity between this theorem and the Fomin–Kirillov theorem, the former cannot be formally deduced from the latter, since there is no term-by-term equality between monomials in the Schubert polynomial  $\mathfrak{S}_w$  (which always lie in the ring  $R_P$ ) and the images  $\pi([F])$  of the faces in the decomposition of the cycle  $X^w$  (which usually do not lie in  $R_P$ ). This can already be seen in the case when  $n = 3$  and  $w = s_2$  (see Remark 2.5).

We note that a Schubert cycle might have a simpler representation by sums of faces than the one given by this theorem (see Example 4.4).

**Example 4.4.** Using Proposition 3.4 and Theorem 4.2, we can express the Schubert divisors  $[X^{s_i}]$  in terms of the elements of the polytope ring corresponding to facets of  $P$ . First, we have

$$[X^{s_{i_0}}] = \mathfrak{S}_{s_{i_0}} \left( -\frac{\partial}{\partial \lambda_1}, -\frac{\partial}{\partial \lambda_2}, \dots \right) = \pi \left( \sum_{j=1}^{i_0} ([\Gamma_j] - [\Gamma_j^-]) \right),$$

where we drop all terms whose indices are out of range. As we have seen, the element  $[\Gamma_j] - [\Gamma_j^-]$  is equal to  $AD_j - AD_{j-1}$ . It follows that

$$[X^{s_{i_0}}] = \pi(AD_{i_0}) = \pi \left( \sum_{j=1}^{i_0} [\Gamma_{i_0-j,j}] \right).$$

We obtain a representation of  $[X^{s_{i_0}}]$  as a sum of  $i_0$  facets. This representation coincides with the one given in Theorem 4.3. Note, however, that  $[X^{s_{n-1}}]$  can be represented by a single facet, namely, we have

$$[X^{s_{n-1}}] = -\sum_{i=1}^{n-1} \frac{\partial}{\partial \lambda_i} = \frac{\partial}{\partial \lambda_n} = \pi[\Gamma_n^-].$$

We have used the equality  $\sum_{i=1}^n \frac{\partial}{\partial \lambda_i} = 0$  in  $R_Q$  because the volume polynomial is translation invariant, in particular, it does not change if we add the same number to all the  $\lambda_i$ .

Theorem 4.3 together with relations in the polytope ring  $R_P$  implies the following dual representation of Schubert cycles by faces. We define *dual Kogan faces* of the Gelfand–Zetlin polytope to be the faces given by equations of the type<sup>4</sup>  $\lambda_{i,j} = \lambda_{i+1,j-1}$  (that is, intersections of facets of the form  $\Gamma_{i,j}^-$ ). In other words, dual Kogan faces are mirror images of Kogan faces with respect to a vertical line. With each dual Kogan face  $F^*$  we can again associate a permutation  $w(F^*)$ . Namely, assign to each equation  $\lambda_{i,j} = \lambda_{i+1,j-1}$  the simple reflection  $s_{n-j+1}$ , and compose these reflections by going from the bottom row to the top row and from right to left in each row. We note that the permutation  $w(F^*)$  is the same as the permutation  $w(F)$  constructed from the Kogan face  $F$  obtained as the mirror image of  $F^*$  with respect to a vertical line (that is, each equation  $\lambda_{i,j} = \lambda_{i+1,j-1}$  is replaced by  $\lambda_{i,n-i-j+1} = \lambda_{i+1,n-i-j+1}$ ).

---

<sup>4</sup>That is, of type  $R$  in the notation of [12], which is the same as an equation of type  $B$  in [15].

**Corollary 4.5.** *The Schubert cycle  $[X^w]$ , regarded as an element of the Gelfand–Zetlin polytope ring, can be represented by the following linear combination of faces:*

$$[X^w] = \pi \left( \sum_{w(F^*)=w_0 w w_0^{-1}} [F^*] \right),$$

where the sum is taken only over the reduced dual Kogan faces.

*Proof.* Consider the linear automorphism of  $\mathbb{R}^d$  that takes a point with coordinates  $\lambda_{i,j}$  to the point with coordinates  $-\lambda_{i,n-i-j+1}$ . It takes a Gelfand–Zetlin polytope  $P_\lambda$  to the Gelfand–Zetlin polytope  $P_{-w_0\lambda}$ , where  $w_0(\lambda_1, \dots, \lambda_n) = (\lambda_n, \dots, \lambda_1)$ . Thus, it induces an automorphism of the space  $V_P$  preserving the lattice  $\Lambda_P$ , and hence an automorphism  $A$  of  $R_P$ . Choose a resolution  $Q$  so that the automorphism  $A$  extends to  $R_Q$ . It is clear that the extended automorphism takes the element  $\pi[F]$  corresponding to a regular face  $F$  of  $P$  to the element  $\pi[F^*]$ , where the face diagram of  $F^*$  is obtained from the face diagram of  $F$  by the mirror reflection with respect to a vertical line. It now suffices to prove that the automorphism  $A$  of  $R_P$  coincides with the automorphism of  $CH^*(X)$  that sends a Schubert cycle  $[X^w]$  to  $[X^{w_0 w w_0^{-1}}]$ . (The latter automorphism is induced by the automorphism of  $X$  that sends a complete flag to the flag of its orthogonal complements.) Indeed, this is easy to verify for Schubert divisors as in Example 4.4 (we basically need to repeat the same computation with dual Kogan faces instead of Kogan faces). The general case now follows, since Schubert divisors are multiplicative generators of the cohomology ring of  $X$ .  $\square$

Note that any Kogan face intersects any dual Kogan face transversely. Hence, we can represent the cycles given by the Richardson varieties as sums of faces.

**Corollary 4.6.** *The product of any two Schubert cycles  $[X^w]$  and  $[X^u]$  can be represented as the following sum of faces:*

$$[X^w] \cdot [X^u] = \pi \left( \sum_{\substack{w(F)=w \\ w(F^*)=w_0 u w_0^{-1}}} [F \cap F^*] \right),$$

where  $F$  and  $F^*$  run over the reduced Kogan faces and the dual Kogan faces, respectively.

### 5. Demazure characters

**5.1. Characters.** For each  $\lambda = (\lambda_1, \dots, \lambda_n)$  we consider the affine hyperplane  $\mathbb{R}^{n-1} \subset \mathbb{R}^n$  with coordinates  $y_1, \dots, y_n$  given by the equation  $y_1 + \dots + y_n + u_0 = 0$ , where  $u_0 = \lambda_1 + \dots + \lambda_n$ . Let  $u_1, \dots, u_{n-1}$  be coordinates in  $\mathbb{R}^{n-1}$  such that  $y_i = u_i - u_{i-1}$  for  $i = 1, \dots, n-1$ , and consider the following linear map  $p: \mathbb{R}^d \rightarrow \mathbb{R}^{n-1}$  from the space  $\mathbb{R}^d$  with coordinates  $\lambda_{i,j}$  to the hyperplane  $\mathbb{R}^{n-1} \subset \mathbb{R}^n$ :

$$u_i = \sum_{j=1}^{n-i} \lambda_{i,j}.$$

In other words, if we arrange the coordinates  $\lambda_{i,j}$  into a triangular table as in (GZ), then  $u_i$  is the sum of all the elements in the  $i$ th row. In what follows we identify  $\mathbb{R}^n$

with the real span of the weight lattice  $\Lambda$  of  $G$  in such a way that the  $i$ th basis vector in  $\mathbb{R}^n$  corresponds to the weight given by the  $i$ th entry of the diagonal torus in  $G$ . Then the hyperplane  $\mathbb{R}^{n-1}$  is the parallel translate of the hyperplane spanned by the roots of  $G$ . It is easy to check that the image of the Gelfand–Zetlin polytope  $P_\lambda \subset \mathbb{R}^d$  under the map  $p$  is the weight polytope of the representation  $V_\lambda$ .

Let  $S$  be a subset of the Gelfand–Zetlin polytope  $P_\lambda$  (in what follows,  $S$  will be a face or a union of faces), and define the *character*  $\chi_S$  of  $S$  as the sum of the formal exponentials  $e^{p(z)}$  over all integer points  $z \in S$ , that is,

$$\chi(S) := \sum_{z \in S \cap \mathbb{Z}^d} e^{p(z)}.$$

The formal exponentials  $e^u$ ,  $u \in \mathbb{Z}^n$ , generate the group algebra of  $\Lambda$ . Thus, the character takes values in this group algebra.

Consider the linear operators  $s_i: \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$  with the point  $s_i(u_1, \dots, u_{n-1})$  differing from the point  $(u_1, \dots, u_{n-1})$  at most in the  $i$ th coordinate, and with the  $i$ th coordinate of  $s_i(u_1, \dots, u_{n-1})$  equal to  $u_{i-1} + u_{i+1} - u_i$ , where  $u_n = 0$ . It is known and easy to verify that the operators  $s_i$  are induced by orthogonal reflections in  $\mathbb{R}^n$  (given by the simple roots) and that they generate an action of the symmetric group  $S_n$  on  $\mathbb{R}^{n-1}$  such that the reflection  $s_i$  corresponds to the elementary transposition  $s_i = (i, i + 1)$  (we use the same notation for a reflection and the corresponding transposition, which is a standard practice when dealing with group actions). We also define the action of  $s_i$  on the group algebra of the weight lattice by setting  $s_i(e^u) := e^{s_i(u)}$ .

In what follows we identify  $\mathbb{R}^{n-1}$  with the real vector space spanned by the roots of  $G$ , and  $s_i$  with the reflections corresponding to simple roots. The simple roots correspond to the standard basis vectors in  $\mathbb{R}^{n-1}$ , that is, the only non-zero coordinate of the simple root  $\alpha_i$  is  $u_i$ , and this coordinate is equal to 1.

Let  $V_{\lambda,w}^-$  be the Demazure  $B^-$ -module defined as the dual space to the space  $H^0(X^w, \mathcal{L}_\lambda|_{X^w})$  of global sections, where  $\mathcal{L}_\lambda := \mathcal{L}_1^{\otimes \lambda_1} \otimes \dots \otimes \mathcal{L}_n^{\otimes \lambda_n}$  is the very ample line bundle on  $X$  corresponding to a strictly dominant weight  $\lambda$ . By the Borel–Weil–Bott theorem,  $V_{\lambda,\text{id}}^-$  is isomorphic to the irreducible representation  $V_\lambda$  of  $G$  with the highest weight  $\lambda$ . We choose a basis of weight vectors in  $V_{\lambda,w}^-$ , and recall that the *Demazure character*  $\chi^w(\lambda)$  of  $V_{\lambda,w}^-$  is the sum, over all weight vectors in this basis, of the exponentials of the corresponding weights, or equivalently,

$$\chi^w(\lambda) := \sum_{\mu \in \Lambda} m_{\lambda,w}(\mu) e^\mu,$$

where  $m_{\lambda,w}(\mu)$  is the multiplicity of the weight  $\mu$  in  $V_{\lambda,w}^-$ .

The main result in this section establishes a relation between the Demazure character of a Schubert variety and the character of the union of the corresponding faces.

**Theorem 5.1.** *For each permutation  $w \in S_n$  the Demazure character  $\chi^w(\lambda)$  is equal to the character of the following union of faces:*

$$\chi^w(\lambda) = \chi\left(\bigcup_{w(F_\lambda)=w} F_\lambda\right),$$

where, as usual,  $F_\lambda$  runs only over the reduced Kogan faces in the Gelfand–Zetlin polytope  $P_\lambda$ .

In contrast to Theorem 4.3, this theorem and its corollaries below (describing the Hilbert functions and the degrees of Schubert varieties in projective embeddings in  $\mathbb{P}(V_\lambda)$ ) use exactly the polytope  $P_\lambda$  and not just any one of the polytopes analogous to  $P_\lambda$ . Whenever the choice of  $\lambda$  matters, we indicate this by using the notation  $F_\lambda$  instead of  $F$  for the faces.

For Kempf permutations, Theorem 5.1 reduces to the form in Corollary 15.2 of [21]. We note that by Proposition 2.3.2 in [15], a permutation  $w$  is Kempf if and only if there is a unique reduced Kogan face  $F$  such that  $w(F) = w$ . Hence,  $\chi^w(\lambda) = \chi(F)$  in this case.

In §5.4 we will reduce this theorem to the purely combinatorial Key Lemma 5.8, the proof of which is given in §6.3.

We now derive some corollaries of Theorem 5.1. First, we can similarly describe the Demazure character of  $B$ -modules. Let  $V_{\lambda,w}^+$  be the Demazure  $B$ -module defined as the dual space to  $H^0(X_w, \mathcal{L}_\lambda|_{X_w})$ , where  $X_w$  is the closure of the  $B$ -orbit of  $w$  in  $X$  (in particular,  $[X_w] = [X^{w_0w}]$  in  $CH^*(X)$ ).

**Corollary 5.2.** *For each permutation  $w \in S_n$  the Demazure character  $\chi_w(\lambda)$  is equal to the character of the following union of faces:*

$$\chi_w(\lambda) = \chi\left(\bigcup_{w(F_\lambda^*)=w w_0} F_\lambda^*\right),$$

where  $F_\lambda^*$  runs over the reduced dual Kogan faces in the Gelfand–Zetlin polytope.

This corollary follows immediately from the proof of Theorem 5.1 together with the definition of dual Kogan faces, since  $\chi_w(\lambda) = w_0\chi^{w_0w}(\lambda)$ .

Another corollary of Theorem 5.1 describes the Hilbert function of the Schubert variety  $X^w$  embedded in  $\mathbb{P}(H^0(X^w, \mathcal{L}_\lambda|_{X^w})^*) \subset \mathbb{P}(V_\lambda)$ .

**Corollary 5.3.** *For any permutation  $w \in S_n$  the space  $H^0(X^w, \mathcal{L}_\lambda|_{X^w})$  has dimension equal to the number of integer points in the union of all reduced Kogan faces with the permutation  $w$ :*

$$\dim H^0(X^w, \mathcal{L}_\lambda|_{X^w}) = \left| \bigcup_{w(F)=w} F_\lambda \cap \mathbb{Z}^d \right|.$$

In particular, the Hilbert function  $H_{\lambda,w}(k) := \dim H^0(X^w, \mathcal{L}_\lambda^{\otimes k}|_{X^w})$  is equal to the Ehrhart polynomial of  $\bigcup_{w(F_\lambda)=w} F_\lambda$ , that is,

$$H_{\lambda,w}(k) = \left| \bigcup_{w(F_\lambda)=w} kF_\lambda \cap \mathbb{Z}^d \right|$$

for all positive integers  $k$ .

This corollary will be essential for the proof of Theorem 4.3.

**5.2. Degrees of Schubert varieties.** To prove Theorem 4.3 we first prove an analogous identity for the *degree polynomials* of Schubert varieties. The *degree polynomial*  $D_w$  on  $\mathbb{R}^n$  is uniquely characterized by the property that  $(d-l(w))! D_w(\lambda) = \text{deg}_\lambda(X^w)$  for all dominant weights  $\lambda \in \mathbb{Z}^n \subset \mathbb{R}^n$ . In particular,  $D_{w_0} = 1$  and  $D_{\text{id}} = (1/d!) \text{deg}_\lambda(X) = \text{Volume}(Q_\lambda)$ . The degree polynomials first appeared in the Bernstein–Gelfand–Gelfand paper [3] and were recently studied by Postnikov and Stanley [21]. Below we prove identities relating the degree polynomial and the volumes of faces of the Gelfand–Zetlin polytope.

Denote by  $\mathbb{R}F \subset \mathbb{R}^d$  the affine span of the face  $F$ . In the formulae of the next theorem, the volume form on  $\mathbb{R}F$  is normalized so that the covolume of the lattice  $\mathbb{Z}^d \cap \mathbb{R}F$  in  $\mathbb{R}F$  is equal to 1.

**Theorem 5.4.** *The following equalities hold:*

$$D_w = \sum_{w(F_\lambda)=w} \text{Volume}(F_\lambda),$$

$$D_w = \sum_{w(F_\lambda^*)=w_0 w w_0^{-1}} \text{Volume}(F_\lambda^*).$$

For Kempf permutations the first equality in this theorem reduces to the last formula in Corollary 15.2 of [21].

*Proof.* Theorem 5.4 follows easily from Corollary 5.3 and Hilbert’s theorem describing the leading monomial of the Hilbert polynomial, by the same arguments as in [11]. Indeed, by Hilbert’s theorem  $\dim(V_{k\lambda,w}^-)$  is a polynomial in  $k$  (for large  $k$ ), and its leading term is equal to  $D_w(\lambda)k^d$ . Next, note that  $\dim(V_{k\lambda,w}^-)$  is the number of integer points in  $\bigcup_{w(F_\lambda)=w} kF_\lambda$  by Corollary 5.3. Finally, use the fact that the volume of each face  $F$  is the leading term in the Ehrhart polynomial of this face (since  $\text{Volume}(kF) = k^n \text{Volume}(F)$  is approximately equal to the number of integer points in  $kF$  for large  $k$ ).  $\square$

*Remark 5.5.* Dual Kogan faces are exactly the faces considered in [16], § 4. We note that the definition of  $w(F^*)$  in [16] is different from ours as well as from that in [15]. Namely, in our notation they associate the permutation  $w_0 w w_0^{-1}$  with a dual Kogan face  $F^*$ . However, since their Schubert cycle  $[X_w]$  is defined so that it coincides with our Schubert cycle  $[X^{w_0 w w_0^{-1}}]$  (see Remark 4.1), their Theorem 8 (describing the toric degeneration of the Schubert variety  $X_w$ ) uses exactly the same faces as in the second equality of our Theorem 5.4, and the latter equality can be deduced from the former by standard arguments from toric geometry.

*Proof of Theorem 4.3.* We now deduce Theorem 4.3 from Theorem 5.4 using Lemma 2.4. Recall that the lattice  $\Lambda_P$  is a sublattice of  $\Lambda_Q$ . In particular, the polytope  $P_\lambda$  can be regarded as an element of  $\Lambda_Q = \text{Sym}^1(\Lambda_Q)$ . Let  $L_\lambda$  denote the image of  $P_\lambda$  under the canonical projection  $\text{Sym}(\Lambda_Q) \rightarrow R_Q$ . It is easy to check that, under the isomorphism in Theorem 3.1, the class  $\pi(L_\lambda)$  corresponds to the first Chern class of the line bundle  $\mathcal{L}_\lambda$ . Hence, we have the following identity in  $R_P$ :

$$[X^w]\pi(L_\lambda)^{d-l} = (\text{deg}_\lambda(X^w))[\text{pt}],$$

where  $d - l = d - l(w)$  is the dimension of the variety  $X^w$  (the product on the left-hand side is taken in  $R_P$ ; according to our usual convention, we identify elements of  $R_P$  with their images in  $M_{Q,P}$ ).

On the other hand, it is easy to check that for each face  $F_\lambda \subset Q_\lambda$  of codimension  $l$  the product  $[F_\lambda]L_\lambda^{d-l}$  in  $R_Q$  is equal to  $(d - l)! \text{Volume}(F_\lambda)$  times the class of a vertex. Hence, by Theorem 5.4 we have

$$[X^w]\pi(L_\lambda)^{d-l} = \pi\left(\sum_{w(F)=w} [F]L_\lambda^{d-l}\right).$$

Since elements of the form  $\pi(L_\lambda)^{d-l}$  span  $R_P^{d-l}$ , we can apply Lemma 2.4 and conclude that  $[X^w] = \pi(\sum_{w(F)=w} [F])$ .  $\square$

**5.3. The Demazure character formula.** To prove Theorem 5.1, we use the Demazure character formula for  $\chi^w(\lambda)$  together with a purely combinatorial argument. Let us recall the Demazure character formula (see [1] for more details). For each  $i = 1, \dots, n - 1$  define the operator  $T_i$  on the group algebra of the weight lattice of  $G$  by the formula

$$T_i(f) = \frac{f - e^{-\alpha_i} s_i(f)}{1 - e^{-\alpha_i}}.$$

Similarly, define the operator  $T_i^-$  by the formula

$$T_i^-(f) = \frac{f - e^{\alpha_i} s_i(f)}{1 - e^{\alpha_i}}.$$

**Theorem 5.6** [1]. *Let  $w = s_{i_1} \cdots s_{i_l}$  be a reduced decomposition of  $w$ . Then the Demazure characters  $\chi_w(\lambda)$  and  $\chi^{w_0 w}(\lambda)$  can be computed as follows:*

$$\chi_w(\lambda) = T_{i_1} \cdots T_{i_l} e^\lambda$$

and

$$\chi^{w_0 w}(\lambda) = T_{n-i_1}^- \cdots T_{n-i_l}^- e^{w_0 \lambda}.$$

The first equality is the standard form of the Demazure character formula. We will use the second equality, which follows immediately from the first one, since  $\chi_w(\lambda) = w_0 \chi^{w_0 w}(\lambda)$  and  $w_0 T_i = T_{n-i}^- w_0$ .

We note that this theorem is similar to Theorem 4.2 (and especially to its  $K$ -theory version ([6]; see also [20], §3)), which describes Schubert cycles using divided difference operators. However, in this theorem we apply the operators  $T_{i_j}$  in the same order as the elementary transpositions  $s_{i_j}$  in a reduced decomposition of  $w$ , while in Theorem 4.2 the order is the opposite (that is, the same as in  $w^{-1}$ ).

**5.4. Mirror mitosis.** *Mitosis* is a combinatorial operation (introduced in [14], [19]) that produces for each Kogan face a set of Kogan faces.<sup>5</sup> If we apply mitosis in the  $i$ th column to the set of all reduced Kogan faces corresponding to a permutation  $w$ , then we obtain all the reduced Kogan faces corresponding to the

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<sup>5</sup>The original definition was in terms of pipe-dreams rather than Kogan faces.

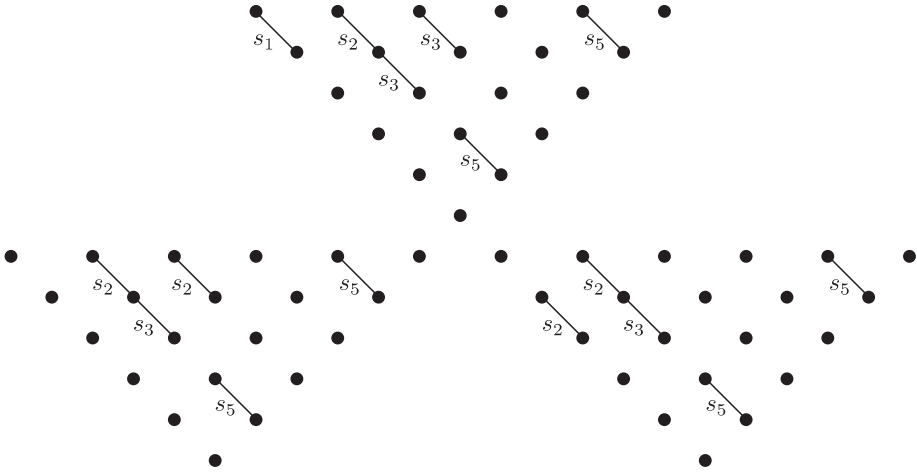


Figure 3. Mirror mitosis applied to the first row of the upper face diagram gives the set consisting of the two lower diagrams.

permutation  $ws_i$  satisfying the condition  $l(ws_i) = l(w) - 1$ . We will need *mirror mitosis*, which is obtained from the usual mitosis by transposition of the face diagrams (interchanging rows and columns). In other words, mirror mitosis for  $w$  is the usual mitosis for  $w^{-1}$ . We use mirror mitosis to deduce Theorem 5.1 from the Demazure character formula. We now give a direct definition of mirror mitosis.

Let  $F$  be a reduced Kogan face of dimension  $l$ . For each  $i = 1, \dots, n - 1$  we construct a set  $M_i^-(F)$  of reduced Kogan faces of dimension  $l + 1$  as follows. For each  $i = 1, \dots, n - 1$  we say that the diagram of  $F$  has an edge in the  $i$ th row if the face  $F$  satisfies the equation  $\lambda_{i-1,j} = \lambda_{i,j}$  for some  $j$ . Similarly, we say that the diagram of  $F$  has an edge in the  $i$ th column if the face satisfies the equation  $\lambda_{j-1,i} = \lambda_{j,i}$  for some  $j$ . We consider the  $i$ th row in the face diagram of  $F$ . If it does not have an edge in the first column, then  $M_i^-(F)$  is empty. Suppose now that the  $i$ th row of  $F$  contains edges in all the columns from the first through the  $k$ th, and does not have an edge in the  $(k + 1)$ th column. Then for each  $j \leq k$  we check whether the  $(i + 1)$ th row has an edge in the  $j$ th column. If it does, we do nothing. The elements of  $M_i^-(F)$  correspond to the values of  $j$  for which there is no edge at the intersection of the  $(i + 1)$ th row and the  $j$ th column. For such a value of  $j$  we delete the  $j$ th edge in the  $i$ th row and shift each edge on the left of it in the same row one step to the southeast (to the  $(i + 1)$ th row) whenever possible. The new reduced Kogan face  $F_{i,j}$  thus obtained is called the  $j$ th *offspring* of  $F$  in the  $i$ th row. The set  $M_i^-(F)$  consists of the offsprings  $F_{i,j}$  for all  $1 \leq j \leq k$ .

The cardinality of  $M_i^-(F)$  is equal to  $k - k'$ , where  $k'$  is the number of edges in the first  $k$  positions in the  $(i + 1)$ th row. This is the same as the number of monomials in  $A_i(x_i^k x_{i+1}^{k'})$ . An illustration of mirror mitosis is given in Figure 3.

The next theorem follows from the properties of the usual mitosis [19].



**Theorem 5.7.** *If  $l(s_i w) = l(w) - 1$ , then*

$$\bigcup_{w(F)=w} M_i^-(F) = \bigcup_{w(E)=s_i w} \{E\}.$$

*Proof of Theorem 5.1.* This now follows by backwards induction on  $l(w)$  from the Demazure character formula (the second equality in Theorem 5.6), Theorem 5.7, and the next lemma.

**Key Lemma 5.8.** *For each permutation  $w \in S_n$  and an elementary transposition  $s_i$  such that  $l(s_i w) = l(w) - 1$ ,*

$$T_i^- \chi \left( \bigcup_{w(F)=w} F_\lambda \right) = \chi \left( \bigcup_{\substack{w(F)=w \\ E \in M_i^-(F)}} E_\lambda \right).$$

The proof of this lemma is purely combinatorial. It is given in §6.3.  $\square$

### 6. Mitosis on parallelepipeds

In this section we reduce the mitosis on faces of the Gelfand–Zetlin polytope to an analogous operation (called *paramitosis*) on faces of a parallelepiped. The latter is easier to study and has a transparent geometric meaning (see Remark 6.7). Paramitosis for parallelepipeds and its applications to exponential sums and Demazure operators are studied in §§6.1 and 6.2. The material therein is self-contained, and all the results are proved by elementary methods. These results are then used in §6.3 to prove Key Lemma 5.8. Another application is Proposition 6.6, which gives a new minimal realization of a simplex as a cubic complex.

**6.1. Parallelepipeds.** We consider integers  $\mu_1, \dots, \mu_m, \nu_1, \dots, \nu_m$  with  $\mu_k \leq \nu_k$  for all  $k = 1, \dots, m$  and define the parallelepiped  $\Pi(\mu, \nu)$  as the convex polytope

$$\Pi(\mu, \nu) = \{y = (y_1, \dots, y_m) \in \mathbb{R}^m \mid \mu_k \leq y_k \leq \nu_k, k = 1, \dots, m\}.$$

For any such parallelepiped  $\Pi = \Pi(\mu, \nu)$  the sum

$$S_\Pi(t) = \sum_{y \in \Pi \cap \mathbb{Z}^m} t^{\sigma(y)}, \quad \text{where } \sigma(y) = \sum_{k=1}^m y_k,$$

is a polynomial in  $t$ . It can be found explicitly; namely, the following proposition holds.

**Proposition 6.1.**

$$S_\Pi(t) = \prod_{k=1}^m \frac{t^{\nu_k+1} - t^{\mu_k}}{t - 1}.$$

*Proof.* Indeed,

$$\sum_{y \in \Pi \cap \mathbb{Z}^m} t^{\sigma(y)} = \left( \sum_{y_1=\mu_1}^{\nu_1} t^{y_1} \right) \left( \sum_{y_2=\mu_2}^{\nu_2} t^{y_2} \right) \dots \left( \sum_{y_m=\mu_m}^{\nu_m} t^{y_m} \right).$$

Each factor on the right-hand side can be computed as the sum of a geometric progression.  $\square$

The following proposition describes a duality property of  $S_{\Pi}(t)$ .

**Proposition 6.2.**

$$S_{\Pi}(t) = t^{\sum_{k=1}^m (\mu_k + \nu_k)} S_{\Pi}(t^{-1}).$$

The proof is a straightforward computation. Proposition 6.2 can be restated in combinatorial terms as follows: the number of ways to represent an integer  $N$  as a sum  $y_1 + \dots + y_m$  in which  $\mu_k \leq y_k \leq \nu_k$  for all  $k = 1, \dots, m$  is equal to the number of ways to represent the integer  $\sum_{k=1}^m (\mu_k + \nu_k) - N$  in the same form.

Fix an integer  $C$ . Consider the following linear operator on the space of Laurent polynomials in  $t$ : with every Laurent polynomial  $f$  we associate the Laurent polynomial  $f^*$  obtained from  $f$  by replacing each power  $t^k$  by  $t^{C-k}$ . In other terms we have  $f^*(t) = t^C f(t^{-1})$ . Clearly,  $f^{**} = f$  for every Laurent polynomial  $f$ . The duality property of  $S_{\Pi}$  can be restated as follows: if  $C = \sum_{k=1}^m (\mu_k + \nu_k)$ , then  $S_{\Pi} = S_{\Pi}^*$ . For the same value of  $C$ , we define the operator  $T_{\Pi}$  by the formula

$$T_{\Pi}(f) = \frac{f - tf^*}{1 - t}.$$

It is not hard to see that for every Laurent polynomial  $f$  the function  $T_{\Pi}(f)$  is also a Laurent polynomial. The operator  $T_{\Pi}$  depends on the parallelepiped  $\Pi$ .

**Proposition 6.3.** *Let  $\Gamma$  be the face of  $\Pi = \Pi(\mu, \nu)$  given by the equation  $y_1 = \mu_1$  (it may coincide with the whole of  $\Pi$  if  $\mu_1 = \nu_1$ ). Then*

$$S_{\Pi} = T_{\Pi}(S_{\Gamma}).$$

*Proof.* We have  $\Gamma = \Pi(\mu_1, \mu_1, \mu_2, \nu_2, \dots, \mu_n, \nu_n)$ . Therefore, by Proposition 6.1,

$$S_{\Gamma}(t) = t^{\mu_1} \prod_{k=2}^m \frac{t^{\nu_k+1} - t^{\mu_k}}{t - 1}$$

and

$$S_{\Pi}(t) = \prod_{k=1}^m \frac{t^{\nu_k+1} - t^{\mu_k}}{t - 1} = \frac{t^{\nu_1+1-\mu_1} S_{\Gamma}(t) - S_{\Gamma}(t)}{t - 1}. \tag{2}$$

Proposition 6.2 applied to  $\Gamma$  gives us that  $S_{\Gamma}(t) = t^{2\mu_1 + \sum_{k=2}^m (\mu_k + \nu_k)} S_{\Gamma}(t^{-1})$ . Substituting this in the right-hand side of equation (2), we get the desired result.  $\square$

Under certain assumptions this proposition remains true if  $\Pi$  and  $\Gamma$  are replaced by their images under an embedding  $\Pi \rightarrow \mathbb{R}^k$  that preserves the sum of the coordinates.

**Proposition 6.4.** *Consider a linear operator  $\Lambda: \mathbb{R}^k \rightarrow \mathbb{R}^m$  defined over the integers such that  $\sigma \circ \Lambda = \sigma$  (the function  $\sigma$  on the right-hand side is the sum of all the coordinate functions on  $\mathbb{R}^k$ ). Let  $\Pi, \Gamma$ , and  $T_{\Pi}$  be as in Proposition 6.3. Assume that  $\Lambda(B \cap \mathbb{Z}^k) = \Pi \cap \mathbb{Z}^m$  and  $\Lambda(A \cap \mathbb{Z}^k) = \Gamma \cap \mathbb{Z}^m$  for some subsets  $A, B \subset \mathbb{R}^k$  such that the restrictions of  $\Lambda$  to  $B \cap \mathbb{Z}^k$  and  $A \cap \mathbb{Z}^k$  are injective. Then*

$$\sum_{z \in B \cap \mathbb{Z}^k} t^{\sigma(z)} = T_{\Pi} \left( \sum_{z \in A \cap \mathbb{Z}^k} t^{\sigma(z)} \right).$$

*Proof.* For each  $z \in B \cap \mathbb{Z}^k$  let  $y = \Lambda(z)$ . Since  $\sigma(z) = \sigma(y)$ , we get that

$$\sum_{z \in B \cap \mathbb{Z}^k} t^{\sigma(z)} = \sum_{y \in \Pi \cap \mathbb{Z}^m} t^{\sigma(y)}$$

(these two sums coincide term by term), and similarly for the right-hand side. Thus, the desired statement follows from Proposition 6.3.  $\square$

**6.2. Combinatorics of parallelepipeds.** Let  $\Pi = \Pi(\mu, \nu)$  be a coordinate parallelepiped in  $\mathbb{R}^m$  of dimension  $m$  with  $\mu_i < \nu_i$  for all  $i = 1, \dots, m$ . We will now discuss the combinatorics of  $\Pi$ . For every point  $y \in \Pi$  with coordinates  $(y_1, \dots, y_m)$  we can define the *paradiagram* (‘para’ from parallelepiped) of  $y$  as the  $m$ -tuple  $(\tilde{y}_1, \dots, \tilde{y}_m)$ , where

$$\begin{aligned} \tilde{y}_i &= 0 & \text{if } y_i = \mu_i, & & \tilde{y}_i &= 1 & \text{if } y_i = \nu_i, \\ \tilde{y}_i &= * & \text{otherwise.} & & & & \end{aligned}$$

A paradiagram is said to be *reduced* if 1 is never followed immediately by 0 in it.

We consider a face  $F$  of  $\Pi$  and note that all points in the relative interior of  $F$  have the same paradiagram. We will call this the paradiagram of  $F$ . A face  $F$  is said to be reduced if its paradiagram is reduced. Define a *parabox* as a sequence of consecutive positions in a paradiagram. A parabox filled with a string (possibly empty) of ones, followed by a single star, followed by a string (possibly empty) of zeros is called an *intron*<sup>6</sup> *parabox*. A parabox that contains the left end of a paradiagram and that is filled with a string (possibly empty) of zeros is called an *initial parabox*. A parabox that contains the right end of a paradiagram and that is filled with a string (possibly empty) of ones is called a *final parabox*. It is not hard to see that any reduced paradiagram consists of an initial parabox followed by several (possibly none) intron paraboxes, followed by a final parabox. Below is an example of how to split a paradiagram into initial, intron, and final paraboxes:

0 0 0	1 1 1 * 0 0	* 0 0	1 1 *	*	1 1 1
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Two reduced faces  $F_1$  and  $F_2$  of  $\Pi$  of the same dimension are said to be related by an *L-move* if their intersection is a non-reduced facet of both  $F_1$  and  $F_2$ . We can also define an *L-move* of a reduced paradiagram. This is an operation that replaces a single string  $* 0$  in a paradiagram by the string  $1 *$ . Note that an *L-move* does not affect the decomposition of a paradiagram into initial, intron, and final paraboxes.

**Proposition 6.5.** *Two faces  $F_1$  and  $F_2$  of the same dimension are related by an L-move if and only if their paradiagrams are related by an L-move.*

*Proof.* Let  $\delta_1$  be the paradiagram of  $F_1$  and  $\delta_2$  the paradiagram of  $F_2$ . Since  $F_1 \cap F_2$  has codimension 1 in  $F_1$ , the paradiagram  $\delta$  of  $F_1 \cap F_2$  is obtained from  $\delta_1$  by replacing one star by either 0 or 1. We consider two cases.

*Case 1:* the star is replaced by 0. Then, since  $F_1 \cap F_2$  is non-reduced, there must be a 1 immediately before this 0. Since  $F_2$  is reduced, this 1 must be replaced by

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<sup>6</sup>The origin of this term is explained in [14], § 3.5.

a star in the paradiagram  $\delta_2$ . Therefore, the paradiagram  $\delta_1$  is obtained from  $\delta_2$  by an  $L$ -move.

*Case 2:* the star is replaced by 1. Then, since  $F_1 \cap F_2$  is non-reduced, there must be a 0 immediately after this 1. Since  $F_2$  is reduced, this 0 must be replaced by a star in the paradiagram  $\delta_2$ . Therefore, the paradiagram  $\delta_2$  is obtained from  $\delta_1$  by an  $L$ -move.  $\square$

Two faces of the same dimension are said to be  $L$ -equivalent if one of them can be obtained from the other by a sequence of  $L$ -moves or inverse  $L$ -moves (on the level of paradiagrams, inverse  $L$ -moves are defined as the inverse operations of  $L$ -moves). For the sake of brevity, we will write  $L$ -classes instead of  $L$ -equivalence classes. Throughout the rest of the subsection, we identify  $L$ -classes of faces and their unions (clearly, an equivalence class can be easily recovered from its union).

**Proposition 6.6.** *The  $L$ -classes form a simplicial cell complex combinatorially equivalent to a standard simplex. More precisely:*

- any  $L$ -class is homeomorphic to a closed disk;
- there is a one-to-one correspondence between  $L$ -classes and the faces of a simplex such that corresponding sets are homeomorphic, and intersections correspond to intersections.

Figure 4 illustrates this proposition for  $m = 3$ .

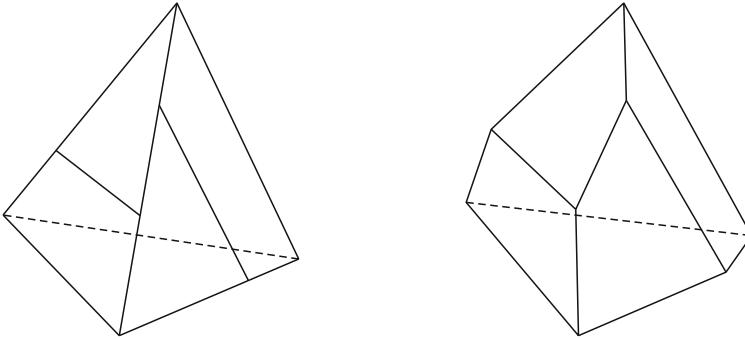


Figure 4. The subdivision of a tetrahedron by two extra edges yields a combinatorial cube.

*Proof.* First consider all reduced vertices. There are exactly  $m + 1$  of them. The paradiagram of a reduced vertex consists of a string of zeros followed by a string of ones. We note that different reduced vertices are never  $L$ -equivalent.

Next, consider any  $L$ -class  $A$  of dimension  $k$ . It has  $k$  intron paraboxes. With the  $L$ -class  $A$  we associate a set  $v(A)$  of  $k + 1$  vertices in the following way: we fill the first  $i \leq k$  intron paraboxes with zeros, and the remaining intron paraboxes with ones. Clearly, the set  $v(A)$  is precisely the set of all reduced vertices contained in the class  $A$ . It follows that  $v(A \cap B) = v(A) \cap v(B)$  for any two classes  $A$  and  $B$ . We note that  $A$  is determined by the positions and sizes of the initial, intron, and final paraboxes, that is, by the set  $v(A)$ . The set  $v(A)$  spans a face of the simplex

$\Sigma$  with vertices at all reduced vertices of  $\Pi$ . Thus, we have an injective map  $v$  from  $L$ -classes of faces of  $\Pi$  to faces of  $\Sigma$ , and it takes intersections to intersections.

The map  $v$  is surjective: any set of reduced vertices has the form  $v(A)$  for some equivalence class  $A$ . Indeed,  $A$  can be defined as the class in which the boundaries of the intron paraboxes are the boundaries between zeros and ones for the vertices in the given set  $v(A)$ .

It remains to prove that any  $L$ -class is homeomorphic to a closed disk. First, note that an  $L$ -class with only one intron parabox is a broken line whose straight line segments are parallel to coordinate axes (every straight line segment of this broken line corresponds to a particular position of the star inside the intron parabox). A broken line is homeomorphic to a line segment. In general, an  $L$ -class is a direct product of broken lines as above, and hence it is homeomorphic to a direct product of line segments, that is, to a closed cube.  $\square$

The most important corollary for us is that the intersection of two  $L$ -classes is again an  $L$ -class.

We can now define *paramitosis*. This is an operation that produces several faces from a single face  $F$ . If the paradiagram of  $F$  has no initial parabox, then the paramitosis of  $F$  is empty. Suppose now that the paradiagram of  $F$  has a non-empty initial parabox. Then we replace it by an intron parabox: the set of all faces obtained in this way (corresponding to all different ways of filling the new intron parabox) is the paramitosis of  $F$ . Below is an example of paramitosis:

$$\boxed{0\ 0\ * \ 0} \xrightarrow{\text{paramitosis}} \boxed{* \ 0\ * \ 0} \quad \text{and} \quad \boxed{1\ * \ * \ 0}$$

The paramitosis of a set of faces is defined as the union of the paramitoses of the individual faces in this set.

*Remark 6.7.* It is easy to describe the paramitosis of an  $L$ -class using the bijection between  $L$ -classes and faces of the standard simplex as defined in Proposition 6.6. Namely, the  $L$ -classes with non-empty initial paraboxes correspond to the faces of the simplex contained in some facet  $H$ . Let  $v$  be the vertex of the simplex that is not contained in  $H$ . Then the paramitosis of a face  $A \subset H$  coincides with the convex hull of  $A$  and  $v$ . It follows that paramitosis of an  $L$ -class is again an  $L$ -class and that paramitosis of the intersection of two  $L$ -classes with non-empty initial paraboxes coincides with the intersection of their paramitoses.

For a subset  $A \subset \Pi$  we define the Laurent polynomial

$$\mathcal{S}(A) = \sum_{y \in A \cap \mathbb{Z}^m} t^{\sigma(y)}.$$

**Proposition 6.8.** *Let  $T_\Pi$  be the operator associated with  $\Pi$  as in §6.1, let the function  $\sigma: \mathbb{R}^m \rightarrow \mathbb{R}$  be the sum of all the coordinates, and let  $A$  be an  $L$ -class of faces in  $\Pi$  with a non-empty initial parabox. Let  $B$  be the paramitosis of  $A$ . Then  $\mathcal{S}(B) = T_\Pi \mathcal{S}(A)$ .*

*Proof.* Consider the paradiagram of a face in  $A$ . Suppose that this paradiagram has  $r$  paraboxes in all, and that the  $\ell$ th parabox starts with index  $j_\ell$  (so that  $j_1 = 1$ ).

Consider the following linear map  $\Lambda_F: \mathbb{R}^m \rightarrow \mathbb{R}^r$ :

$$\Lambda_F(y_1, \dots, y_m) = \left( \sum_{j=j_1}^{j_2-1} y_j, \sum_{j=j_2}^{j_3-1} y_j, \dots, \sum_{j=j_r}^m y_j \right).$$

We have the equality  $\sigma \circ \Lambda_F = \sigma$ , where  $\sigma$  is the function computing the sum of all coordinates.

We can now apply Proposition 6.4 to the map  $\Lambda_F$ .  $\square$

A similar statement holds for unions of  $L$ -classes.

**Proposition 6.9.** *Let  $A_1, \dots, A_k$  be  $L$ -classes with non-empty initial paraboloes, and suppose that the  $L$ -classes  $B_1, \dots, B_k$  are obtained from  $A_1, \dots, A_k$  by paramitosis. Then*

$$\mathcal{S}(B_1 \cup \dots \cup B_k) = T_{\Pi} \mathcal{S}(A_1 \cup \dots \cup A_k) = T_{\Pi} \mathcal{S}(B_1 \cup \dots \cup B_k).$$

*Proof.* We will use the inclusion-exclusion formula:

$$\mathcal{S}(A_1 \cup \dots \cup A_k) = \sum_{I \neq \emptyset} (-1)^{|I|-1} \mathcal{S}(A_I),$$

where the summation is over all non-empty subsets  $I \subset \{1, \dots, k\}$ , and  $A_I$  is the intersection of all the  $A_i$ ,  $i \in I$ . The same formula holds for  $B_I$ , and  $T_{\Pi}$  is linear, hence it suffices to show that  $\mathcal{S}(B_I) = T_{\Pi} \mathcal{S}(A_I)$ . But  $A_I$  is also an  $L$ -class with non-empty initial parabox. Thus, the first equality follows from Proposition 6.8.

The second equality follows from the first one, since  $T_{\Pi} \circ T_{\Pi} = T_{\Pi}$ .  $\square$

Let  $M(A)$  denote the paramitosis of  $A$ .

**Proposition 6.10.** *Suppose that  $A_1, \dots, A_k$  are  $L$ -classes with non-empty initial parabox, and  $B_1, \dots, B_r$  are  $L$ -classes with empty initial parabox. Suppose also that  $B_i = M((A_1 \cup \dots \cup A_k) \cap B_i)$  for all  $i \in \{1, \dots, r\}$ . Then*

$$\mathcal{S}M(A_1 \cup \dots \cup A_k \cup B_1 \cup \dots \cup B_r) = T_{\Pi} \mathcal{S}(A_1 \cup \dots \cup A_k \cup B_1 \cup \dots \cup B_r).$$

*Proof.* By the inclusion-exclusion formula, we have for the right-hand side (RHS):

$$\begin{aligned} \text{RHS} &= T_{\Pi} \mathcal{S}(A_1 \cup \dots \cup A_k \cup B_1 \cup \dots \cup B_r) = T_{\Pi} \mathcal{S}(A_1 \cup \dots \cup A_k) \\ &\quad + T_{\Pi} \mathcal{S}(B_1 \cup \dots \cup B_r) - T_{\Pi} \mathcal{S}((A_1 \cup \dots \cup A_k) \cap (B_1 \cup \dots \cup B_r)). \end{aligned}$$

Put  $A'_i = (A_1 \cup \dots \cup A_k) \cap B_i$  for every  $i \in \{1, \dots, r\}$ . Since  $B_i = M(A'_i)$ , we get that  $T_{\Pi} \mathcal{S}(B_1 \cup \dots \cup B_r) = T_{\Pi} \mathcal{S}(A'_1 \cup \dots \cup A'_r)$  by the second equality in Proposition 6.9. Hence,

$$T_{\Pi} \mathcal{S}(B_1 \cup \dots \cup B_r) = T_{\Pi} \mathcal{S}((A_1 \cup \dots \cup A_k) \cap (B_1 \cup \dots \cup B_r)),$$

and  $\text{RHS} = T_{\Pi} \mathcal{S}(A_1 \cup \dots \cup A_k)$ .

It remains to note that the left-hand side coincides with  $\mathcal{S}M(A_1 \cup \dots \cup A_k)$ , because  $M(B_1 \cup \dots \cup B_r)$  is empty. The desired statement now follows from the first equality in Proposition 6.9.  $\square$

*Remark 6.11.* We note that the condition  $B = M((A_1 \cup \dots \cup A_k) \cap B)$  in Proposition 6.10 is satisfied whenever  $B = M(A)$  for some  $L$ -class  $A \subset A_1 \cup \dots \cup A_k$ . Indeed, if  $B = M(A)$ , then  $A = H \cap B$  by the definition of paramitosis, where  $H$  is the hyperplane  $y_1 = \mu_1$ . Since  $H$  contains all the  $A_i$ , we always have the inclusion  $(A_1 \cup \dots \cup A_k) \cap B \subset H \cap B$ . On the other hand, the condition  $A \subset A_1 \cup \dots \cup A_k$  implies the opposite inclusion  $H \cap B \subset (A_1 \cup \dots \cup A_k) \cap B$ .

**6.3. Fibre diagrams, ladder moves, and the proof of Key Lemma 5.8.**

We now apply the general results for parallelepipeds to mitosis on faces of the Gelfand–Zetlin polytopes  $P_\lambda$ . Fix some  $i$ . We will consider mirror mitosis in the  $i$ th row (in what follows, mitosis will always mean mirror mitosis). Let  $q_i: \mathbb{R}^d \rightarrow \mathbb{R}^{d-(n-i)}$  denote the linear projection that forgets all entries in the  $i$ th row, that is, it forgets the values of all the coordinates  $\lambda_{i,j}$  with first index  $i$ . The fibres of  $P_\lambda$  are defined as the fibres of this projection restricted to the Gelfand–Zetlin polytope  $P_\lambda$ .

We fix the values of all coordinates  $\lambda_{i',j}$  with  $i' \neq i$ . This determines a fibre of  $P_\lambda$ . The fibre can be given in the coordinates  $y_j = \lambda_{i,j}$  by the following inequalities:

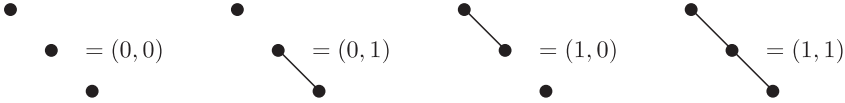
$$\begin{array}{ccccccc}
 \lambda_{i-1,1} & & \lambda_{i-1,2} & & \lambda_{i-1,3} & & \dots & & \lambda_{i-1,n-i+1} \\
 & y_1 & & y_2 & & \dots & & & y_{n-i} \\
 & & \lambda_{i+1,1} & & \dots & & \lambda_{i+1,n-i-1} & & 
 \end{array}$$

Let  $\mu'_j = \max(\lambda_{i-1,j}, \lambda_{i+1,j-1})$  and  $\nu'_j = \min(\lambda_{i-1,j+1}, \lambda_{i+1,j})$ , where  $\lambda_{i+1,0} = -\infty$  (or a sufficiently large negative number) and  $\lambda_{i+1,n-i} = +\infty$  (or a sufficiently large positive number). Therefore, the fibre can be identified with the coordinate parallelepiped  $\Pi(\mu', \nu') \subset \mathbb{R}^{n-i}$ .

Let  $F$  be any reduced Kogan face of  $P_\lambda$ . We define a fibre of the face  $F$  as the intersection of  $F$  with a fibre of  $P_\lambda$ . It will be convenient to represent a fibre of  $F$  by the  $i$ th fibre diagram of  $F$ , that is, by the restriction of the face diagram of  $F$  to the union of rows  $i - 1$ ,  $i$ , and  $i + 1$ . We note that the mitosis in the  $i$ th row can be seen on the level of the fibre diagram—it does not affect other parts of the face diagram. With the fibre diagram of each Kogan face we can associate the paradiagram of a face of the parallelepiped  $\Pi(\mu', \nu')$  as follows. A fibre of each Kogan face is a face of  $\Pi(\mu', \nu')$ , and we take the paradiagram of this face (note that the length of this paradiagram, which is equal to the dimension of  $\Pi(\mu', \nu')$ , may be strictly less than  $n - i$ ). It is easy to check that the paradiagram of a reduced Kogan face is also reduced, and that mirror mitosis on the level of fibre diagrams coincides with the paramitosis on the associated paradiagrams.

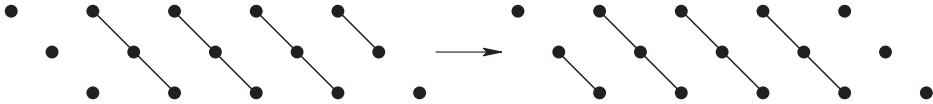
For convenience of the reader we now recall the definition of a ladder move from [2] in the language of reduced Kogan faces. Consider the rows  $i - 1$ ,  $i$ , and  $i + 1$  in the face diagram of  $F$ . We define a diagonal as a collection of 3 dots in rows  $i - 1$ ,  $i$ , and  $i + 1$  that are aligned in the direction from northwest to southeast, together with all the line segments joining pairs from these 3 dots and belonging to the face diagram. Diagonals can be of four possible types:  $(0, 0)$ ,  $(0, 1)$ ,  $(1, 0)$ , and  $(1, 1)$ . The first entry is 1 if the diagonal contains a segment connecting rows  $i - 1$  and  $i$ , otherwise the first entry is 0. The second entry is 1 if the diagonal contains

a segment connecting rows  $i$  and  $i + 1$ , otherwise the second entry is 0:



We can now describe the correspondence between fiber diagrams and paradiagrams in combinatorial terms as follows: diagonals of type  $(0, 0)$ ,  $(0, 1)$ ,  $(1, 0)$  are replaced by  $*$ ,  $1$ ,  $0$ , respectively, and diagonals of type  $(1, 1)$  are ignored (each such diagonal decreases by one the dimension of the parallelepiped  $\Pi(\mu', \nu')$ , that is, the length of the paradiagram). For instance, the first fiber diagram of the upper face in Figure 3 yields the paradiagram  $\boxed{0\ 0\ * \ 0}$ .

Define a *box* as any sequence of consecutive diagonals in a fibre diagram. In our pictures a box will look like a parallelogram with angles  $45^\circ$  and  $135^\circ$ . By definition, a *ladder-movable box* is a box whose first (left-most) diagonal is of type  $(0, 0)$ , followed by any number of diagonals of type  $(1, 1)$ , and, finally, by a single diagonal of type  $(1, 0)$ . Symbolically, we represent such a box as a sum  $(0, 0) + k(1, 1) + (1, 0)$ , where  $k$  is the number of type- $(1, 1)$  diagonals. The *ladder move* in the paper [2] makes this ladder-movable box into the box  $(0, 1) + k(1, 1) + (0, 0)$ :



Note that ladder moves do not change the permutation associated with a face. Moreover, they take reduced faces to reduced faces. Finally, note that, under the correspondence between fibre diagrams and paradiagrams, the ladder moves are exactly the  $L$ -moves of the previous subsection.

We are now ready to prove Key Lemma 5.8. Denote by  $\Gamma''$  the set  $\bigcup_{w(F)=w} F$  and by  $\Pi''$  the union of all the faces that can be obtained from faces in  $\Gamma''$  by mirror mitosis in the  $i$ th row. These are the sets considered in Key Lemma 5.8, and to prove the lemma we have to show that

$$T_i^-(\chi(\Gamma'')) = \chi(\Pi'').$$

Let  $\Gamma'$  and  $\Pi'$  denote fibres of  $\Gamma''$  and  $\Pi''$ , respectively, in the  $i$ th row, that is, the pre-images of a point  $z \in \mathbb{R}^{d-(n-i)}$  under restriction of the map  $q_i$  to  $\Gamma''$  and  $\Pi''$ , respectively.

Then Key Lemma 5.8 can be deduced from the next result.

**Lemma 6.12.** *Let  $T_\Pi$  be the operator associated with the coordinate parallelepiped  $\Pi(\mu', \nu')$  as in §6.1. If  $\Gamma'$  and  $\Pi'$  are identified with subsets of  $\Pi(\mu', \nu')$ , then*

$$\sum_{y \in \Pi' \cap \mathbb{Z}^{n-i}} t^{\sigma(y)} = T_\Pi \left( \sum_{y \in \Gamma' \cap \mathbb{Z}^{n-i}} t^{\sigma(y)} \right).$$

*Proof of Key Lemma 5.8 using Lemma 6.12.* Note that  $q_i(\Pi')$  is a single point  $z \in \mathbb{R}^{d-(n-i)}$  (that is, all the coordinates in all the rows except for row  $i$  are fixed).



For a point  $x \in \Pi'$ , denote by  $y = (y_1, \dots, y_{n-i})$  the coordinates of  $x$  in row  $i$ . Let  $\sigma_j(z) = \sigma_j(x)$  (for  $j \neq i$ ) be the sum of the coordinates in row  $j$ . By the definitions of  $T_i^-$  and  $T_\Pi$ , the following identity holds for all  $x \in \Pi'$  after substituting  $t = e^{\alpha_i}$ :

$$T_i^- e^{p(x)} = \prod_{j \neq i} e^{\sigma_j(z)\alpha_j} T_\Pi(t^{\sigma(y)}).$$

In Lemma 6.12 we replace  $t$  by  $e^{\alpha_i}$  and multiply both sides by the product

$$\prod_{j \neq i} e^{\sigma_j(z)\alpha_j}.$$

To obtain Key Lemma 5.8, it now suffices to take the sum over all fibres  $\Pi'$  of  $\Pi''$  in the  $i$ th row corresponding to the integer points  $z \in \mathbb{R}^{d-(n-i)}$ .  $\square$

*Proof of Lemma 6.12.* Lemma 6.12 will follow from Proposition 6.10 once we check that  $\Gamma'$  satisfies the conditions of the latter. We know that  $\Gamma'$  is closed under  $L$ -moves, since  $\Gamma''$  is closed under ladder moves. We can split  $\Gamma'$  into a union  $A_1 \cup \dots \cup A_k \cup B_1 \cup \dots \cup B_r$  of  $L$ -classes in which  $A_i$  and  $B_i$  have non-empty and empty initial parabox, respectively. By Remark 6.11 it suffices to show for each  $i \in \{1, \dots, r\}$  that  $B_i = M(A'_i)$  for some  $A'_i \subset (A_1 \cup \dots \cup A_r)$ . This follows from the next lemma.

**Lemma 6.13.** *Let  $F$  be a reduced Kogan face such that  $w(F) = w$  and the  $i$ th fibre diagram of  $F$  begins with  $\varkappa$  consecutive diagonals of type  $(1, 1)$  followed by the diagonal of type  $(0, 0)$ . If  $l(s_i w) = l(w) - 1$ , then there exists another reduced Kogan face  $F'$  such that  $w(F') = w$ , the  $i$ th fibre diagram of  $F'$  begins with  $\varkappa$  consecutive diagonals of type  $(1, 1)$  followed by the diagonal of type  $(1, 0)$ , and  $F \cap \Pi(\mu', \nu')$  is contained in  $M(F' \cap \Pi(\mu', \nu'))$ .*

*Proof.* We recall that the face diagram of  $F$  defines a reduced decomposition  $w = s_{i_1} \cdots s_{i_l}$  which by definition splits into two reduced words  $w_1$  and  $w_2$  as follows. The word  $w_1 = s_{i_1} \cdots s_{i_p}$  is obtained by composing elementary transpositions corresponding to points on a path from the bottom row to the  $i$ th row inclusive, and  $w_2 = s_{i_{p+1}} \cdots s_{i_l}$  is obtained by going from the  $(i - 1)$ th row to the top row. In particular, the word  $w_1$  contains only  $s_j$  with  $j \geq i$ .

If  $\varkappa = 0$ , then  $w_1$  contains only transpositions  $s_j$  with  $j > i$ . In particular,  $w_1(i) = i$  and  $(i + 1, w_1^{-1}(i + 1))$  is an inversion for  $w_1$  except for the case when  $w_1(i + 1) = i + 1$ . Hence, the assumption that  $l(s_i w) < l(w)$  (which is equivalent to  $w^{-1}(i) > w^{-1}(i + 1)$ ) implies that  $l(s_i w_2) < l(w_2)$ . Indeed,

$$w^{-1}(i) = w_2^{-1}(i) > w_2^{-1} w_1^{-1}(i + 1) \geq w_2^{-1}(i + 1)$$

(the last inequality holds because the word  $w = w_1 w_2$  is reduced). If  $\varkappa > 0$ , then  $w_1$  can be further decomposed as  $w'_1 s_{i+1} s_{i+2} \cdots s_{i+\varkappa} s_i s_{i+1} \cdots s_{i+\varkappa-1} w''_1$ , where  $w'_1$  contains only transpositions  $s_j$  with  $j > i$  and  $w''_1$  contains only  $s_j$  with  $j > i + \varkappa$ . By similar arguments we deduce that  $l(s_{i+\varkappa} w_2) < l(w_2)$  (use the identity  $s_i(s_{i+1} s_{i+2} \cdots s_{i+\varkappa} s_i s_{i+1} \cdots s_{i+\varkappa-1}) = (s_{i+1} s_{i+2} \cdots s_{i+\varkappa} s_i s_{i+1} \cdots s_{i+\varkappa-1}) s_{i+\varkappa}$ ).

By applying the exchange property to the word  $w_2 = s_{i_{p+1}} \cdots s_{i_l}$  we can replace it by a reduced word  $w'_2 = s_{i+\varkappa} s_{i_{p+1}} \cdots \hat{s}_{i_r} \cdots s_{i_l}$ . We now replace  $F$  by a reduced

face  $F'$  with the same permutation and with non-empty initial parabox as follows. In the face diagram of  $F$  we delete the edge corresponding to  $s_{i_r}$  and add the new edge  $\lambda_{i, \kappa+1} = \lambda_{i-1, \kappa+1}$ . The resulting face diagram defines the face  $F'$ . By construction,  $M(F' \cap \Pi(\mu', \nu'))$  contains  $F \cap \Pi(\mu', \nu')$ .  $\square$

Let us now return to the proof of Lemma 6.12. We apply Lemma 6.13 to a reduced Kogan face  $F \in \Gamma''$  whose paradiagram  $B$  lies in  $B_i$  and begins with the symbol  $*$ . We get a face  $F' \in \Gamma''$  whose paradiagram  $A$  lies in  $A_1 \cup \dots \cup A_r$  and is such that  $M(A) = B$ . Hence, the  $L$ -equivalence class  $A'_i$  of  $A$  also lies in  $A_1 \cup \dots \cup A_r$  and satisfies  $M(A'_i) = B_i$ , as required. Lemma 6.12 is proved.

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