PARTITIONS: FROM EULER TO RAMANUJAN

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Abstract. Notes from a course given at the Summer School “Modern Mathematics”, Dubna, Russia, from July 21 to 25, 2023.

How many ways are there to partition a natural number into a sum of several terms, if sums that differ only in the order of the terms are considered the same? It turns out that there is no simple answer to this seemingly elementary question. However, the theory that begins with this question is very interesting, and its results find applications in various areas of mathematics and mathematical physics.

1. First Lecture, July 21, 2023

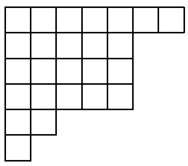
1.1. **Partitions and Young Diagrams.** A partition of a natural number is its representation as a sum of natural terms. The order of the terms does not matter: for example, $2+3$ and $3+2$ are the same partition of the number 5. Therefore, these terms can be considered non-strictly decreasing. Here is a formal definition:

Definition 1.1. *A partition* of a natural number n is a set of natural numbers $\lambda = (\lambda_1, \ldots, \lambda_k)$, for which $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ and $\lambda_1 + \cdots + \lambda_k = n$.

Sometimes it is convenient to consider a partition as a non-increasing sequence of non-negative integers $(\lambda_1, \ldots, \lambda_k, \ldots)$, where all λ_i starting from some index are zero. In other words, an infinite “tail” of zeros is appended to the finite set of λ_i.

A partition can be represented graphically using *Young diagrams*. A Young diagram of a partition $(\lambda_1, \ldots, \lambda_k)$ is a subset of the fourth quadrant of the plane consisting of unit squares. The squares are arranged in consecutive rows aligned to the left, with the number of squares in the i-th row equal to λ_i (so the length of each subsequent row does not exceed the length of the previous one).

Example 1.2. The diagram shows the Young diagram corresponding to the partition $(7, 5, 5, 5, 2, 1)$ of the number 25.



Let’s reflect the Young diagram λ across the diagonal (i.e., the line $x + y = 0$). We get a new Young diagram, which we will denote by $\lambda' = (\lambda'_1, \ldots, \lambda'_m)$ and call the *conjugate* of λ (sometimes it is also called *transposed*). It is clear that λ'_i is equal to the number of components of the original partition λ that are greater than or equal to i.

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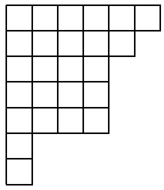
Example 1.3. The diagram $(6, 5, 4, 4, 4, 1, 1)$ is the conjugate of the diagram from the previous example.



We will denote the number of partitions of a number n by $p(n)$. We also agree that $p(0) = 1$.

It is easy to find $p(n)$ for small values of n (say, for $n \leq 5$). They are given in the following table:

n	0	1	2	3	4	5	6
$p(n)$	1	1	2	3	5	7	11

Exercise 1.4. Verify this and draw all Young diagrams corresponding to the partitions of the number n for each $n \leq 5$.

A natural question arises: is there a formula to find $p(n)$ for a given n? It turns out that there is no simple closed formula for the number of partitions (like, for example, for binomial coefficients). However, something can be said about this sequence: namely, its *generating function* can be written. We will deal with this now.

1.2. **Reminder about Generating Functions.** At the beginning of this section, we will briefly recall some information about generating functions and formal power series and discuss a few simple examples of their use. A more detailed explanation can be found in many textbooks on combinatorics, for example, in the books [**?**] and [**?**].

Let $a_0, a_1, \dots, a_n, \dots$ be an arbitrary numerical sequence. Consider a *formal power series* in the variable q:

$$a_0 + a_1 q + a_2 q^2 + \dots + a_n q^n + \dots. \qquad (*)$$

It is called the *generating function* for the original sequence.

Remark 1.5. We will work with generating functions exactly as *formal* power series — expressions of the form $(*)$, which can be thought of as "polynomials of infinite degree". Such expressions can be added and multiplied with each other. Note that these operations are correctly defined: of course, to add or multiply two series, an infinite number of operations must be performed, but the number of operations required to find each coefficient in the sum or product is finite. At this point, we will not be interested in questions of convergence of these series for certain numerical values of q.

Exercise 1.6. Let $A(q) = a_0 + a_1 q + a_2 q^2 + \dots$ be a formal power series, where $a_0 \neq 0$. Prove that there exists a series $B(q)$ inverse to $A(q)$ — i.e., such a series that $A(q) \cdot B(q) = 1$. What happens if $a_0 = 0$?

Sometimes the generating function expressed as a formal power series can be written in some other form (for example, as a rational function of q), which often allows us to obtain some new information about the sequence $(a_0, \dots, a_n, \dots)$.

1.3. **Partitions into Distinct Terms.** Let $p_D(n)$ be the number of partitions of n into pairwise distinct terms. For example, $p_D(8) = 6$: the corresponding partitions are (8), $(7, 1)$, $(6, 2)$, $(5, 3)$, $(5, 2, 1)$, and $(4, 3, 1)$.

Proposition 1.7. *The generating function $P_D(q) = \sum_{n\geq 0} p_D(n)q^n$ is represented as an infinite product*

$$P_D(q) = (1+q)(1+q^2)(1+q^3)\cdots = \prod_{k=1}^{\infty}(1+q^k).$$

Proof. Expand the brackets in the previous expression and do not combine like terms. Each term will then have the form $q^{k_1}q^{k_2}\dots q^{k_r}$, where $k_1 > k_2 > \dots > k_r$. However, to calculate the next (say, k-th) term of this series, we need to take only a finite number (in this case, k) of the first factors — the rest will not affect the coefficient at q^k in any way. □

Similarly, the famous *Euler's formula* for the generating function of the number of partitions is proven.

Theorem 1.8 (L. Euler)**.** *The generating function $P(q)$ for the number of partitions of a number n is given by the following infinite product:*

$$P(q) = \sum p(n)q^n = \frac{1}{(1-q)(1-q^2)(1-q^3)\dots} = \prod_{k=1}^{\infty}(1-q^k)^{-1}.$$

Proof. The expression for $P(q)$ can be rewritten as

$$P(q) = (1+q+q^2+\dots)(1+q^2+q^4+\dots)(1+q^3+q^6+\dots)\dots$$

Similarly to the previous case, if we expand the brackets in the right-hand side and do not combine like terms, each term will have the form $q^{k_1 m_1}q^{k_2 m_2}\dots q^{k_r m_r}$, where $k_1 > k_2 > \dots > k_r$. □

1.4. **Partition into odd and distinct summands.** Here is another problem that can be solved by means of generating functions. Let us look at the partitions of a number into *odd* parts. The number of ways to do this will be denoted by $p_O(n)$.

For example, $p_O(8) = 6$, since there are 6 such partitions for the number 8:

$$7+1 = 5+3 = 5+1+1+1 = 3+3+1+1 = 3+1+1+1+1+1 = 1+1+1+1+1+1+1+1.$$

We have seen that the number of partitions of the number 8 into distinct terms will be the same. It turns out that this is true for any n.

Proposition 1.9. $p_D(n) = p_O(n)$ *for any n.*

Proof. We will prove that the generating functions $P_O(q) = \sum p_O(n)q^n$ and $P_D(q) = \sum p_D(n)q^n$ are equal.

Reasoning exactly as we did for arbitrary partitions, we find that the generating function $P_O(q) = \sum p_O(n)q^n$ equals

$$P_O(q) = \frac{1}{1-q}\cdot\frac{1}{1-q^3}\cdot\frac{1}{1-q^5}\cdot\dots.$$

As we have just shown,

$$P_D(q) = (1+q)(1+q^2)(1+q^3)\dots$$

The rest involves purely algebraic transformations. Multiply and divide the series $P_O(q)$ by the infinite product $(1-q^2)(1-q^4)\dots$:

$$P_O(q) = \frac{1}{1-q}\cdot\frac{1}{1-q^3}\cdot\frac{1}{1-q^5}\cdots = \frac{1}{1-q}\cdot\frac{1-q^2}{1-q^2}\frac{1}{1-q^3}\cdot\frac{1-q^4}{1-q^4}\cdots =$$
$$= \frac{(1-q^2)(1-q^4)(1-q^6)\dots}{(1-q)(1-q^2)(1-q^3)\dots} = (1+q)(1+q^2)(1+q^3)\cdots = P_D(q).$$

□

Another question can be posed: given that we already know there are as many partitions into odd parts as into distinct parts, can we construct a "natural" bijective mapping (bijection) between the sets of such partitions? In other words, how can we associate each partition of a number n into odd parts with its corresponding partition into distinct parts?

Several such bijections can be constructed. Let's describe one using an example.

Suppose we are given a partition of a number into odd parts, such as:

$$23 = 7+5+5+3+1+1+1$$

Consider a "centered Young diagram": draw a diagram symmetric about the vertical axis (see the figure on the left), with 7 boxes in the first row, 5 in the second and third, and so on.

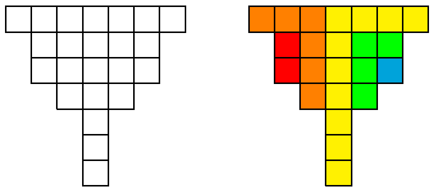


FIGURE 1.1. Partitioning Young diagrams into hooks

Now, partition this diagram into hooks as shown in the figure on the right. We obtain a partition of the number 23 into *distinct* parts:

$$23 = 10+6+4+2+1.$$

Exercise 1.10. Verify that this correspondence is indeed a bijection (it is called *Sylvester's bijection*).

1.5. **Pentagonal Numbers.** Consider the infinite product inverse to $P(q)$. This product of an infinite number of binomials is:

$$P(q)^{-1} = \prod_{k=1}^{\infty}(1-q^k).$$

Exercise 1.11. Calculate the first 8 terms (up to q^7 inclusive) of this infinite product.

Euler calculated the first few dozen terms of this product, obtaining:

$$P(q)^{-1} = 1-q-q^2+q^5+q^7-q^{12}-q^{15}+q^{22}+q^{26}-q^{35}-q^{40}+\dots$$

Here, several interesting observations can be made. First, it is clear that all coefficients of this series are either 0 or ±1, with non-zero terms becoming rarer as the degree increases. Second, all terms except the constant term come in pairs: two negative, two positive, then two negative again, and so on. The difference between the degrees in each pair equals the pair number: initially one (q and q^2), then two (q^5 and q^7), then three (q^{12} and q^{15}), and so forth.

Finally, the sequence of degrees 1, 5, 12, 22, 35, etc., was well-known to Euler these are the so-called *pentagonal numbers*, which equal the number of points in a pentagon with sides of length 1, 2, 3, etc., respectively (see Figure 1.2).

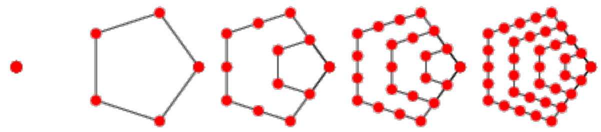


FIGURE 1.2. Pentagonal numbers

It is easy to see that the m-th pentagonal number is $m(3m-1)/2$.

It turns out that the following result holds. Its proof will be presented in the next lecture.

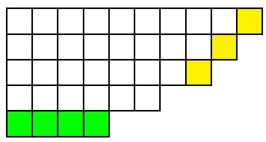
Theorem 1.12 (Euler's Pentagonal Theorem)**.** *The series inverse to* $P(q) = \sum p(n)q^n$ *has the form*

$$P(q)^{-1} = 1 + \sum_{m=1}^{\infty} (-1)^m \left(q^{\frac{m(3m-1)}{2}} + q^{\frac{m(3m+1)}{2}} \right).$$

We will prove this theorem in two ways. The first will be combinatorial; it belongs to Sylvester's student Franklin.

Proof. The expression $(1-q)(1-q^2)\dots$ can be viewed as the generating function for partitions into distinct parts, where each partition is counted with a weight of 1 for partitions with an even number of parts and -1 for those with an odd number of parts. The statement of the theorem then asserts that there are almost as many of these partitions: their number differs by one if the weight of the partition is of the form $\frac{n(3n\pm1)}{2}$, and is equal otherwise.

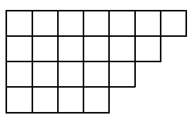
Let's try to construct a bijection between such partitions with an even and odd number of parts. Consider a Young diagram with rows of varying lengths and introduce three characteristics: ℓ the number of rows, b the length of the bottom row (marked green), and d the length of the "diagonal," i.e., the longest sequence of cells that can be reached from the rightmost cell of the first row by moving like a knight (mark these cells yellow).

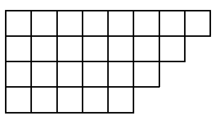


Now, if dd. Note that for the resulting diagram, $b' \leq d'$.

Similarly, we can construct a mapping that sends a diagram with $b \leq d$ to one with $b' > d'$: simply cut off the bottom row and attach it as a diagonal. Thus, the resulting mapping will be an involution.

It will be defined for "almost all" diagrams. The diagrams for which it is not defined correspond to cases where $\ell = d = b$ and $\ell = d = b-1$ (when $\ell = 4$, they are shown in the figure below).





Clearly, these diagrams consist of $\ell^2 + \frac{\ell(\ell-1)}{2} = \frac{\ell(3\ell-1)}{2}$ and $\ell^2 + \frac{\ell(\ell+1)}{2} = \frac{\ell(3\ell+1)}{2}$ cells, respectively, and enter with a weight equal to $(-1)^\ell$. This proves Euler's Pentagonal Theorem. $\square$

The second method will be discussed in the next session. For it, we will need Jacobi's identity for the triple product.

2. Second Lecture, July 22, 2023

2.1. **Counting the Number of Partitions Using Euler's Pentagonal Theorem.** As a consequence of Euler's Pentagonal Theorem, we will show how to derive a recursive formula for the number of partitions.

By multiplying the series $P(q)$ and $P(q)^{-1}$, we obtain a unit. Therefore,

$$\left(\sum p(k)q^k\right)\left(1+\sum_{m=1}^{\infty}(-1)^m\left(q^{\frac{m(3m-1)}{2}}+q^{\frac{m(3m+1)}{2}}\right)\right)=1.$$

On one hand, the coefficient of q^n for $n>0$ in the product $P(q)P(q)^{-1}$ is zero.

On the other hand, if $\sum a_n q^n$ and $\sum b_m q^m$ are two power series, the coefficient of q^k in their product is given by $a_k b_0+a_{k-1}b_1+\cdots+a_0 b_k=\sum a_{k-i}b_i$. Let's express the coefficient of q^n in the left-hand side and thereby find the relation for the number of partitions $p(n)$:

$$p(n)+\sum_{m=1}^{\infty}(-1)^m\left(p(n-m(3m-1)/2)+p(n-m(3m+1)/2)\right)=0$$

(assuming $p(k)=0$ for $k<0$). Moving all terms except the first to the right-hand side, we get:

$$p(n)=\sum_{m=1}^{\infty}(-1)^{m+1}\left(p(n-m(3m-1)/2)+p(n-m(3m+1)/2)\right).$$

This is a recursive relation whose depth constantly increases. Let's write it for $6\le n\le 12$ and use it to find the corresponding $p(n)$.

$$\begin{aligned}
p(6) &= p(5)+p(4)-p(1)=7+5-1=11;\\
p(7) &= p(6)+p(5)-p(2)-p(0)=11+7-2-1=15;\\
p(8) &= p(7)+p(6)-p(3)-p(1)=15+11-3-1=22;\\
p(9) &= p(8)+p(7)-p(4)-p(2)=22+15-5-2=30;\\
p(10) &= p(9)+p(8)-p(5)-p(3)=30+22-7-3=42;\\
p(11) &= p(10)+p(9)-p(6)-p(4)=42+30-11-5=56;\\
p(12) &= p(11)+p(10)-p(7)-p(5)+p(0)=56+42-15-7+1=77.
\end{aligned}$$

2.2. **Jacobi's Identity for the Triple Product.**

Theorem 2.1 (Jacobi's Identity for the Triple Product)**.**

$$\prod_{k=1}^{\infty}(1+xq^k)(1+x^{-1}q^{k-1})(1-q^k)=\sum_{j=-\infty}^{+\infty}q^{\frac{j(j+1)}{2}}x^j.$$

Proof. Consider the infinite product

$$f(x)=\prod_{k=1}^{\infty}(1+xq^k)(1+x^{-1}q^{k-1}).$$

It can be viewed as a Laurent series in x:

$$f(x)=\sum_{n=-\infty}^{\infty}a_n(q)x^n,$$

where the coefficients $a_n(q)$ are formal power series in q.

It is easy to see that $f(xq) = x^{-1}q^{-1}f(x)$ (this is a routine check).

From this, it follows that $a_n(q)q^{n+1} = a_{n+1}(q)$ for any $n \in \mathbb{Z}$. Thus, all terms $a_n(q)$ can be expressed knowing $a_0(q)$: indeed,

$$a_1(q) = qa_0(q); \quad a_2(q) = q^2 a_1(q) = q^3 a_0(q), \ldots, \quad a_n(q) = q^n a_{n-1}(q) = \cdots = q^{1+2+\cdots+n} a_0(q)$$

for positive n; for negative n, the same equality is checked similarly. So, we have the equality

$$f(x) = a_0(q) \sum_{n \in \mathbb{Z}} q^{n(n+1)/2} x^n.$$

Now, let's compute $a_0(q)$. This coefficient is the constant term in the expression

$$(1 + xq)(1 + xq^2)(1 + xq^3) \ldots (1 + x^{-1})(1 + x^{-1}q)(1 + x^{-1}q^2) \ldots .$$

Let $a_0(q)$ be

$$a_0(q) = \sum_{n \geq 0} b_n q^n.$$

Similarly to the reasoning from the previous session, the coefficient b_n of this series is the number of ways to represent the number n as a sum of several distinct elements from the set $\{1, 2, 3, \ldots\}$ and the same number of distinct elements from the set $\{0, 1, 2, \ldots\}$.

It remains to note that b_n is nothing but the number of partitions $p(n)$. Indeed, such a representation of the number n can be associated with a Young diagram, where the "arms" (fragments of rows from the diagonal to the right edge of the diagram, including the diagonal) are elements from $\{1, 2, 3, \ldots\}$, and the "legs" (fragments of columns from the diagonal to the bottom edge, excluding the cell on the diagonal) are elements from $\{0, 1, 2, \ldots\}$. Therefore,

$$a_0(q) = \prod_{k=1}^{\infty} (1 - q^k)^{-1} = P(q),$$

and consequently,

$$\prod_{k=1}^{\infty} (1 + xq^k)(1 + x^{-1}q^{k-1}) = \prod_{k=1}^{\infty} (1 - q^k)^{-1} \sum_{n=-\infty}^{\infty} x^n q^{\frac{n(n+1)}{2}}.$$

Dividing both sides of the equality by $P(q)$ gives Jacobi's identity for the triple product. □

2.3. **Second Proof of Euler's Pentagonal Theorem.** In this section, we will derive Euler's Pentagonal Theorem from Jacobi's identity for the triple product. To do this, we will substitute q^3 for q in Jacobi's identity:

$$\prod_{k=1}^{\infty} (1 + xq^{3k})(1 + x^{-1}q^{3k-3})(1 - q^{3k}) = \sum_{j=-\infty}^{+\infty} q^{\frac{3j^2+3j}{2}} x^j.$$

Now, let's set $x = -q^{-1}$. We get:

$$\prod_{k=1}^{\infty} (1 - q^{3k-1})(1 - q^{3k-2})(1 - q^{3k}) = \sum_{j=-\infty}^{+\infty} (-1)^j q^{\frac{3j^2+j}{2}}.$$

And this is Euler's Pentagonal Theorem:

$$\prod_{k=1}^{\infty} (1 - q^k) = 1 + \sum_{j=1}^{\infty} (-1)^j \left(q^{\frac{j(3j-1)}{2}} + q^{\frac{j(3j+1)}{2}} \right).$$

2.4. **Jacobi's Identity.** The following equality is another consequence of the identity for the triple product. It also belongs to Jacobi.

Theorem 2.2. *The following equality holds:*

$$\prod_{k=1}^{\infty}(1-q^k)^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{n(n+1)}{2}}$$

Proof. There is a temptation to substitute $x=-1$ into Jacobi's identity. In the left-hand side, we would get something very similar to the product $\prod(1-q^k)$, but not quite: there would still be an additional factor $(1-q^0)$, which equals zero.

So, we will proceed slightly differently. Divide both sides of Jacobi's identity by $1+x^{-1}$. The left-hand side becomes

$$\frac{1}{1+x^{-1}}\prod_{k=1}^{\infty}(1+xq^k)(1+x^{-1}q^{k-1})(1-q^k) = \prod_{k=1}^{\infty}(1+xq^k)(1+x^{-1}q^k)(1-q^k).$$

The right-hand side will be equal to

$$\frac{1}{1+x^{-1}}\sum_{n=-\infty}^{\infty} q^{\frac{n(n+1)}{2}}x^n = \frac{1}{1+x^{-1}}\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}(x^n+x^{-n-1}) =$$
$$= \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}\frac{x^n+x^{-n-1}}{1+x^{-1}} = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}(x^n-x^{n-1}+\dots+x^{-n}).$$

Now, we can substitute $x=-1$. Since the last sum equals $(-1)^n(2n+1)$, we obtain the desired equality. □

3. Third Lecture, July 24, 2023

3.1. **Ramanujan's Congruence Modulo 5.** Using Euler's and Jacobi's theorems, we will show that $p(5k+4)\equiv 0 \mod 5$ for any k. This was first observed by Ramanujan.

Theorem 3.1. *For any k, the congruence $p(5k+4)\equiv 0 \mod 5$ holds.*

Proof. Let's introduce the following notation:

$$E(q) = \prod_{k\geq 1}(1-q^k) = \sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{n(3n-1)}{2}}$$

and

$$J(q) = \prod_{k\geq 1}(1-q^k)^3 = \sum_{n\geq 0}(-1)^n(2n+1)q^{\frac{n(n+1)}{2}}.$$

Note that $\frac{n(3n\pm1)}{2}$ gives remainders of 0, 1, or 2 modulo 5. Therefore,

$$E(q) = E_0(q)+E_1(q)+E_2(q),$$

where $E_k(q)$ is the sum of all monomials a_nq^n in $E(q)$ for which $n\equiv k \mod 5$.

Similarly, $\frac{n(n+1)}{2}$ gives remainders of 0, 1, and 2 modulo 5, and thus $J(q)=J_0+J_1+J_2$. However, when $\frac{n(n+1)}{2}\equiv 2 \mod 5$, we have $2n+1\equiv 0 \mod 5$. Therefore, if we reduce the coefficients of $J(q)$ modulo 5, the term J_2 becomes zero.

So, let's express $P(q)$ as follows:

$$P(q) = \frac{1}{E(q)} = \frac{E(q)J(q)}{E^5(q)}.$$

However, by the "lazy binomial theorem," $(1-q)^5 = 1-q^5$ and similarly $(1-q^k)^5 = 1-q^{5k}$. Therefore, $E^5(q) = E(q^5)$. This means,

$$P(q) = \frac{(E_0 + E_1 + E_2)(J_0 + J_1)}{E(q^5)} = \frac{E_0J_0 + E_0J_1 + E_1J_0 + E_1J_1 + E_2J_0 + E_2J_1}{E(q^5)}.$$

Thus, the coefficients of q with degrees congruent to 4 modulo 5 are zero. □

Alternatively, we can start with $P(q) = E^9(q)/E^{10}(q) = J^3(q)/E^2(q^5)$ and note that $J^3 = J_0^3 + 3J_0^2J_2 + 3J_0J_1^2 + J_1^3$ also contains only terms whose degrees are congruent to 0, 1, 2, and 3 modulo 5.

Problem 3.1. Prove Ramanujan's second congruence: $p(7k + 5) \equiv 0 \mod 7$.

Problem 3.2 (*)**.** Prove Ramanujan's third congruence: $p(11k + 6) \equiv 0 \mod 11$.

The second problem requires significantly more computations, but both can be solved using the same methods as for 5.

3.2. **Asymptotic Behavior of** $p(n)$**.** The asymptotic behavior of the function $p(n)$ is described by the following theorem, proven in 1918 by G. Hardy and S. Ramanujan, and independently in 1920 by the Russian-American mathematician Ya. V. Uspensky:

Theorem 3.2. *There is an asymptotic equality*

$$p(n) \sim \frac{1}{4\pi\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}} \qquad \text{as} \quad n \to \infty.$$

This is a complex result requiring a subtle application of complex analysis methods. Its "elementary" (i.e., not requiring complex-analytic methods) proof was proposed in 1942 by P. Erds; it is also quite involved. You can read it, for example, in the book by Melvin B. Nathanson, *Elementary methods in number theory* (Springer, 2000).

We will prove a significantly weaker version of this theorem. Our proof follows the review by Igor Pak: Igor Pak. *Partition bijections: a survey* (Section 9.6).

Theorem 3.3. *There exist positive numbers* $0 < a < c$ *such that*

$$e^{a\sqrt{n}} < p(n) < e^{c\sqrt{n}}.$$

Moreover, we can take $c = \pi\sqrt{\frac{2}{3}}$.

Proof. The lower bound is easy: we can take rough estimate for $p(n)$ by a binomial coefficient. Let $p_k(n)$ be the number of partitions of n into at most k parts. We have the following inequality:

$$k!p_k(n) > \binom{n+k-1}{k-1}.$$

Indeed, the right-hand side is the number of partitions of n into k *ordered* parts, which are allowed to be zero (in combinatorics this formula is often referred to as "balls and urns"). The left-hand side is all possible orderings of k parts of a partition of n.

Replace the binomial with the leading term of the corresponding polynomial in n, and $p_k(n)$ with $p(n)$:

$$k!p(n) > \frac{n^{k-1}}{(k-1)!}.$$

Take $k = \lfloor\sqrt{n}\rfloor$. Show that $p(n) > n^{k-1}/(k!)^2$. To do this, use Stirling's formula:

$$k! \simeq \sqrt{2\pi k}\frac{k^k}{e^k}.$$

We obtain that the inequality we need to prove is

$$p(n) > \frac{k^{2k-2}e^{2k}}{2\pi k^{2k+1}} = \frac{e^{2\sqrt{n}}}{n^{3/2}},$$

from which the required lower bound follows.

For the upper bound, start with the following equalities:

$$np(n) = \sum_{r=1}^{n} r \sum_{\lambda \vdash n} m_r(\lambda) = \sum_{r=1}^{n} r \sum_{m=1}^{\lfloor n/r \rfloor} p(n-mr).$$

Here, $m_r(\lambda)$ denotes the number of rows of length r in the partition λ.

The first equality is proven by counting the total number of cells in all diagrams corresponding to partitions of n in two ways. These cells are $np(n)$; on the other hand, we can sum the lengths of rows of a given length over all diagrams and take the sum over row lengths. The second equality is proven as follows:

$$\begin{aligned}\sum_{\lambda \vdash n} m_r(\lambda) = &\left|\{\lambda \vdash n : m_r(\lambda) = 1\}\right| + 2\left|\{\lambda \vdash n : m_r(\lambda) = 2\}\right| \\ &+ 3\left|\{\lambda \vdash n : m_r(\lambda) = 3\}\right| + \dots \\ = &\left|\{\lambda \vdash n : m_r(\lambda) \geq 1\}\right| + \left|\{\lambda \vdash n : m_r(\lambda) \geq 2\}\right| \\ &+ \left|\{\lambda \vdash n : m_r(\lambda) \geq 3\}\right| + \dots \\ =& p(n-r) + p(n-2r) + p(n-3r)\dots\end{aligned}$$

Having obtained this recursive relation, use induction on n. Assume that $p(k) < e^{c\sqrt{k}}$ for all $k < n$, where $c = \pi\sqrt{\frac{2}{3}}$. Use this relation to prove the inductive step:

$$np(n) < \sum_{r=1}^{n} \sum_{m=1}^{\lfloor n/r \rfloor} r e^{c\sqrt{n-mr}} < e^{c\sqrt{n}} \sum_{r=1}^{\infty} \sum_{m=1}^{\infty} r e^{(-cm/2\sqrt{n})r}$$

Here, we used the estimate for the square root:

$$c\sqrt{n-mr} = c\sqrt{n}\sqrt{1-\frac{mr}{n}} < c\sqrt{n}\left(1-\frac{mr}{2n}\right) = c\sqrt{n} - \frac{cmr}{2\sqrt{n}}.$$

Note that $\sum_1^\infty rt^r = t/(1-t)^2$ and $e^{-x}/\left(1-e^{-x}\right)^2 < \frac{1}{x^2}$ for all $x \in \mathbb{R}$. From this, we get:

$$p(n) < \frac{e^{c\sqrt{n}}}{n} \sum_{m=1}^{\infty} \frac{e^{-cm/2\sqrt{n}}}{\left(1-e^{-cm/2\sqrt{n}}\right)^2} < \frac{e^{c\sqrt{n}}}{n} \sum_{m=1}^{\infty} \frac{4n}{c^2 m^2} = e^{c\sqrt{n}} \frac{4}{c^2}\left(\frac{\pi^2}{6}\right) = e^{c\sqrt{n}}.$$

Here, we used the known equality for $\zeta(2)$, namely: $\sum_{m=1}^\infty \frac{1}{m^2} = \frac{\pi^2}{6}$. The upper bound is proven. □

4. Fourth Lecture, July 25, 2023

4.1. **Rogers–Ramanujan Identities.** The goal of this lecture is to prove the following two identities. They are called the *Rogers–Ramanujan identities.* They were first proved by Rogers in the 19th century, later rediscovered by Ramanujan, and in 1919 they published a joint paper with Rogers. Independently, these identities were proved by Issai Schur in 1917; the combinatorial proof presented here is due to him. Essentially, it is a modification of Franklin's bijection (see the first lecture).

We will follow the aforementioned survey by Igor Pak. Rogers' proof of the Rogers–Ramanujan identities can be found, for example, in the book: David M. Bressoud, *Proofs and confirmations* (AMS, 2000; a Russian translation will soon be published by MCCME).

Theorem 4.1 (Rogers–Ramanujan, Schur)**.** *The following identities hold:*

$$1+\sum_{k=1}^{\infty}\frac{q^{k^2}}{(1-q)\,(1-q^2)\cdots(1-q^k)}=\prod_{i=0}^{\infty}\frac{1}{(1-q^{5i+1})\,(1-q^{5i+4})},\qquad (*)$$

$$1+\sum_{k=1}^{\infty}\frac{q^{k(k+1)}}{(1-q)\,(1-q^2)\cdots(1-q^k)}=\prod_{i=0}^{\infty}\frac{1}{(1-q^{5i+2})\,(1-q^{5i+3})}.\qquad (**)$$

These two identities are very similar, so we will prove the first one, leaving the second as an exercise for the reader. But first, let us clarify what they mean.

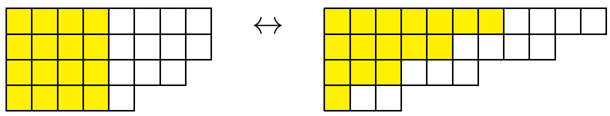
The right-hand side of identity $(*)$ is the generating function for the number of partitions into parts congruent to ± 1 modulo 5. Denote the set of such partitions of n by $\mathcal{A}_n$.

Next, let us introduce two more sets of partitions. Let $\mathcal{B}_n$ denote the set of partitions of n into parts where any two parts differ by at least 2 (we will say that such a diagram has *significantly distinct* rows). Finally, let $\mathcal{C}_n$ be the set of partitions λ of n for which the last row $b(\lambda)$ is at least as large as the number of rows $\ell(\lambda)$.

Clearly, the generating function for the cardinalities of the sets $\mathcal{C}_n$ is the left-hand side of equality $(*)$.

Lemma 4.2. *The number of partitions in the sets $\mathcal{B}_n$ and $\mathcal{C}_n$ is equal.*

Proof. We will construct an explicit bijection between these sets. It is illustrated in the figure below.



□

Lemma 4.3. *The following equality holds:*

$$\prod_{r=0}^{\infty}\frac{1}{(1-q^{5r+1})\,(1-q^{5r+4})}=\sum_{m=-\infty}^{\infty}(-1)^m q^{\frac{m(5m-1)}{2}}\prod_{i=1}^{\infty}\frac{1}{(1-q^i)}.$$

Proof. This is a consequence of Jacobi's triple product identity. Indeed, make the substitution $q\mapsto q^5$ and set $x=-q^{-3}$. We obtain:

$$\prod_{r=1}^{\infty}(1-q^{5r-3})(1-q^{5r-2})(1-q^{5r})=\sum_{m=-\infty}^{\infty}(-q^2)^m q^{\frac{5m(m+1)}{2}}.$$

Multiplying both sides by the Eulerian generating function $\prod(1-q^k)^{-1}$, we obtain the desired result. □

Thus, it turns out that identity $(*)$ is equivalent to the following equality:

$$\left(\prod_{i=1}^{\infty}(1-q^i)\right)\left(1+\sum_{k=1}^{\infty}\frac{q^{k^2}}{(1-q)\,(1-q^2)\cdots(1-q^k)}\right)=\sum_{m=-\infty}^{\infty}(-1)^m q^{\frac{m(5m-1)}{2}}.\qquad (***)$$

And this we will now prove combinatorially.

4.2. **Schur's Bijection.** First, let us introduce some notation. Let $\mathcal{D}_n$ be the set of partitions of n into distinct parts, and let $\mathcal{D} = \cup_{n=1}^{\infty}\mathcal{D}_n$. Next, let $\mathcal{B} = \cup_{n=1}^{\infty}\mathcal{B}_n$. Finally, let $\mathcal{R} = \mathcal{D}\times\mathcal{B}$ be the set of pairs consisting of a Young diagram with distinct rows λ and a diagram with significantly distinct rows μ, and let $\mathcal{R}_n = \{(\lambda,\mu) \mid |\lambda| + |\mu| = n\}$ be the set of such pairs of diagrams with total weight n.

The *sign* of a pair (λ,μ) is defined as $(-1)^{\ell(\lambda)}$, i.e., the parity of the number of rows in the diagram λ.

Our task is to construct a bijection α on the set $\mathcal{R}_n$ that changes the sign for all pairs that are not fixed points. First, we define the set of fixed points of the bijection as follows: these will be pairs of diagrams (λ,μ), where $\lambda = (2m-1, 2m-2,\dots,m)$ and $\mu = (2m-1, 2m-3,\dots,3,1)$, as well as $\lambda = (2m, 2m-1,\dots,m+1)$ and $\mu = (2m-1, 2m-3,\dots,3,1)$ (in the figure below, these pairs are shown for $m=4$). Note that λ in this bijection is precisely the fixed points of Franklin's bijection.

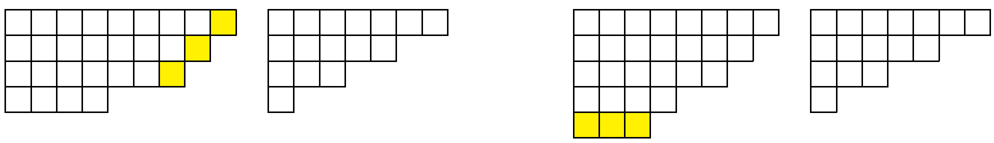
Let us introduce the following notation. Denote by $a(\lambda)$, $\ell(\lambda)$, $b(\lambda)$, and $d(\lambda)$ the number of columns of the diagram λ, its number of rows, the length of the bottom row, and the length of the diagonal (starting from the rightmost cell and moving down-left like a bishop), respectively. Further, let $u(\mu)$ denote the "skew diagonal" of the diagram μ, also starting from the rightmost cell of the first row and moving down-left, but like a knight.

Now we will construct the bijection for the remaining pairs of diagrams (λ,μ). For this, we consider several cases.

First, suppose $a(\lambda) > a(\mu)+2$. Then we take the first row λ_1 of the diagram λ, cut it off, and attach it to μ at the top. Clearly, this operation does not change the total area of the diagrams, but the parity of the number of rows in λ changes, since the number of rows decreases by 1. Conversely, if $a(\lambda) < a(\mu)$, we cut off the first row of the diagram μ and attach it to λ.

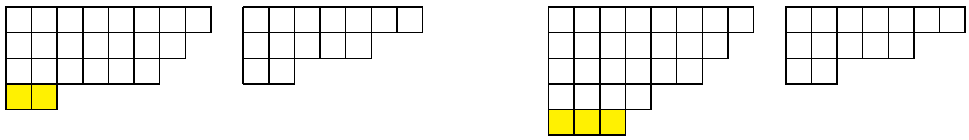
The remaining cases are when $a(\lambda) = a(\mu)$ and $a(\lambda) = a(\mu)+1$. Denote these by $\mathcal{R}_n^0$ and $\mathcal{R}_n^1$, respectively, and we will construct a bijection between these sets (excluding the fixed points defined above). We will construct a mapping from $\mathcal{R}_n^1$ to $\mathcal{R}_n^0$. So, let $(\lambda,\mu) \in \mathcal{R}_n^1$. Our mapping will equalize the lengths of the first rows of these diagrams and change the number of rows in λ by one. Consider three numbers: $b(\lambda)$, $d(\lambda)$, and $u(\mu)$. Take the smallest of them; these three possibilities will determine three cases.

Case 1. Suppose $d(\lambda) < b(\lambda)$ and $d(\lambda) \le u(\mu)$. Then we apply Franklin's bijection to λ: cut off the diagonal of λ and reattach it as the bottom row. The number of rows in λ will increase by one, and the lengths of the first rows of λ and μ will become equal.

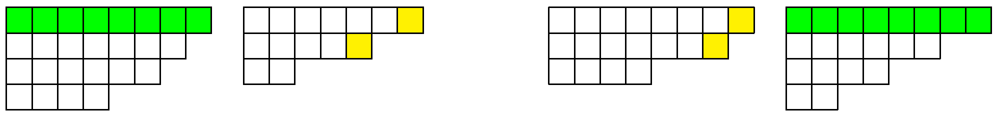


Case 2. Suppose $b(\lambda) \le d(\lambda)$ and $b(\lambda) \le u(\mu)$. Then the bottom row of λ can be cut off and reattached as a skew diagonal in μ. The number of rows in λ will decrease

by one, and the lengths of the first rows of λ and μ will become equal.



Case 3. Suppose $u(\mu) < d(\lambda)$ and $u(\mu) \le b(\lambda)$. Then we take the first row from λ and the skew diagonal from μ. Then simultaneously reattach the first row from λ as the first row in μ, and the skew diagonal from μ as the diagonal to λ (this will be possible because after cutting off the first row, $d(\lambda)$ will decrease by one, and the inequality $u(\mu) < d(\lambda)$ is strict). The resulting pairs of diagrams will have the following characteristics: $a(\lambda') = a(\mu') = a(\lambda)$, $d(\lambda') = u(\mu)$, $u(\mu') > u(\mu)$.



This mapping is easy to invert, so it will be a bijection between $\mathcal{R}_n^0$ and $\mathcal{R}_n^1$. Thus, we have constructed an involution on $\mathcal{R}$, whose fixed points are pairs of diagrams with total weight $\frac{m(5m-1)}{2}$ and $\frac{m(5m+1)}{2}$, taken with weight $(-1)^m$. This proves equality $(***)$, which, as we have seen, is equivalent to the first Rogers–Ramanujan identity $(*)$.

Problem 4.1. Following a similar approach, prove the second Rogers–Ramanujan identity $(**)$.

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