European Journal of Combinatorics 107 (2023) 103613

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Pipe dreams for Schubert polynomials of the classical groups

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ARTICLE INFO

Article history:
Received 2 November 2020
Accepted 17 August 2022
Available online xxxx

ABSTRACT

Schubert polynomials for the classical groups were defined by S. Billey and M. Haiman in 1995; they are polynomial representatives of Schubert classes in a full flag variety over a classical group. We provide a combinatorial description for these polynomials, as well as their double versions, by introducing analogues of pipe dreams, or RC-graphs, for Weyl groups of classical types.

1. Introduction

1.1. Flag varieties and Schubert polynomials

Let G be a reductive group over $\mathbb{C}$. Let us fix a Borel subgroup B in G and the corresponding maximal torus $T \subset B$. A classical result of Borel [1] states that the cohomology ring $H^*(G/B, \mathbb{Z})$ is isomorphic as a graded ring to the *coinvariant* ring of the Weyl group W of G, i.e. to the quotient of the polynomial ring in $\dim T$ variables modulo the ideal generated by W-invariants of positive degree.

The cohomology ring of G/B has a nice additive basis formed by *Schubert classes* σ_w: the classes of Schubert varieties, i.e. of the closures of B-orbits in G/B. It is indexed by the elements of the Weyl group $w \in W$. A natural question is to construct a system of polynomial representatives of Schubert classes.

This question was answered in the case $G = \mathrm{GL}_n$ by I. N. Bernstein, I. M. Gelfand and S. I. Gelfand [2] who showed that the polynomials representing classes of Schubert varieties can be obtained from a polynomial representing the top class (the class of a point) by a sequence of *divided difference operators*. A. Lascoux and M.-P. Schützenberger [3] considered one particularly nice choice

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https://doi.org/10.1016/j.ejc.2022.103613
0195-6698/

of the top class and defined a system of polynomials $\mathfrak{S}_w \in \mathbb{Z}[z_1, \dots, z_n]$, the *Schubert polynomials*, with many good combinatorial and geometric properties. Here $w \in \mathcal{S}_n$ runs over the group of permutations in n letters, i.e. the Weyl group of GL_n,

This system of polynomials is *stable* in the following sense: for the embedding $\mathcal{S}_n \hookrightarrow \mathcal{S}_{n+1}$, the representative $\mathfrak{S}_w$ of a given permutation $w \in \mathcal{S}_n \subset \mathcal{S}_{n+1}$ does not depend upon n. So the polynomials $\mathfrak{S}_w$ can be viewed as elements of the ring $\mathbb{Z}[z_1, z_2, \dots]$ in countably many variables, indexed by the *finitary* permutations $w \in \mathcal{S}_\infty = \varinjlim \mathcal{S}_n$. In [3], it is shown that these polynomials form a $\mathbb{Z}$-*basis* of this ring (also cf. [4]).

Geometrically this can be interpreted as follows. The standard embedding $\mathrm{GL}_n \hookrightarrow \mathrm{GL}_{n+1}$ together with the embedding of Borel subgroups $B_n \hookrightarrow B_{n+1}$ defines an embedding of flag varieties $G_n/B_n \hookrightarrow G_{n+1}/B_{n+1}$. This gives a surjective map $H^*(G_{n+1}/B_{n+1}, \mathbb{Z}) \to H^*(G_n/B_n, \mathbb{Z})$. This map is compatible with the Schubert classes: for each $w \in \mathcal{S}_\infty$, there exists a *stable* Schubert class $\sigma_w^{(\infty)} = \varprojlim \sigma_w^{(n)} \in \varprojlim H^*(G_n/B_n, \mathbb{Z})$. *A priori* this class is a homogeneous power series in $z_1, z_2, \dots$. However it turns out that it can be represented by a unique *polynomial*, which is the Schubert polynomial $\mathfrak{S}_w$. The elements $\{\mathfrak{S}_w\}$ are thus obtained as the *unique* solutions of an infinite system of equations involving divided difference operations.

The polynomials $\mathfrak{S}_w$ have nonnegative integer coefficients; their nonnegativity is completely unobvious from their definition involving divided difference operators. Their combinatorial description was obtained independently by S. Fomin and An. Kirillov [5] and S. Billey and N. Bergeron [6]. The monomials in $\mathfrak{S}_w$ are indexed by certain diagrams, called *pipe dreams* or *rc-graphs*, see Section 2.4. A geometric interpretation of pipe dreams was obtained by A. Knutson and E. Miller in [7]: they showed that the pipe dreams for a permutation w are in bijection with the irreducible components of a certain flat Gröbner degeneration of the corresponding matrix Schubert variety $\overline{X_w}$.

There is an obvious analogy between Schubert and Schur polynomials: the monomials of the latters are indexed by semistandard Young tableaux. In fact, if w is a Grassmannian permutation, i.e. it has a unique descent, the corresponding Schubert polynomial equals the Schur polynomial $s_\lambda(w)$, where $\lambda(w)$ is a partition obtained from w (see Section 6.1 for details). This equality can be easily obtained from the following geometric argument: the Schur polynomials represent the classes of Schubert varieties in Grassmannians G/P, where P is a maximal parabolic group in G, and the map $H^*(G/P) \to H^*(G/B)$ sends them to Schubert classes of Grassmannian permutations in a full flag variety. There is also a purely combinatorial proof: one can establish a bijection between the Young tableaux indexing the monomials in $s_{\lambda(w)}$ and the pipe dreams indexing the monomials in $\mathfrak{S}_w$. We recall this proof in Theorem 6.4; see also [8].

In [6] the authors introduced the notion of the *bottom pipe dream* for each Schubert polynomial. It is a maximal pipe dream according to some partial order on pipe dreams, defined in combinatorial terms; such a pipe dream exists and is unique for each permutation. This allowed them to show that the basis change matrix between the Schubert polynomials and the monomial basis in $\mathbb{Z}[z_1, z_2, \dots]$ is unitriangular, and hence Schubert polynomials form a $\mathbb{Z}$-basis in $\mathbb{Z}[z_1, z_2, \dots]$.

The construction of Schubert polynomials can be extended as follows. Instead of the cohomology ring $H^*(G/B, \mathbb{Z})$ we can consider the T-equivariant cohomology ring $H_T^*(G/B, \mathbb{Z})$. The map $G/B \to \mathrm{pt}$ defines the module structure on $H_T^*(G/B, \mathbb{Z})$ with respect to the polynomial ring $\mathbb{Z}[t_1, \dots, t_n] = H_T^*(\mathrm{pt})$. One can be interested in polynomial representatives of the T-equivariant classes of $[X_w]$. These are also classical objects, known as *double Schubert polynomials* $\mathfrak{S}_w(z, t)$; these are homogeneous polynomials in $2n$ variables: $z_1, \dots, z_n$ and $t_1, \dots, t_n$. The specialization $t_i = 0$ gives us the usual Schubert polynomials $\mathfrak{S}_w(x)$ in n variables $z_1, \dots, z_n$. Double Schubert polynomials also have a description in terms of pipe dreams.

1.2. The case of symplectic and orthogonal groups

It is interesting to replace GL_n by another reductive complex algebraic group and ask the same series of questions. We will be interested in the classical groups of types B_n, C_n, and D_n: these are SO_{2n+1}, Sp_{2n}, and SO_{2n}, respectively. The Weyl group W for such a group is a hyperoctahedral group, or the group of signed permutations on n letters, for the types B and C, or, in the type D, the subgroup of this group consisting only of signed permutations with even number of sign changes.

The (generalized) full flag variety G/B also has a cellular decomposition $G/B = \bigsqcup_{w\in W} BwB/B$, with the cells indexed by the elements of the Weyl group W; and the cell closures $X_w = \overline{BwB/B}$ define a basis $\sigma_w \in H^*(G/B, \mathbb{Z})$.

As in the case of GL_n, one can consider the embeddings $G_n \hookrightarrow G_{n+1}$, where G_n and G_{n+1} are classical groups of the same type (B, C or D) and of the rank n and $n+1$, respectively. We can fix an element $w \in W_\infty = \bigcup W_n$ of the "limit Weyl group" and take the projective limit of Schubert cycles $\sigma_w^{(\infty)} \in \varprojlim H^*(G_n/B_n, \mathbb{Z})$. The question is as follows: what is a "nice" representative of this class in the ring of power series $\mathbb{Z}[[z_1, z_2, \dots]]$?

In [9] S. Billey and M. Haiman define *Schubert polynomials for the classical groups*. They show that σ_w^∞ is represented by a unique element $\mathfrak{F}_w$ of the subring $\mathbb{Z}[z_1, z_2, \dots, p_1(\mathbf{z}), p_3(\mathbf{z}), \dots]$, where $p_k(\mathbf{z}) = \sum_{i=1}^\infty z_i^k$ are the Newton power sums. Note that we need to take only the odd power sums; the images of the even power sums vanish in each of the coinvariant rings for each of the types B, C, and D. Similarly to $\mathfrak{S}_w$, the functions $\mathfrak{F}_w$ are obtained as unique solutions of an infinite system of equations involving divided difference operators. They also form a basis of the corresponding ring.

Schubert polynomials for the classical groups can be also expressed via the usual Schubert polynomials and the *Stanley symmetric functions*. This implies that they are nonnegative integer combinations of monomials in $z_1, z_2, \dots$ and $p_1, p_3, \dots$.

Like in the previous case, we can consider the Schubert classes coming from cycles on Lagrangian/orthogonal Grassmannians. The results of P. Pragacz [10] imply that the corresponding Schubert polynomials are equal to Schur's P- and Q-functions; the details follow in Section 6.2.

One also can introduce double versions of Schubert polynomials, which represent the classes of $[X_w]$ in the T-equivariant cohomology ring $H_T^*(G/B, \mathbb{Z})$, where $T \cong (\mathbb{C}^*)^n$ is a maximal torus in G. They depend upon $z_1, z_2, \dots, p_1, p_3, \dots$, and another set of variables $t_1, t_2, \dots$ Formulas for double Schubert polynomials $\mathfrak{F}_w(\mathbf{z}, \mathbf{p}, \mathbf{t})$ were found by T. Ikeda, L. Mihalcea, and H. Naruse in [11].

Finally, in the paper [12], An. Kirillov and H. Naruse provide a combinatorial construction of Schubert polynomials for classical Weyl groups outside the type A. They define analogues of pipe dreams whose generating functions are equal to (ordinary or equivariant) Schubert polynomials.

Let us also mention an earlier paper [13] by S. Fomin and An. Kirillov, where the authors were constructing Schubert polynomials of the type B in a completely different way: they were looking for a family of polynomials indexed by the hyperoctahedral group and satisfying certain five properties, similar to those of Schubert polynomials in the type A (stability, nonnegativity of coefficients etc.). They have shown that such a family of polynomials does not exist; then they considered families of polynomials defined by all but one of these properties. This led them to several different families of polynomials. We should note that they are also different from the Schubert polynomials of type B defined by Billey and Haiman and considered in this paper.

1.3. Our results

This paper is devoted to combinatorial study of Schubert polynomials for groups of the types B, C, and D. We provide an alternative construction of pipe dreams for these cases; it is given in Section 3. The pipe dreams are configurations of strands, similar to those in the type A; some of the strands may be equipped with an additional element, called *faucet*, which represents the sign change. To such a pipe dream we can associate a permutation $w \in W$, called the *shape* of pipe dream, and a monomial; our main results, Theorems 3.7, 3.4 and 3.11, state that the Schubert polynomial $\mathfrak{F}_w(\mathbf{z}, \mathbf{p})$ equals the sum of monomials for all pipe dreams of shape $w \in W$ of type B, C or D, respectively. We also give a similar description of double Schubert polynomials of these types in Corollary 3.18.

Our construction of pipe dreams is similar to the one by Kirillov and Naruse; the main difference is that in their setting each pipe dream corresponds to a couple of terms in the Schubert polynomial (namely, to the product of several binomials), as opposed to one monomial in our setting. We provide a detailed comparison of these two constructions in Section 4.

Then we introduce a notion of admissible moves on pipe dreams of a given type; these are certain transformations not changing the shape of a pipe dream, that turn it into a poset. We show that each pipe dream for a permutation of each of the types B–D can be reduced to a certain canonical form, referred to as the bottom pipe dream, by a sequence of admissible moves, similar to the ladder

moves in the type A; thus the poset of pipe dreams corresponding to a given Weyl group element is shown to have a unique minimal element. The existence of bottom pipe dreams is established in Theorems 5.7, 5.8, and 5.9. This allows us to give a new proof of Theorem 5.10, which states that the Schubert polynomials (of each type) form a basis of the ring $\mathbb{Q}[\mathbf{z}, p_1, p_3, \dots]$.

In the last section we study Grassmannian permutations; we give a bijective proof of the fact that the Schubert polynomial of such a permutation is Schur's P- or Q-function (Theorem 6.7). For this we recall the definition of the latter functions involving shifted Young tableaux and establish a bijection between these tableaux and the pipe dreams of the corresponding Grassmannian permutation.

1.4. Related questions: bumpless pipe dreams and specializations of Schubert polynomials

Recently, T. Lam, S. J. Lee, and M. Shimozono [14] introduced objects called *bumpless pipe dreams*. They also can be used to obtain a formula for double Schubert polynomials; however, there is no weight-preserving bijection between them and the usual pipe dreams, so the two presentations are genuinely different.

In her recent preprint [15], A. Weigandt defines a generalization of bumpless pipe dreams, which provide a similar description of Grothendieck polynomials, and shows that they correspond to alternating sign matrices. It would be interesting to find analogues of these results, as well as the notion of bumpless pipe dreams, for the groups of classical types outside the type A.

Another series of questions concerns specializations of Schubert polynomials. In [4], I. Macdonald conjectured a formula for the principal specialization of a Schubert polynomial; this formula was proven algebraically by S. Fomin and R. Stanley [16]. Recently, a combinatorial proof was found by S. Billey, A. Holroyd, and B. Young [17]. Taking an appropriate limit of Schubert polynomials gives the so-called *backstable Schubert polynomials*, also defined in the aforementioned paper [14]. Similarly to Schubert polynomials of types B, C and D, they depend upon two sets of variables, being polynomials in the first set and symmetric functions in the second set of variables. They also admit a similar expression for their principal specializations. E. Marberg and B. Pawlowski [18] prove analogues of Macdonald's formula for Schubert polynomials of types B, C, and D. A natural (however challenging) question is to find a combinatorial proof of their results by means of pipe dreams, either those defined in this paper or the bumpless ones.

1.5. Structure of the paper

This text is organized as follows. In Section 2 we recall the main definitions and notions concerning the Weyl groups of classical types, the Schubert polynomials for these groups, as well as the construction of pipe dreams in the type A. In Section 3 we describe the constructions of pipe dreams for Schubert polynomials of the types B, C, and D (Sections 3.1–3.4). In Section 3.5 we are dealing with the double Schubert polynomials; we provide a generalization of the previous construction to this case. Finally, in Section 3.7 we compute several examples of Schubert polynomials. Section 5 is devoted to the proof of existence of bottom pipe dreams. Section 4 is devoted to comparing our construction with the "excited extended Young diagrams" by Kirillov and Naruse: we recall their construction in Section 4.1 and relate it to the ours in Section 4.2. In Section 5.1 we introduce the admissible moves, and in Section 5.2 we recall the situation in the type A. The next three subsections are devoted to the cases B, C, and D respectively. Section 6 is devoted to Grassmannian permutations; we show that the Schubert polynomial of such a permutation is a P- or Q-Schur function. In the appendix we provide more examples: namely, we list all the elements of the group $\mathcal{BC}_2$ and some elements of the group $\mathcal{D}_3$ and for each of these elements list all its pipe dreams.

2. Preliminaries

2.1. Weyl groups of the classical types

Let $\mathbb{Q}[\mathbf{z}] = \mathbb{Q}[z_1, z_2, \dots]$ be the ring of polynomials in countably many variables $z_1, z_2, \dots$. Let $p_k = z_1^k + z_2^k + \dots$ denote the kth power sum; this is not a polynomial, but rather a symmetric

function in the z_i's. Consider the ring of power series $\mathbb{Q}[\mathbf{z}, p_1, p_3, \dots] = \mathbb{Q}[z_1, z_2, \dots, p_1, p_3, \dots]$ which are polynomial in the z_i and the p_k for k odd. All its generators are algebraically independent, so this ring can be viewed just as the polynomial ring in $z_1, z_2, \dots$ and $p_1, p_3, \dots$.

Denote by $\mathcal{S}_n$ the symmetric group in n variables, and let $\mathcal{S}_\infty = \varinjlim \mathcal{S}_n$ be the group of finitary permutations of $\mathbb{Z}_{>0}$, i.e. the group of permutations fixing all but finitely many points. It is generated by the simple transpositions $s_i = (i \leftrightarrow i+1)$. The group S_∞ acts on $\mathbb{Q}[\mathbf{z}, \dots, p_1, p_3, \dots]$ by permuting the z_i's.

Let us also consider the group of signed permutations $\mathcal{BC}_n$ (also called the *hyperoctahedral group*). It can be viewed as the group of permutations w of the $2n$-element set $\{1, -1, \dots, n, -n\}$ satisfying the condition $w(i) = j$ iff $w(-i) = -j$. The groups $\mathcal{BC}_n$ are embedded one into another in the obvious way, so we can consider the injective limit $\mathcal{BC}_\infty$, which is the group of finitary signed permutations. The standard generators for $\mathcal{BC}_\infty$ are $s_i = (i \leftrightarrow i+1)$ for $i \geq 1$ and $s_0 = (1 \leftrightarrow -1)$. The group $\mathcal{BC}_\infty$ acts on the formal power series ring $\mathbb{Q}[[z_1, z_2, \dots]]$ by letting s_i interchange z_i and z_{i+1} for $i \geq 1$, and by letting s_0 send z_1 into $-z_1$. We can restrict this action to the rings $\mathbb{Q}[\mathbf{z}]$ and $\mathbb{Q}[\mathbf{z}, p_1, p_3, \dots]$ and see that the s_i for $i \geq 1$ fix the p_k, while $s_0 p_k = p_k - 2z_1^k$.

We denote by $\mathcal{D}_n$ the subgroup of $\mathcal{BC}_n$ of signed permutations with an even number of sign changes. The union of these subgroups is denoted by $\mathcal{D}_\infty$. The standard generators for these groups are s_i for $i \geq 1$ and an additional generator $s_{\hat{1}} = s_0 s_1 s_0$ which replaces z_1 with $-z_2$ and z_2 with $-z_1$.

These groups satisfy the relations $s_i^2 = \mathrm{id}$ and the *Coxeter relations* (cf., for instance, [19] or [20]):

$\mathcal{S}_\infty$: $s_i s_j = s_j s_i$ for $|i-j| \geq 2$, $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for $i \geq 1$;

$\mathcal{BC}_\infty$: $s_i s_j = s_j s_i$ for $|i-j| \geq 2$, $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for $i \geq 1$, $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$.

$\mathcal{D}_\infty$: $s_i s_j = s_j s_i$ for $|i-j| \geq 2$, $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for $i \geq 1$ (here $i, j \in \{\hat{1}, 1, 2, \dots\}$; while performing arithmetic operations we treat $\hat{1}$ as 1).

We will use the standard terminology for Coxeter groups. Let $\mathcal{F}_\infty$ be one of the groups $\mathcal{S}_\infty$, $\mathcal{BC}_\infty$, or $\mathcal{D}_\infty$. A *word* $(s_{i_1}, \dots, s_{i_k})$ is a finite sequence of its generators. We say that this word *represents* an element $w \in \mathcal{F}_\infty$, if $w = s_{i_1} \dots s_{i_k}$. The minimal number k such that there exists a k-element word representing w is called the *length* of w and denoted by $k = \ell(w)$.

Often we will use the one-line notation (not to be confused with the cycle notation) for elements $w \in \mathcal{S}_n$, writing a permutation w as a sequence $w(1)w(2)\dots w(n)$. For instance, $4132 \in \mathcal{S}_4$ maps 1 into 4, 2 into 1, 3 into itself and 4 into 2.

For signed permutations $w \in \mathcal{BC}_n$ we will write negative numbers $-m$ as $\overline{m}$ in one-line notation. For example, $3\bar{2}\bar{4}1 \in \mathcal{BC}_4$ maps 1 into 3, 2 into -2, 3 into -4, and 4 into 1.

2.2. Schubert polynomials for the classical groups

Let us recall the definition of Schubert polynomials from [3] and the definition of Schubert polynomials for the classical groups from [9].

Definition 2.1. Define the *divided difference operators* on the rings $\mathbb{Q}[\mathbf{z}]$ and $\mathbb{Q}[\mathbf{z}, p_1, p_3, \dots]$ as follows:

$$\partial_i f = \frac{f - s_i f}{z_i - z_{i+1}} \quad \text{for} \quad i \geq 1;$$
$$\partial_0 f = \frac{f - s_0 f}{-2z_1};$$
$$\partial_0^B f = 2\partial_0 f = \frac{f - s_0 f}{-z_1};$$
$$\partial_{\hat{1}} f = \frac{f - s_{\hat{1}} f}{-z_1 - z_2}.$$

Definition 2.2. Schubert polynomials (of type A) are homogeneous polynomials $\mathfrak{S}_w(\mathbf{z}) \in \mathbb{Q}[\mathbf{z}]$ indexed by the permutations $w \in \mathcal{S}_\infty$ and satisfying the relations

$$\begin{aligned} \mathfrak{S}_{\mathrm{id}} &= 1; \\ \partial_i \mathfrak{S}_w &= \begin{cases} \mathfrak{S}_{ws_i} & \text{if} \quad \ell(ws_i) < \ell(w); \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

for each $i \geq 1$.

Definition 2.3. Schubert polynomials of type C are elements $\mathfrak{C}_w(\mathbf{z}, \mathbf{p}) \in \mathbb{Q}[\mathbf{z}, p_1, p_3, \dots]$ that are homogeneous in $\mathbf{z}$, indexed by the permutations $w \in \mathcal{BC}_\infty$ and satisfy the relations

$$\begin{aligned} \mathfrak{C}_{\mathrm{id}} &= 1; \\ \partial_i \mathfrak{C}_w &= \begin{cases} \mathfrak{C}_{ws_i} & \text{if} \quad \ell(ws_i) < \ell(w); \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{1}$$

for each $i \geq 0$.

Definition 2.4. Schubert polynomials of type B are defined as $\mathfrak{B}_w = 2^{-s(w)}\mathfrak{C}_w$, where $s(w)$ is the number of entries from $\{1, 2, \dots\}$ changing their sign under the action of w. They also can be defined by the relations obtained from 2.3 by replacing ∂_0 by ∂_0^B.

Definition 2.5. Schubert polynomials of type D are elements $\mathfrak{D}_w(\mathbf{z}, \mathbf{p}) \in \mathbb{Q}[\mathbf{z}, p_1, p_3, \dots]$ that are homogeneous in $\mathbf{z}$, indexed by the permutations $w \in \mathcal{D}_\infty$ and satisfy the relations

$$\begin{aligned} \mathfrak{D}_{\mathrm{id}} &= 1; \\ \partial_i \mathfrak{D}_w &= \begin{cases} \mathfrak{D}_{ws_i} & \text{if} \quad \ell(ws_i) < \ell(w); \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

for each $i = \hat{1}, 1, 2, \dots$.

It was shown by S. Billey and M. Haiman [9] that such polynomials exist and are uniquely determined by these properties. We recall these results below as Theorems 2.7–2.9.

For convenience we replace the ring $\mathbb{Q}[\mathbf{z}, p_1(\mathbf{z}), p_3(\mathbf{z}), \dots]$ by an isomorphic ring $\mathbb{Q}[\mathbf{z}, p_1(\mathbf{x}), p_3(\mathbf{x}), \dots]$. Here $p_k(\mathbf{x}) = \sum_{i=1}^{\infty} x_i^k$, and the isomorphism is the unique ring isomorphism with $z_i \mapsto z_i$ and $p_k(\mathbf{x}) \mapsto -p_k(\mathbf{z})/2$. Then the Schubert polynomials of types B, C, and D are polynomials in $z_1, z_2, \dots$ and symmetric functions in $x_1, x_2, \dots$.

It is easy to see that the generators of the groups $\mathcal{BC}_\infty$ and $\mathcal{D}_\infty$ act on the ring $\mathbb{Q}[\mathbf{z}, p_1(\mathbf{x}), p_3(\mathbf{x}), \dots]$ in the following way:

$$\begin{aligned} &s_i f(z_1, \dots, z_i, z_{i+1}, \dots; x_1, x_2, \dots) = f(z_1, \dots, z_{i+1}, z_i, \dots; x_1, x_2, \dots) \quad \text{for } i \geq 1; \\ &s_0 f(z_1, z_2, z_3, \dots; x_1, x_2, \dots) = f(-z_1, z_2, z_3 \dots; z_1, x_1, x_2, \dots); \\ &s_{\hat{1}} f(z_1, z_2, z_3 \dots; x_1, x_2, \dots) = f(-z_2, -z_1, z_3 \dots; z_1, z_2, x_1, x_2, \dots). \end{aligned}$$

2.3. Stanley symmetric functions and Schubert polynomials

In this subsection we discuss the expression of Schubert polynomials via Stanley symmetric functions. We start with introducing some notation.

- Let w be an element of $\mathcal{S}_\infty$, $\mathcal{BC}_\infty$ or $\mathcal{D}_\infty$. Denote the set of all reduced words for w by $R(w)$.
- Let $\mathbf{a} = s_{a_1}s_{a_2}\dots s_{a_l} \in R(w)$ be a reduced word for $w \in \mathcal{BC}_\infty$ or $w \in \mathcal{D}_\infty$. We define the set of *peaks* of $\mathbf{a}$ as the set
$$P(\mathbf{a}) = \{i \in \{2, 3, \dots, l-1\} \mid a_{i-1} < a_i > a_{i+1}\}.$$
For the group $\mathcal{D}_\infty$ we assume that $\hat{1} = 1 < 2 < 3 < \cdots$
- A weakly decreasing sequence of nonnegative integers $j_1 \geq j_2 \geq \cdots \geq j_l$ is said to be *x-admissible* for the word $\mathbf{a}$, if $j_{i-1} = j_i = j_{i+1}$ implies that $i \notin P(\mathbf{a})$.
To each x-admissible sequence $j_1 \geq j_2 \geq \cdots \geq j_l$ we can assign a monomial $x_{j_1}x_{j_2}\dots x_{j_l} = x_1^{\alpha_1}\dots x_m^{\alpha_m} = \mathbf{x}^\alpha$. Such monomials are also called *x-admissible*. We denote the set of all x-admissible monomials for $\mathbf{a}$ by $\mathscr{A}_x(\mathbf{a})$.
- Let $w \in \mathcal{S}_n$ and let $\mathbf{a} = s_{a_1}s_{a_2}\dots s_{a_l} \in R(w)$ be a reduced word for w. A weakly increasing sequence of positive integers $j_1 \leq j_2 \leq \cdots \leq j_l$ is said to be *z-admissible* for $\mathbf{a}$ if for each i we have $j_i \leq a_i$, and the equality $j_i = j_{i+1}$ implies that $a_i > a_{i+1}$.
To each z-admissible sequence $j_1 \leq j_2 \leq \cdots \leq j_l$ we can assign a monomial $z_{j_1}z_{j_2}\dots z_{j_l} = z_1^{\beta_1}\dots z_m^{\beta_m} = \mathbf{z}^\beta$. Such monomials are also called *z-admissible*; we denote the set of z-admissible monomials for $\mathbf{a}$ by $\mathscr{A}_z(\mathbf{a})$.
- Denote by $i(\alpha)$ the number of distinct variables occuring in $\mathbf{x}^\alpha$ in nonzero powers.
- For the group $\mathcal{D}_\infty$ we denote by $o(\mathbf{a})$ the total number of occurrences of the letters s_1 and $s_{\hat{1}}$ in $\mathbf{a}$.

Definition 2.6. The *Stanley symmetric functions* are defined as follows. For $w \in \mathcal{BC}_\infty$ we define
$$F_w(\mathbf{x}) = \sum_{\substack{\mathbf{a}\in R(w)\\ \mathbf{x}^\alpha \in \mathscr{A}_x(\mathbf{a})}} 2^{i(\alpha)}\mathbf{x}^\alpha.$$
For $w \in \mathcal{D}_\infty$ we define
$$E_w(\mathbf{x}) = \sum_{\substack{\mathbf{a}\in R(w)\\ \mathbf{x}^\alpha \in \mathscr{A}_x(\mathbf{a})}} 2^{i(\alpha)-o(\mathbf{a})}\mathbf{x}^\alpha.$$

The following theorem, describing the relation of Schubert polynomials and Stanley symmetric functions, is due to S. Fomin and R. Stanley [16] and S. Billey, W. Jockush and R. Stanley [21].

Theorem 2.7 ([21, Thm. 1.1])**.** *For each $w \in \mathcal{S}_\infty$ we have*
$$\mathfrak{S}_w(\mathbf{z}) = \sum_{\substack{\mathbf{a}\in R(w)\\ \mathbf{z}^\beta \in \mathscr{A}_z(\mathbf{a})}} \mathbf{z}^\beta.$$

Its analogues for the cases of $\mathcal{BC}_\infty$ and $\mathcal{D}_\infty$ were obtained by S. Billey and M. Haiman [9].

Theorem 2.8 ([9, Thm 3A])**.** *Let $w \in \mathcal{BC}_\infty$. The Schubert polynomial $\mathfrak{C}_w$ satisfies the equality*
$$\mathfrak{C}_w(\mathbf{z};\mathbf{x}) = \sum_{\substack{uv=w\\ \ell(u)+\ell(v)=\ell(w)\\ v\in\mathcal{S}_n}} F_u(\mathbf{x})\mathfrak{S}_v(\mathbf{z}) = \sum_{\substack{uv=w\\ \ell(u)+\ell(v)=\ell(w)\\ v\in\mathcal{S}_n}} \sum_{\substack{\mathbf{a}\in R(u)\\ \mathbf{x}^\alpha\in\mathscr{A}_x(\mathbf{a})}} \sum_{\substack{\mathbf{b}\in R(v)\\ \mathbf{z}^\beta\in\mathscr{A}_z(\mathbf{b})}} 2^{i(\alpha)}\mathbf{x}^\alpha\mathbf{z}^\beta.$$

Theorem 2.9 ([9, Thm 4A])**.** *Let $w \in \mathcal{D}_\infty$. The Schubert polynomial $\mathfrak{D}_w$ satisfies the equality*
$$\mathfrak{D}_w(\mathbf{z};\mathbf{x}) = \sum_{\substack{uv=w\\ \ell(u)+\ell(v)=\ell(w)\\ v\in\mathcal{S}_n}} E_u(\mathbf{x})\mathfrak{S}_v(\mathbf{z}) = \sum_{\substack{uv=w\\ \ell(u)+\ell(v)=\ell(w)\\ v\in\mathcal{S}_n}} \sum_{\substack{\mathbf{a}\in R(u)\\ \mathbf{x}^\alpha\in\mathscr{A}_x(\mathbf{a})}} \sum_{\substack{\mathbf{b}\in R(v)\\ \mathbf{z}^\beta\in\mathscr{A}_z(\mathbf{b})}} 2^{i(\alpha)-o(\mathbf{a})}\mathbf{x}^\alpha\mathbf{z}^\beta.$$

2.4. Pipe dreams in type A

In this subsection we briefly recall the construction of pipe dreams for the symmetric group $\mathcal{S}_n$. We fix a positive integer n and consider the staircase Young diagram with rows of length

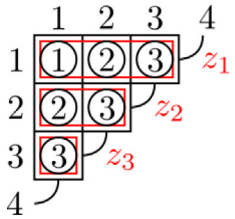


Fig. 1. Base diagram for pipe dreams of size $n = 4$.

$n-1,\dots,2,1$. We will refer to this diagram as the *base* $B_{\mathcal{S}_n}$. Let us index all the boxes in the ith row by the variable z_i (cf. Fig. 1).

Now let us fill this diagram by elements of two types: *elbow joints* and *crosses* . We also put a single elbow to the right of the rightmost box of each row. We obtain a diagram consisting of n strands, or *pipes* (hence the name "pipe dream").

For each box $\square$ we introduce the following notation:

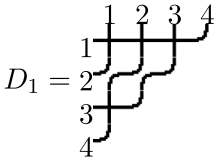
- wt($\square$) is the weight of an element in this box. It is equal to 1 for a cross and to 0 for an elbow joint .
- $\sigma(\square)\in \mathcal{S}_n$ is an element equal to s_i for a cross in the box with the number i and to the identity permutation id for an elbow.
- var($\square$) is the variable corresponding to this box, i.e. z_i, where i is the row number.

To each pipe dream D we can assign:

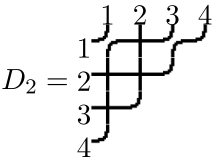
- a monomial $\mathbf{z}^{\beta(D)} = \prod_{\square\in B_{\mathcal{S}_n}} \mathrm{var}(\square)^{\mathrm{wt}(\square)}$ (i.e. the product of all variables z_i over all the crosses in D);
- A word word(D) obtained by writing out all the letters $\sigma(\square)$ while reading all the boxes in the base from right to left, from top to bottom;
- If word(D) is a reduced word for the permutation $w \in \mathcal{S}_n$, then the pipe dream D is said to be *reduced*. We shall say that $w = w(D)$ is the *shape* of D.

This definition can be restated as follows: a pipe dream is said to be reduced if each two strands cross at most once. Its shape is the permutation $w \in \mathcal{S}_n$ such that the strands connect the numbers $1, 2, \dots, n$ on the left-hand side of the diagram with the numbers $w(1), w(2), \dots, w(n)$ on its top.

Example 2.10. Here are two examples of pipe dreams of size $n = 4$.



$$\mathbf{z}^{\beta(D_1)} = z_1^3 z_3$$
$$\mathrm{word}(D_1) = s_3 s_2 s_1 s_3$$
$$w(D_1) = 4132 \in \mathcal{S}_4$$



$$\mathbf{z}^{\beta(D_2)} = z_1 z_2^2 z_3$$
$$\mathrm{word}(D_2) = s_2 s_3 s_2 s_3$$
$$D_2 \text{ is non-reduced.}$$

Let us show that for a reduced pipe dream D of shape w the monomial $\mathbf{z}^{\beta(D)} = z_{j_1} z_{j_2} \dots z_{j_l}$ is z-admissible for the word word(D) $= s_{a_1} s_{a_2} \dots s_{a_l}$.

Indeed, the ith cross is positioned in the j_ith row and a_ith diagonal (crosses are ordered from right to left, from top to bottom, so $j_1 \le j_2 \le \dots \le j_l$). Since the numbers of the diagonals decrease

while reading each row right to left, then $j_i = j_{i+1}$ implies $a_i > a_{i+1}$. Since the row number of the box does not exceed the number of the corresponding diagonal, we have $j_i \leq a_i$.

Let $w \in \mathcal{S}_n$, $\mathbf{a} = s_{a_1} s_{a_2} \dots s_{a_l} \in R(w)$, and $\mathbf{z}^\beta = z_{j_1} z_{j_2} \dots z_{j_l} \in \mathscr{A}_z(\mathbf{a})$. There exists a unique pipe dream D such that word$(D) = \mathbf{a}$ and $\mathbf{z}^{\beta(D)} = \mathbf{z}^\beta$. It can be obtained by placing a cross at the intersection of the j_ith row and the a_ith diagonal for each i.

It is clear that the set of all reduced pipe dreams of shape $w \in S_n$ does not depend upon n, i.e. is stable under the embedding $S_n \hookrightarrow S_{n+1}$. Denote this set by $\mathrm{PD}_A(w)$. Theorem 2.7 can be restated as follows.

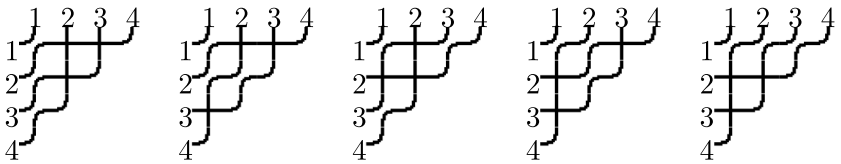
Theorem 2.11 ([5,6]). *For a permutation $w \in \mathcal{S}_n$, we have*

$$\mathfrak{S}_w(\mathbf{z}) = \sum_{D \in \mathrm{PD}_A(w)} \mathbf{z}^{\beta(D)}.$$

Example 2.12. Let $w = s_i \in \mathcal{S}_n$, $i < n$. If $D \in \mathrm{PD}_A(w)$ is a reduced pipe dream, then D contains exactly one cross. It is located in the ith diagonal; the row number of this cross can be equal to $1, 2, \dots, (i-1), i$. So

$$\mathfrak{S}_{s_i}(\mathbf{z}) = z_1 + z_2 + \dots + z_i.$$

Example 2.13. Let $w = 1432 \in \mathcal{S}_4$. There are five reduced pipe dreams of shape w:



The Schubert polynomial is thus equal to

$$\mathfrak{S}_{1432}(\mathbf{z}) = z_1^2 z_2 + z_1^2 z_3 + z_1 z_2^2 + z_1 z_2 z_3 + z_2^2 z_3.$$

3. Construction of pipe dreams in the types *B*, *C*, and *D*

3.1. Type C

The Schubert polynomials of the types B, C, and D are elements of the ring $\mathbb{Q}[\mathbf{z}, p_1(\mathbf{x}), p_3(\mathbf{x}), \dots]$, i.e. they are polynomials in $\mathbf{z}$ and symmetric functions in $\mathbf{x}$. For our purposes, it will be sometimes more convenient to deal with polynomials both in $\mathbf{z}$ and $\mathbf{x}$. For this let us give the following technical definition.

Definition 3.1. Let $F \in \mathbb{Q}[\mathbf{z}, p_1(\mathbf{x}), p_3(\mathbf{x}), \dots]$, and let $k \geq 0$ be a nonnegative integer. We define *k-truncation* of F as follows:

$$F^{[k]}(\mathbf{z}, x_1, \dots, x_k) = F(\mathbf{z}, x_1, \dots, x_k, 0, 0, \dots).$$

It is a symmetric polynomial in $x_1, \dots, x_k$.

Let $w \in \mathcal{BC}_n$ be a signed permutation of n variables. Consider the Schubert polynomial $\mathfrak{C}_w(\mathbf{z}, \mathbf{x})$ of the type C. As above, denote by $\mathfrak{C}_w^{[k]}(\mathbf{z}, \mathbf{x})$ its k-truncation.

The pipe dreams for $\mathfrak{C}_w^{[k]}$ are obtained by putting elements inside boxes of the shape $B^k_{\mathcal{C}_n}$ shown in Fig. 2. It consists of a *staircase block*, which is the staircase Young diagram $(n-1, n-2, \dots, 2, 1)$,

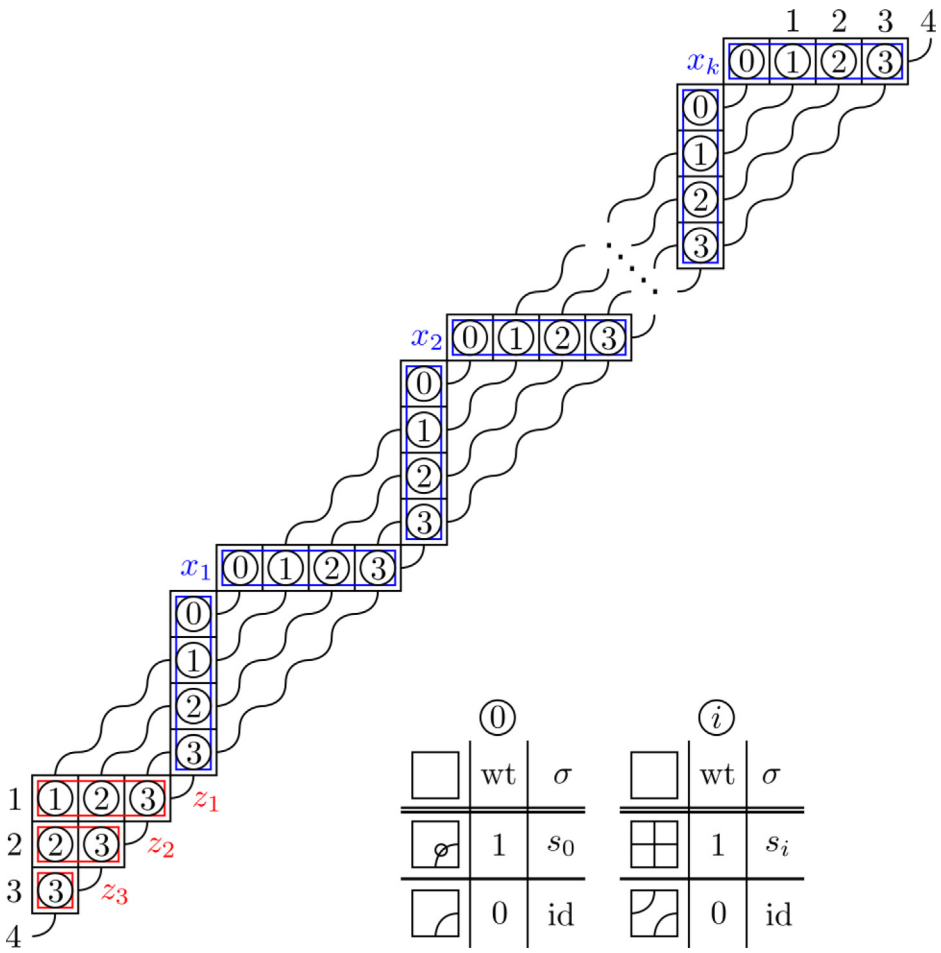


Fig. 2. The base $B^k_{C_4}$ for c-signed pipe dreams.

and of k consecutive blocks of a different shape, which we call Γ-*blocks*. Each Γ-block consists of an $n \times 1$ column and a $1 \times n$ row; all the boxes outside these blocks are filled by elbow joints (this part is sometimes referred to as the *sea of elbows*). The boxes in each row and each column of Γ-blocks are indexed by the integers from 0 to $n-1$, from top to bottom and from left to right respectively.

The topmost box of each column and the leftmost box of each row can be filled by the elements of two different kinds: the single elbow joint or by a new kind of element, *an elbow joint with a faucet*. All the remaining boxes are filled either with elbow joints or crosses. We will refer to crosses and to elbows with faucets as *significant elements*. The diagram obtained by this construction is called a *c-signed pipe dream*. It consists of n strands, or pipes, connecting the left edge of the diagram with its top edge. We index the left and the top edges of the pipes by $1, \ldots, n$; as in the case of classical pipe dreams, this defines a permutation. Moreover, some of the pipes have faucets on them; each faucet is responsible for the sign change for the corresponding variable.

In more formal terms, we can assign to each box $\square$ the following data:

- the weight wt($\square$) of the element inside it. It is equal to 1 for crosses and elbows with faucets (i.e. for significant elements) and 0 for elbows.

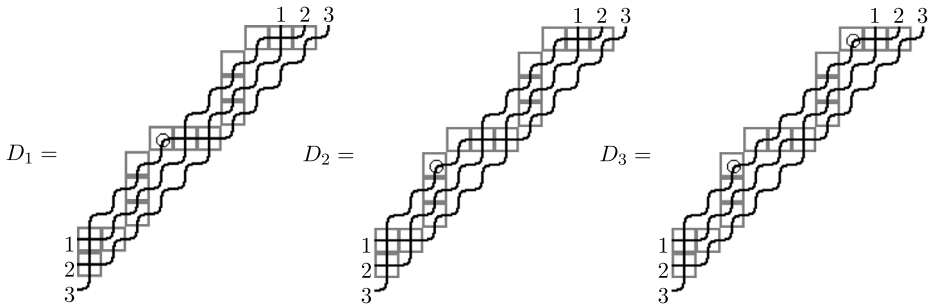
- A letter $\sigma(\square) \in \mathcal{C}_n$. It is equal to s_i for a cross inside a box indexed by $i > 0$, to s_0 for an elbow with a faucet in a box indexed by zero, and to nothing (the identity permutation id) for elbow joints.
- A variable $\mathrm{var}(\square)$. It is equal to z_i for a box in the ith row of the staircase, counted from above, and to x_i for a box in the ith Γ-block, counted from below.

To each c-signed pipe dream D we can assign the following data:

- A monomial $\mathbf{x}^{\alpha(D)}\mathbf{z}^{\beta(D)} = \prod_{\square \in B^k_{\mathcal{C}_n}} \mathrm{var}(\square)^{\mathrm{wt}(\square)}$ (i.e. the product of the variables for all significant elements in D).
- A word $\mathrm{word}(D)$ obtained as the result of reading the letters $\sigma(\square)$ while reading the boxes of the base from right to left, top to bottom.
- The word $\mathrm{word}(D)$ is subdivided into two parts, $\mathrm{word}_x(D)$ and $\mathrm{word}_z(D)$. The first of them consists of the letters $\sigma(\square)$ from the Γ-blocks, while the second one corresponds to the staircase part of the pipe dream.
- If $\mathrm{word}(D)$ is a reduced word for a permutation $w = w(D) \in \mathcal{C}_n$, then the pipe dream D is said to be *reduced*, and the permutation w is called the *shape* of D.
  The last definition can be restated as follows. A c-signed pipe dream D is reduced, if:
  - each strand has no more than one faucet on it;
  - if two strands intersect twice, then exactly one of these strands has a faucet located between these two intersections.

  The shape of a reduced c-signed pipe dream is the permutation $w \in \mathcal{C}_n$ such that each strand connects the number i on the left of the diagram with $w(i)$ or $-w(i)$ above it. If the strand has a faucet on it, this corresponds to the minus sign; no faucet corresponds to plus.
- If $\mathrm{word}(D)$ is a reduced word for $w \in \mathcal{C}_n$, then $\mathrm{word}_x(D)$ and $\mathrm{word}_z(D)$ are reduced words for the permutations $u = u(D) \in \mathcal{C}_n$ and $v = v(D) \in \mathcal{S}_n$, respectively. Note that $uv = w$ and $\ell(u) + \ell(v) = \ell(w)$.

Example 3.2. Here are three examples of c-signed pipe dreams of size $n = 3$ with $k = 2$ Γ-blocks.



$\mathbf{x}^{\alpha(D_1)}\mathbf{z}^{\beta(D_1)} = z_1 z_2 x_1^3 x_2$
$\mathrm{word}(D_1) = s_1 s_2 s_1 s_0 s_1 s_2$
$w(D_1) = 21\overline{3} \in \mathcal{BC}_3$

$\mathbf{x}^{\alpha(D_2)}\mathbf{z}^{\beta(D_2)} = z_1^2 z_2 x_1^3 x_2$
$\mathrm{word}(D_2) = s_1 s_2 s_1 s_0 s_2 s_1 s_2$
D_2 is non-reduced.

$\mathbf{x}^{\alpha(D_3)}\mathbf{z}^{\beta(D_3)} = z_1 z_2 x_1^2 x_2^2$
$\mathrm{word}(D_3) = s_1 s_0 s_2 s_0 s_1 s_2$
D_3 is non-reduced.

If D is a reduced c-signed pipe dream, the monomial $\mathbf{z}^{\beta(D)}$ is z-admissible for $\mathrm{word}_z(D)$ (the proof is similar to the case of $\mathcal{S}_n$, cf. Section 2.4), while the monomial $\mathbf{x}^{\alpha(D)} = x_{j_1} x_{j_2} \dots x_{j_l}$ is x-admissible for the word $\mathrm{word}_x(D) = s_{a_1} s_{a_2} \dots s_{a_l}$. Indeed, $j_1 \geq j_2 \geq \dots \geq j_l$ are exactly the numbers of Γ-blocks that contain significant elements. While reading each Γ-block right to left, top to bottom the numbers a_i inside the boxes first decrease and then increase. This means that the equality $j_{i-1} = j_i = j_{i+1}$ (three neighboring significant elements are located in the same Γ-block) implies that either $a_{i-1} > a_i$, or $a_i < a_{i+1}$, and hence $i \notin P(\mathrm{word}_x(D))$.

Proposition 3.3. *Let $w \in \mathcal{C}_n$ be decomposed into a product $uv = w$, where $v \in \mathcal{S}_n$, $\ell(u)+\ell(v) = \ell(w)$. Let $\mathbf{a} = s_{a_1}s_{a_2}\dots s_{a_l} \in R(u)$ and $\mathbf{b} \in R(v)$ be reduced words, and let $\mathbf{x}^\alpha = x_{j_1}x_{j_2}\dots x_{j_l} \in \mathscr{A}_x(\mathbf{a})$, $\mathbf{z}^\beta \in \mathscr{A}_z(\mathbf{b})$ be admissible monomials, such that $k \geq j_1 \geq j_2 \dots \geq j_l$. Then there exist exactly $2^{i(\alpha)}$ reduced c-signed pipe dreams D, such that* $\mathrm{word}_x(D) = \mathbf{a}$, $\mathrm{word}_z(D) = \mathbf{b}$, $\mathbf{x}^{\alpha(D)} = \mathbf{x}^\alpha$, $\mathbf{z}^{\beta(D)} = \mathbf{z}^\beta$.

Proof. There exists exactly one placement of crosses in the staircase block (the proof is similar to the case of $\mathcal{S}_n$) and exactly two ways to place a nonzero number of significant elements in each of the Γ-blocks. Indeed, let $j_i = j_{i+1} = \dots = j_{i+m}$. Then the sequence $a_i, a_{i+1}, \dots, a_{i+m}$ has no peaks, which means that it starts with a strictly decreasing segment (since a reduced word cannot contain two consecutive identical letters) and then strictly increases. Let a_{i+t} be the smallest number in this sequence.

The significant element for a_{i+t} can be placed in two possible ways: in the horizontal or the vertical part of the j_ith Γ-block. The significant elements for $a_i, \dots, a_{i+t-1}$ can be placed only in its horizontal part, and the remaining elements corresponding to $a_{i+t+1}, \dots, a_{i+m}$ are necessarily placed in the vertical part.

Since there are exactly $i(\alpha)$ blocks containing significant elements, and each of them can be filled in two possible ways, there are $2^{i(\alpha)}$ pipe dreams for such a word. □

Denote by $\mathrm{PD}^k_{\mathcal{C}_n}(w)$ the set of all reduced c-signed pipe dreams with k Γ-blocks of shape $w \in \mathcal{C}_n$. Then the previous discussion can be summarized as the following theorem.

Theorem 3.4. *Let $w \in \mathcal{BC}_n$, $k \geq 0$. Then the k-truncated Schubert polynomial $\mathfrak{C}^k_w(\mathbf{z}, \mathbf{x})$ equals the sum of monomials over all reduced c-signed pipe dreams in $\mathrm{PD}^k_{\mathcal{C}_n}(w)$:*

$$\sum_{D \in \mathrm{PD}^k_{\mathcal{C}_n}(w)} \mathbf{x}^{\alpha(D)}\mathbf{z}^{\beta(D)} = \sum_{\substack{uv=w \\ \ell(u)+\ell(v)=\ell(w) \\ v \in \mathcal{S}_n}} \sum_{\substack{\mathbf{a} \in R(u) \\ \mathbf{x}^\alpha \in \mathscr{A}_x(\mathbf{a}) \\ \alpha_{k+1}=\alpha_{k+2}=\dots=0}} \sum_{\substack{\mathbf{b} \in R(v) \\ \mathbf{z}^\beta \in \mathscr{A}_z(\mathbf{b})}} 2^{i(\alpha)}\mathbf{x}^\alpha\mathbf{z}^\beta = \mathfrak{C}^{[k]}_w(\mathbf{z}, \mathbf{x}).$$

3.2. Type B

The Schubert polynomials $\mathfrak{B}_w$ of type B differ from $\mathfrak{C}_w$ only by a multiplicative coefficient: $\mathfrak{B}_w = 2^{-s(w)}\mathfrak{C}_w$, where $s(w)$ is the number of integers from 1 to n that change their sign under the action of w. The number of sign changes in w is equal to the number of entries of s_0 in each reduced word $\mathbf{a} \in R(w)$.

We can use the formula from Theorem 2.8 to get a similar formula for the polynomials $\mathfrak{B}_w$:

$$\mathfrak{B}_w(\mathbf{z}; \mathbf{x}) = \sum_{\substack{uv=w \\ \ell(u)+\ell(v)=\ell(w) \\ v \in \mathcal{S}_n}} \sum_{\substack{\mathbf{a} \in R(u) \\ \mathbf{x}^\alpha \in \mathscr{A}_x(\mathbf{a})}} \sum_{\substack{\mathbf{b} \in R(v) \\ \mathbf{z}^\beta \in \mathscr{A}_z(\mathbf{b})}} 2^{i(\alpha)-s(w)}\mathbf{x}^\alpha\mathbf{z}^\beta.$$

Here we describe a construction of b-signed pipe dreams. Let us modify the base for c-signed pipe dreams in the following way: in each Γ-block we make the vertical and the horizontal part overlap, so that the whole block would consist of $2n-1$ boxes. A base with k blocks will be denoted by $B^k_{\mathcal{B}_n}$. We will fill the boxes of this base in a similar way. The diagram we obtain is called a *b-signed pipe dream* (cf. Fig. 3).

For each b-signed pipe dream D the words $\mathrm{word}_x(D)$ and $\mathrm{word}_z(D)$, the monomial $\mathbf{x}^{\alpha(D)}\mathbf{z}^{\beta(D)}$, as well as the notion of reducibility, the shape $w(D)$ and the permutations $u(D)$, $v(D)$ are defined similarly to c-signed pipe dreams.

In the same way we can show that for a reduced b-signed pipe dream D the monomial $\mathbf{x}^{\alpha(D)}$ is x-admissible for the permutation $\mathrm{word}_x(D)$, and the monomial $\mathbf{z}^{\beta(D)}$ is z-admissible for $\mathrm{word}_z(D)$.

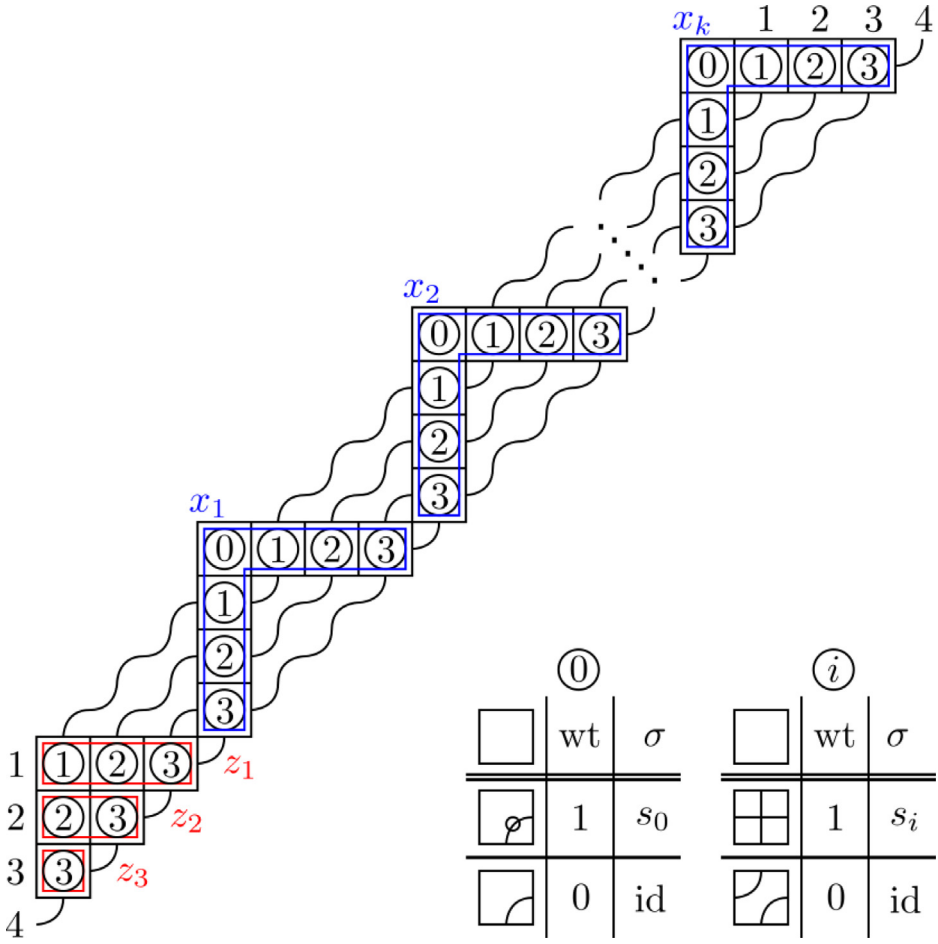
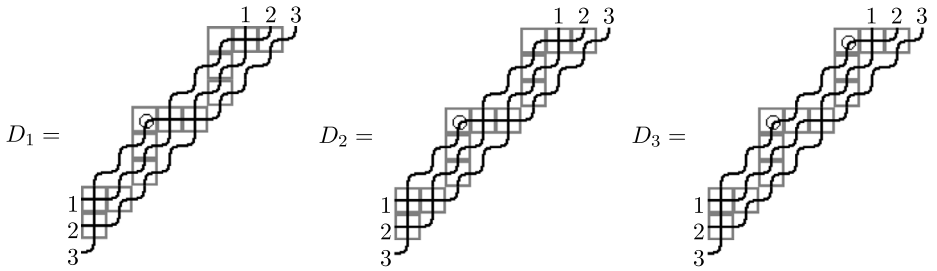


Fig. 3. The base $B^k_{\mathcal{B}_4}$ for b-signed pipe dreams.

Example 3.5. Here are three examples of b-signed pipe dreams of size $n = 3$ with $k = 2$ Γ-blocks.



$\mathbf{x}^{\alpha(D_1)}\mathbf{z}^{\beta(D_1)} = z_1 z_2 x_1^3 x_2$
$\mathrm{word}(D_1) = s_1 s_2 s_1 s_0 s_1 s_2$
$w(D_1) = 21\overline{3} \in \mathcal{BC}_3$

$\mathbf{x}^{\alpha(D_2)}\mathbf{z}^{\beta(D_2)} = z_1^2 z_2 x_1^3 x_2$
$\mathrm{word}(D_2) = s_1 s_2 s_1 s_0 s_2 s_1 s_2$
D_2 is non-reduced.

$\mathbf{x}^{\alpha(D_3)}\mathbf{z}^{\beta(D_3)} = z_1 z_2 x_1^2 x_2^2$
$\mathrm{word}(D_3) = s_1 s_0 s_2 s_0 s_1 s_2$
D_3 is non-reduced.

Proposition 3.6. *Let $w \in \mathcal{BC}_n$ be decomposed into a product $uv = w$, where $v \in \mathcal{S}_n$, $\ell(u)+\ell(v) = \ell(w)$. Let $\mathbf{a} = s_{a_1}s_{a_2}\dots s_{a_l} \in R(u)$, $\mathbf{b} \in R(v)$ be reduced words, and let $\mathbf{x}^\alpha = x_{j_1}x_{j_2}\dots x_{j_l} \in \mathscr{A}_x(\mathbf{a})$, $\mathbf{z}^\beta \in \mathscr{A}_z(\mathbf{b})$ be admissible monomials satisfying $k \geq j_1 \geq j_2 \dots \geq j_l$.*

Then there exist exactly $2^{i(\alpha)-s(w)}$ reduced b-signed pipe dreams D such that $\mathrm{word}_x(D) = \mathbf{a}$, $\mathrm{word}_z(D) = \mathbf{b}$, $\mathbf{x}^{\alpha(D)} = \mathbf{x}^\alpha$, $\mathbf{z}^{\beta(D)} = \mathbf{z}^\beta$.

Proof. We need to find the number of ways to put significant elements in all Γ-blocks. Let $j_i = j_{i+1} = \dots = j_{i+m}$. The sequence $a_i, a_{i+1}, \dots, a_{i+m}$ contains no peaks, so it first decreases and then increases. Let a_{i+t} be the minimal element of this sequence. If $a_{i+t} = 0$, we need to place an elbow with a faucet in the upper-left box of the corresponding block; otherwise $a_{i+t} \neq 0$ and the cross corresponding to this letter can be placed either in the horizontal, or in the vertical part of the Γ-block. The elements corresponding to $a_i, \dots, a_{i+t-1}$ can be placed only in the horizontal part of the block, and those corresponding to $a_{i+t+1}, \dots, a_{i+m}$ can be only in the vertical part. Hence for each nonempty block with a faucet there is a unique way to place the elements, and for blocks without faucets there are two ways. The number of nonempty blocks without faucets is equal to $i(\alpha) - s(w)$. The proposition is proved. □

Denote by $\mathrm{PD}^k_{\mathcal{B}_n}(w)$ the set of all reduced b-signed pipe dreams with k Γ-blocks of shape $w \in \mathcal{B}_n$. Then the following theorem holds.

Theorem 3.7. *Let $w \in \mathcal{BC}_n$ and $k \geq 0$. The k-truncation $\mathfrak{B}^{[k]}_w(\mathbf{z}, \mathbf{x})$ of the Schubert polynomial $\mathfrak{B}_w(\mathbf{z}, \mathbf{x})$ can be obtained as the sum of monomials over all reduced b-signed pipe dreams in $\mathrm{PD}^k_{\mathcal{B}_n}(w)$:*

$$\sum_{D \in \mathrm{PD}^k_{\mathcal{B}_n}(w)} \mathbf{x}^{\alpha(D)}\mathbf{z}^{\beta(D)} = \sum_{\substack{uv=w \\ \ell(u)+\ell(v)=\ell(w) \\ v \in \mathcal{S}_n}} \sum_{\substack{\mathbf{a} \in R(u) \\ \mathbf{x}^\alpha \in \mathscr{A}_x(\mathbf{a}) \\ \alpha_{k+1}=\alpha_{k+2}=\dots=0}} \sum_{\substack{\mathbf{b} \in R(v) \\ \mathbf{z}^\beta \in \mathscr{A}_z(\mathbf{b})}} 2^{i(\alpha)-s(w)}\mathbf{x}^\alpha\mathbf{z}^\beta = \mathfrak{B}^{[k]}_w(\mathbf{z}, \mathbf{x}).$$

3.3. Type D

The base $B^k_{\mathcal{D}_n}$ consists of a staircase block with $n-1$ stairs and k Γ-blocks of height and width $n-1$. We join the blocks by elbows  . In each box of the staircase we write the number of its diagonal. The upper left corner of each Γ-block is indexed by the symbol $1'$, while the remaining boxes are indexed by the integers $2, \dots, n$ from top to bottom and from left to right (cf. Fig. 4).

Let us fill the boxes indexed by $1'$ by elements of four possible types: elbows , crosses  , crosses with a faucet and elbows with two faucets . All the remaining boxes will be filled by crosses and elbows . The object we obtain will be called a *d-signed pipe dream*. As in the previous cases, it consists of strands that connect the left edge with the top one. Let us index these edges by the integers from 1 to n. To each box □ of the base we assign the following data:

- the weight $\mathrm{wt}(\square)$ of the element inside it. It is equal to 2 for elbows with two faucets , 1 for crosses and crosses with a faucet , and 0 for elbows .
- zero, one or two letters $\sigma(\square) \in \mathcal{BC}_n$. This set of letters is equal to s_i for a cross in a box indexed by i, to s_1 for a cross in a box indexed by $1'$, to $s_{\hat{1}}$ for a cross with a faucet , to a pair of letters $s_1s_{\hat{1}} = s_{\hat{1}}s_1$ for an elbow with two crosses and nothing (the identity permutation id) for elbows .
- a variable $\mathrm{var}(\square)$. It is equal to z_i for a box in the ith row of the staircase (counted from above) and to x_i for a box in the ith Γ-block (counted from below).

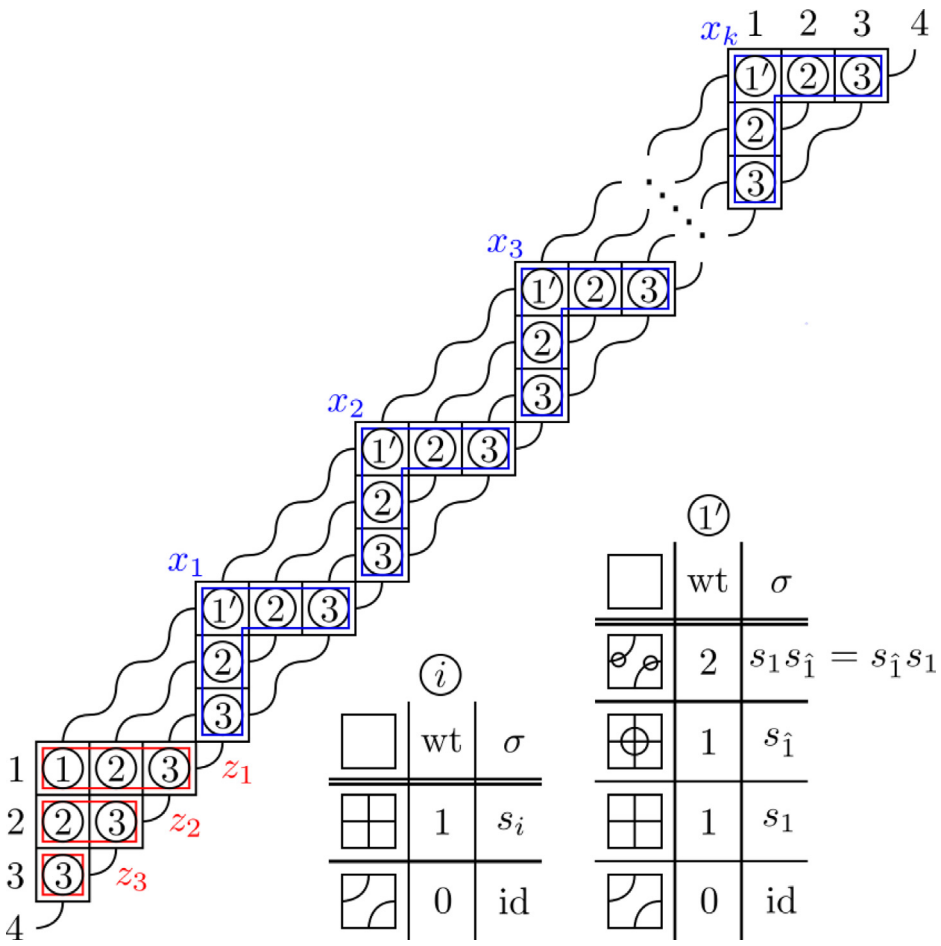


Fig. 4. The base $B^k_{\mathcal{D}_4}$ for d-signed pipe dreams.

To each d-signed pipe dream D we assign:

- a monomial $\mathbf{x}^{\alpha(D)}\mathbf{z}^{\beta(D)} = \prod_{\square \in B^k_{\mathcal{D}_n}} \mathrm{var}(\square)^{\mathrm{wt}(\square)}$;
- the number $r(D)$ of elbow joints with two faucets occuring in D;
- a set of words $\mathrm{word}^1(D), \mathrm{word}^2(D), \ldots, \mathrm{word}^{2^{r(D)}}(D)$. Each of these words is obtained by consecutive reading of all the letters $\sigma(\square)$, where $\square$ runs over the set of all boxes of the base from right to left, top to bottom. For each elbow with two faucets there are two possibilities to read the pair of commuting letters $s_1, s_{\hat{1}}$, so the total number of words obtained is equal to $2^{r(D)}$.
- Each of these words $\mathrm{word}^p(D)$ can be split into two subwords $\mathrm{word}^p_x(D)$ and $\mathrm{word}_z(D)$ (the latter word is the same for all p). The first word is obtained by reading the letters of Γ-blocks, and the second one corresponds to the staircase block.
- If $\mathrm{word}^p(D)$ (and hence all the other words) is a reduced word for the permutation $w = w(D) \in \mathcal{D}_n$, the pipe dream D is said to be *reduced*, and the permutation w is called the *shape* of this pipe dream.

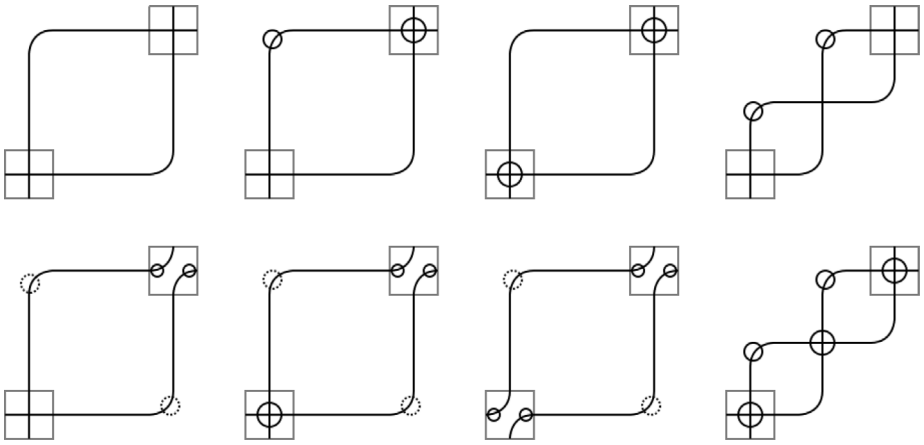


Fig. 5. Forbidden patterns not occuring in the reduced d-signed pipe dreams (dashed faucets may either be or not be there).

- If $\mathrm{word}^p(D)$ is a reduced word for $w \in \mathcal{D}_n$, then $\mathrm{word}^p_x(D)$ and $\mathrm{word}_z(D)$ are reduced words for the permutations $u = u(D) \in \mathcal{D}_n$ and $v = v(D) \in \mathcal{S}_n$ respectively. Note that $uv = w$ and $\ell(u) + \ell(v) = \ell(w)$.

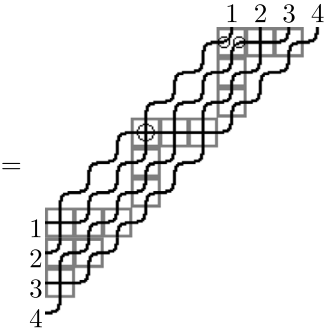
Remark 3.8. Let us restate the definition of a reduced d-signed pipe dream in terms of the intersection of the strands; it is somewhat more involved than in the previous cases. Consider two strands that intersect twice, in the boxes a and b. A d-signed pipe dream is *nonreduced* if it contains any of the eight patterns shown in Fig. 5 (the boxes a and b are highlighted by gray frames):

- the boxes a and b contain crosses $+$, and the segments of both strands between the intersection do not have faucets on them;
- the boxes a and b contain crosses $+$, and the segments of both strands between the intersection have faucets on them;
- the boxes a and b contain crosses with faucets, and the segments of both strands between the intersection do not have faucets on them;
- the boxes a and b contain crosses with faucets, and the segments of both strands between the intersection have faucets on them;
- one of the boxes a, b contains a cross $+$, and the other contains a cross with a faucet. Exactly one of the two strands has a faucet between these two boxes;
- one of the boxes a, b contains a cross $+$, and the other contains an elbow with two faucets;
- one of the boxes a, b contains a cross with a faucet, and the other contains an elbow with two faucets;
- each of the boxes a and b contains an elbow with two faucets.

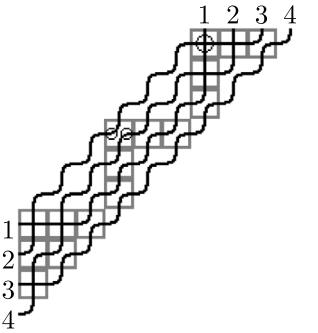
In the last three cases the existence of an extra faucet on the segments situated between a and b is not important. If a d-signed pipe dream does not contain any of these eight "forbidden" patterns, such a pipe dream is *reduced*.

The shape of a reduced d-signed pipe dream D can be read in a similar way as in the cases B and C: it is the permutation $w = w(D) \in \mathcal{D}_n$ such that for each $i = 1, \dots, n$ the number i on the left-hand side is joined with $(-1)^{c_i(D)} w(i)$ on the top side, where $c_i(D)$ is the number of crosses on the ith strand.

Example 3.9. Here are two examples of d-signed pipe dreams of size $n = 4$ with $k = 2$ Γ-blocks.

$D_1 =$ 

$$\mathbf{x}^{\alpha(D_1)}\mathbf{z}^{\beta(D_1)} = z_1 z_3 x_1^3 x_2^3$$
$$\mathrm{word}_1(D_1) = s_2 s_1 s_{\hat{1}} s_3 s_2 s_{\hat{1}} s_1 s_3$$
$$\mathrm{word}_2(D_1) = s_2 s_{\hat{1}} s_1 s_3 s_2 s_{\hat{1}} s_1 s_3$$
$$w(D_1) = 1\overline{4}2\overline{3} \in \mathcal{D}_4$$

$D_2 =$ 

$$\mathbf{x}^{\alpha(D_2)}\mathbf{z}^{\beta(D_2)} = z_1^2 z_3 x_1^2 x_2^3$$
$$\mathrm{word}_1(D_2) = s_2 s_{\hat{1}} s_2 s_1 s_{\hat{1}} s_2 s_1 s_3$$
$$\mathrm{word}_2(D_2) = s_2 s_{\hat{1}} s_2 s_{\hat{1}} s_1 s_2 s_1 s_3$$
D_2 is non-reduced

Similarly to the previous cases we can show that if D is a reduced d-signed pipe dream, the monomial $\mathbf{z}^{\beta(D)}$ is z-admissible for $\mathrm{word}_z(D)$, and the monomial $\mathbf{x}^{\alpha(D)}$ is x-admissible for each of the words $\mathrm{word}_x^1(D), \ldots, \mathrm{word}_x^{2^{r(D)}}(D)$.

Proposition 3.10. *Let $w \in \mathcal{D}_n$ be decomposed into a product $uv = w$, where $v \in \mathcal{S}_n$ and $\ell(u)+\ell(v) = \ell(w)$. Let $\mathbf{a} = s_{a_1} s_{a_2} \ldots s_{a_l} \in R(u)$ and $\mathbf{b} \in R(v)$ be reduced words, and let $\mathbf{x}^\alpha = x_{j_1} x_{j_2} \ldots x_{j_l} \in \mathscr{A}_x(\mathbf{a})$ and $\mathbf{z}^\beta \in \mathscr{A}_z(\mathbf{b})$ be admissible monomials, and $k \geq j_1 \geq j_2 \ldots \geq j_l$.*

Let $r(\mathbf{a}, \alpha)$ denote the number of i such that $j_i = j_{i+1}$, $a_i = 1$, $a_{i+1} = \hat{1}$ or, vice versa, $a_i = \hat{1}$, $a_{i+1} = 1$.

Then there exist exactly $2^{i(\alpha)-o(\mathbf{a})+r(\mathbf{a},\alpha)}$ reduced d-signed pipe dreams D such that $\mathrm{word}_x^p(D) = \mathbf{a}$ for some p, $\mathrm{word}_z(D) = \mathbf{b}$, $\mathbf{x}^{\alpha(D)} = \mathbf{x}^\alpha$, and $\mathbf{z}^{\beta(D)} = \mathbf{z}^\beta$. For each of these pipe dreams D we have $r(D) = r(\mathbf{a}, \alpha)$.

Proof. Similarly to the previous cases we can show that there exists a unique way to place the crosses in the staircase block. We need to find the number of ways of placing the elements in each of the Γ-blocks. Let $j_i = j_{i+1} = \cdots = j_{i+m}$. We distinguish between the following cases:

- the sequence $a_i, \ldots, a_{i+m}$ does not contain elements equal to 1 or $\hat{1}$. There are two possibilities for putting the cross corresponding to the smallest element a_{i+t} of the sequence: it can be located in the horizontal or in the vertical part of the block. The remaining elements are placed in a unique way.
- The sequence $a_i, \ldots, a_{i+m}$ contains 1, but does not contain $\hat{1}$. Then the top left corner of the j_ith block contains a cross  ; all the remaining elements are placed uniquely.
- The sequence $a_i, \ldots, a_{i+m}$ contains $\hat{1}$ and does not contain 1. Then we put a cross with a faucet  into the corner of the j_ith block; the remaining elements are placed uniquely.
- The sequence $a_i, \ldots, a_{i+m}$ contains both $\hat{1}$ and 1. Then we place an elbow with two faucets  into the corner of the j_ith block; the positions of the remaining elements are determined uniquely.

This means that for $i(\alpha) - o(\mathbf{a}) + r(\mathbf{a}, \alpha)$ blocks there are two possibilities to place the elements, and all the remaining blocks are filled uniquely. This concludes the proof. □

Denote by $\mathrm{PD}^k_{\mathcal{D}_n}(w)$ the set of all reduced d-signed pipe dreams with k Γ-blocks of shape $w \in \mathcal{D}_n$. Since each pipe dream D corresponds to $2^{r(D)}$ pairs $(\mathbf{a}, \mathbf{x}^\alpha)$, where $\mathbf{a}$ is a reduced word, and $\mathbf{x}^\alpha$ is an

admissible monomial, each such pair corresponds to $2^{i(\alpha)-o(\mathbf{a})+r(\mathbf{a},\alpha)}$ pipe dreams, and $r(D) = r(\mathbf{a}, \alpha)$, this gives us the following theorem.

Theorem 3.11. *Let $w \in \mathcal{D}_n$ and $k \geq 0$. Then the k-truncation $\mathfrak{D}_w^{[k]}(\mathbf{z}, \mathbf{x})$ of the Schubert polynomial $\mathfrak{D}_w(\mathbf{z}, \mathbf{x})$ is equal to the sum of monomials over all reduced d-signed pipe dreams in $\mathrm{PD}^k_{\mathcal{D}_n}(w)$:*

$$\sum_{D\in \mathrm{PD}^k_{\mathcal{D}_n}(w)} \mathbf{x}^{\alpha(D)}\mathbf{z}^{\beta(D)} = \sum_{\substack{uv=w\\ \ell(u)+\ell(v)=\ell(w)\\ v\in S_n}} \sum_{\substack{\mathbf{a}\in R(u)\\ \mathbf{x}^\alpha \in \mathscr{A}_X(\mathbf{a})\\ \alpha_{k+1}=\alpha_{k+2}=\cdots=0}} \sum_{\substack{\mathbf{b}\in R(v)\\ \mathbf{z}^\beta \in \mathscr{A}_Z(\mathbf{b})}} 2^{i(\alpha)-o(\mathbf{a})}\mathbf{x}^\alpha\mathbf{z}^\beta = \mathfrak{D}_w^{[k]}(\mathbf{z}, \mathbf{x}).$$

3.4. Infinite pipe dreams

Let $\mathcal{F}_n$ denote either $\mathcal{BC}_n$ or $\mathcal{D}_n$. Denote by $\mathfrak{F}_w(\mathbf{x}, \mathbf{z})$ the Schubert polynomial of type B, C or D respectively for $w \in \mathcal{F}_n$.

We can consider bases of pipe dreams with countably many Γ-blocks, going infinitely in the northeastern direction:

$$B_{\mathcal{F}_n} := \bigcup_{k\in\mathbb{N}} B^k_{\mathcal{F}_n}.$$

These bases can be filled with elements according to the same rules as in the finite case. Reducibility is defined the same way as above. Note that if a signed pipe dream D of arbitrary type is reduced, it contains finitely many significant elements. This means that for a reduced infinite pipe dreams D the monomial $\mathbf{x}^{\alpha(D)}\mathbf{z}^{\beta(D)}$ and the shape $w = w(D) \in \mathcal{F}_n$ are well defined.

Denote by $\mathrm{PD}_{\mathcal{F}_n}(w)$ the set of reduced infinite signed pipe dreams of shape $w \in \mathcal{F}_n$:

$$\mathrm{PD}_{\mathcal{F}_n}(w) = \bigcup_{k\in\mathbb{N}} \mathrm{PD}^k_{\mathcal{F}_n}(w)$$

(finite pipe dreams are assumed to have an infinite "tail" of elbows ). Since Schubert polynomials equal the projective limits of their truncations

$$\mathfrak{F}_w(\mathbf{x}, \mathbf{z}) = \varprojlim_{k\to\infty} \mathfrak{F}_w^{[k]}(\mathbf{x}, \mathbf{z}),$$

they are equal to the sums of monomials over infinite signed pipe dreams:

Corollary 3.12. *For each $w \in \mathcal{F}_n$, we have*

$$\mathfrak{F}_w(\mathbf{x}, \mathbf{z}) = \sum_{D\in \mathrm{PD}_{\mathcal{F}_n}(w)} \mathbf{x}^{\alpha(D)}\mathbf{z}^{\beta(D)}.$$

3.5. Double Schubert polynomials

Double Schubert polynomials for the classical groups were introduced in [11]. They are elements of the ring $\mathbb{Q}[\mathbf{z}, p_1(\mathbf{x}), p_3(\mathbf{x}), \ldots, \mathbf{t}]$, where $\mathbf{t} = t_1, t_2, \ldots$ is another countable set of variables, responsible for the action of a maximal torus of the corresponding classical group.

This ring is equipped with an action of the group $\mathcal{BC}_\infty \times \mathcal{BC}_\infty$ and hence of its index 4 subgroup $\mathcal{D}_\infty \times \mathcal{D}_\infty$. The first copy of $\mathcal{BC}_\infty$ acts as before, permuting the variables z_i:

$$\begin{aligned} s_i^z f(z_1, \ldots, z_i, z_{i+1}, \ldots; x_1, x_2, \ldots; t_1, t_2, \ldots) &= f(z_1, \ldots, z_{i+1}, z_i, \ldots; x_1, x_2, \ldots; t_1, t_2, \ldots) \text{ for } i \geq 1;\\ s_0^z f(z_1, z_2, z_3, \ldots; x_1, x_2, \ldots; t_1, t_2, \ldots) &= f(-z_1, z_2, z_3 \ldots; z_1, x_1, x_2, \ldots; t_1, t_2, \ldots);\\ s_{\hat{1}}^z f(z_1, z_2, z_3 \ldots; x_1, x_2, \ldots; t_1, t_2, \ldots) &= f(-z_2, -z_1, z_3 \ldots; z_1, z_2, x_1, x_2, \ldots; t_1, t_2, \ldots). \end{aligned}$$

(recall that $s_{\hat{1}} = s_0 s_1 s_0$).

The second copy of $\mathcal{BC}_\infty$ acts by permuting the variables t_i:

$$s_i^t f(z_1, z_2 \dots; x_1, x_2, \dots; t_1, \dots, t_i, t_{i+1}, \dots) = f(z_1, z_2 \dots; x_1, x_2, \dots; t_1, \dots, t_{i+1}, t_i, \dots) \text{ for } i \geq 1;$$
$$s_0^t f(z_1, z_2, \dots; x_1, x_2, \dots; t_1, t_2, t_3, \dots) = f(z_1, z_2, \dots; -t_1, x_1, x_2, \dots; -t_1, t_2, t_3, \dots);$$
$$s_{\hat{1}}^t f(z_1, z_2, \dots; x_1, x_2, \dots; t_1, t_2, t_3 \dots) = f(z_1, z_2, z_3 \dots; -t_1, -t_2, x_1, x_2, \dots; -t_2, -t_1, t_3, \dots).$$

It is easy to check explicitly that this defines an action, and that the actions of the two copies of $\mathcal{BC}_\infty$ commute.

This allows us to define two families of divided difference operators. Following [11], we call them ∂_i and δ_i.

Definition 3.13 (*[11, Sec. 2.5]*)**.** Let us define divided difference operators ∂ and δ acting on the ring $\mathbb{Q}[\mathbf{z}, p_1(\mathbf{x}), p_3(\mathbf{x}), \dots, \mathbf{t}]$ as follows:

$$\partial_i f = \frac{f - s_i^z f}{z_i - z_{i+1}}, \quad \delta_i f = \frac{f - s_i^t f}{t_{i+1} - t_i}$$

for $i \geq 1$;

$$\partial_0 f = \frac{f - s_0^z f}{-2z_1}, \quad \delta_0 f = \frac{f - s_0^t f}{2t_1};$$
$$\partial_0^B f = 2\partial_0 f = \frac{f - s_0^z f}{-z_1}, \quad \delta_0^B f = 2\delta_0 f = \frac{f - s_0^t f}{t_1};$$
$$\partial_{\hat{1}} f = \frac{f - s_{\hat{1}}^z f}{-z_1 - z_2}, \quad \delta_{\hat{1}} f = \frac{f - s_{\hat{1}}^t f}{t_1 + t_2}.$$

Definition 3.14. Double Schubert polynomials $\mathfrak{B}_w(\mathbf{z}, \mathbf{x}, \mathbf{t})$, $\mathfrak{C}_w(\mathbf{z}, \mathbf{x}, \mathbf{t})$ and $\mathfrak{D}_w(\mathbf{z}, \mathbf{x}, \mathbf{t})$ (as before, we denote by $\mathfrak{F}_w(\mathbf{z}, \mathbf{x}, \mathbf{t})$ any of these three polynomials) are elements of the ring $\mathbb{Q}[\mathbf{z}, p_1(\mathbf{x}), p_3(\mathbf{x}), \dots, \mathbf{t}]$ that are indexed by the permutations $w \in \mathcal{BC}_\infty$ or $w \in \mathcal{D}_\infty$ and satisfy the equations

$$\mathfrak{F}_{id} = 1,$$

$$\partial_i \mathfrak{F}_w = \begin{cases} \mathfrak{F}_{ws_i}, & \text{if } \ell(ws_i) < \ell(w), \\ 0, & \text{if } \ell(ws_i) > \ell(w), \end{cases} \quad \delta_i \mathfrak{F}_w = \begin{cases} \mathfrak{F}_{s_i w}, & \text{if } \ell(s_i w) < \ell(w), \\ 0, & \text{if } \ell(s_i w) > \ell(w), \end{cases}$$

for each $i = 1, 2, \dots$,

$$\partial_0 \mathfrak{C}_w = \begin{cases} \mathfrak{C}_{ws_0}, & \text{if } \ell(ws_0) < \ell(w), \\ 0, & \text{if } \ell(ws_0) > \ell(w), \end{cases} \quad \delta_0 \mathfrak{C}_w = \begin{cases} \mathfrak{C}_{s_0 w}, & \text{if } \ell(s_0 w) < \ell(w), \\ 0, & \text{if } \ell(s_0 w) > \ell(w), \end{cases}$$
$$\partial_0^B \mathfrak{B}_w = \begin{cases} \mathfrak{B}_{ws_0}, & \text{if } \ell(ws_0) < \ell(w), \\ 0, & \text{if } \ell(ws_0) > \ell(w), \end{cases} \quad \delta_0^B \mathfrak{B}_w = \begin{cases} \mathfrak{B}_{s_0 w}, & \text{if } \ell(s_0 w) < \ell(w), \\ 0, & \text{if } \ell(s_0 w) > \ell(w), \end{cases}$$
$$\partial_{\hat{1}} \mathfrak{D}_w = \begin{cases} \mathfrak{D}_{ws_{\hat{1}}}, & \text{if } \ell\left(ws_{\hat{1}}\right) < \ell(w), \\ 0, & \text{if } \ell\left(ws_{\hat{1}}\right) > \ell(w), \end{cases} \quad \delta_{\hat{1}} \mathfrak{D}_w = \begin{cases} \mathfrak{D}_{s_{\hat{1}} w}, & \text{if } \ell\left(s_{\hat{1}} w\right) < \ell(w), \\ 0, & \text{if } \ell\left(s_{\hat{1}} w\right) > \ell(w). \end{cases}$$

3.6. Double Schubert polynomials as sums over pipe dreams

The goal of this subsection is to express double Schubert polynomials of types B, C, and D as sums over the set of pipe dreams. We start with recalling the following theorem, which expresses double Schubert polynomials via the ordinary ones.

Theorem 3.15 (*Cauchy Expansion Formula, [11, Cor. 8.10]*)**.** *Double Schubert polynomials $\mathfrak{F}_w(\mathbf{z}, \mathbf{x}, \mathbf{t})$ satisfy the equation*

$$\mathfrak{F}_w(\mathbf{z}, \mathbf{x}, \mathbf{t}) = \sum_{\substack{vu = w \\ \ell(v) + \ell(u) = \ell(w) \\ v \in \mathcal{S}_n}} \mathfrak{S}_{v^{-1}}(-\mathbf{t}) \mathfrak{F}_u(\mathbf{z}, \mathbf{x}).$$

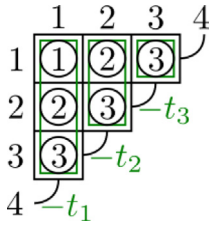


Fig. 6. The base for pipe dreams of size $n = 4$ indicating the variables var'.

The following easy proposition describes the pipe dreams for the inverse permutation.

Proposition 3.16. *Let $v \in \mathcal{S}_n$, and let $\square$ be a box in the base $B_{\mathcal{S}_n}$. Let $\mathrm{var}'(\square) = -t_j$, where j is the number of the column containing this box. To a pipe dream D we can assign a monomial*

$$(-\mathbf{t})^{\beta'(D)} = \prod_{\square \in B_{\mathcal{S}_n}} \mathrm{var}'(\square)^{\mathrm{wt}(\square)}$$

(the product of all the variables $-t_j$ over all the crosses in D). Then

$$\mathfrak{S}_{v^{-1}}(-\mathbf{t}) = \sum_{D \in \mathrm{PD}_A(v)} (-\mathbf{t})^{\beta'(D)}.$$

Proof. Denote by D^T the *transposed* pipe dream: the pipe dream obtained by reflecting D with respect to the NW-SE diagonal. It is easy to see that $D \in \mathrm{PD}_A(v)$ iff $D^T \in \mathrm{PD}_A(v^{-1})$. The proposition follows from this assertion and Theorem 2.11. $\square$

As before, let us denote the truncations of Schubert polynomials as follows:

$$\mathfrak{F}_w^{[k]}(\mathbf{z}, \mathbf{x}, \mathbf{t}) = \mathfrak{F}_w(\mathbf{z}, x_1, x_2, \ldots, x_k, 0, 0, \ldots, \mathbf{t}).$$

Theorems 3.4, 3.7, 3.11 and Proposition 3.16 imply that the formula from Theorem 3.15 can be rewritten as follows:

Corollary 3.17. *Let $w \in \mathcal{F}_n$ and $k \geq 0$. Then we have*

$$\mathfrak{F}_w^{[k]}(\mathbf{z}, \mathbf{x}, \mathbf{t}) = \sum_{\substack{vu = w \\ \ell(v)+\ell(u)=\ell(w) \\ v \in \mathcal{S}_n}} \sum_{D' \in \mathrm{PD}^k_{\mathcal{F}_n}(u)} \sum_{D'' \in \mathrm{PD}_A(v)} \mathbf{x}^{\alpha(D')} \mathbf{z}^{\beta(D')} (-\mathbf{t})^{\beta'(D'')}.$$

The base for double Schubert polynomials $DB^k_{\mathcal{F}_n}$ can be obtained by adding another staircase of size $(n-1, \ldots, 2, 1)$ (shown in Fig. 6) in the northeastern part of the base $B^k_{\mathcal{F}_n}$ and joining them by elbows  (cf. Fig. 7). The boxes of the "double" base $DB^k_{\mathcal{F}_n}$ are filled according to the same rules as in the case of $B^k_{\mathcal{F}_n}$; the object obtained is called a *b-, c- or d-double signed pipe dream*.

For a double signed pipe dream D of any of the three types we can define similarly to the ordinary case the following data:

- a monomial $\mathbf{x}^{\alpha(D)} \mathbf{z}^{\beta(D)} (-\mathbf{t})^{\beta'(D)}$;
- the reducibility property;
- for D reduced, the shape $w = w(D) \in \mathcal{F}_n$.

We can subdivide the base $DB^k_{\mathcal{F}_n}$ into the base for usual signed pipe dreams $B^k_{\mathcal{F}_n}$ and the staircase $B'_{\mathcal{S}_n}$ in the northeastern part. So, if D is reduced and its shape is equal to $w \in \mathcal{F}_n$, it is subdivided into $D' \in \mathrm{PD}^k_{\mathcal{F}_n}(u)$ and $D'' \in \mathrm{PD}_A(v)$. Obviously, $\ell(u) + \ell(v) = \ell(w)$ and $vu = w$.

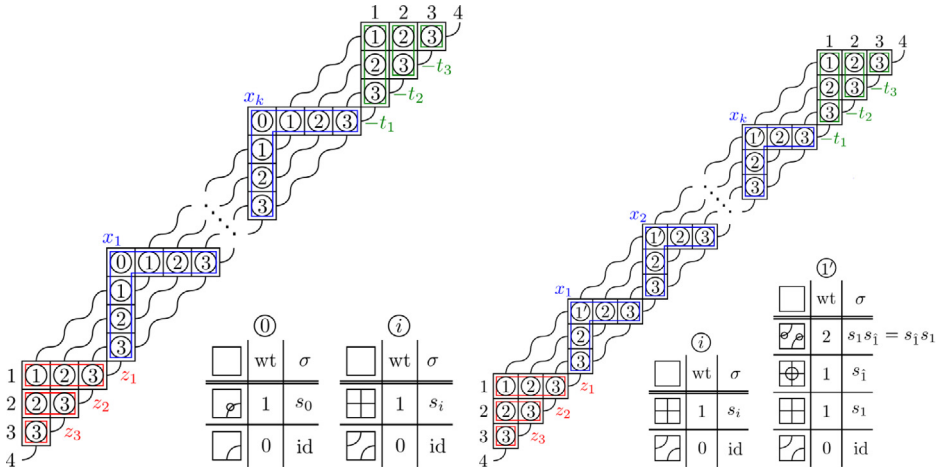


Fig. 7. Bases $DB^k_{\mathcal{B}_4}$ and $DB^k_{\mathcal{D}_4}$ for b- and d-double signed pipe dreams.

The converse is also true: if $\ell(u) + \ell(v) = \ell(w)$ and $vu = w, v \in \mathcal{S}_n$, then the pipe dreams $D' \in \mathrm{PD}^k_{\mathcal{F}_n}(u)$ and $D'' \in \mathrm{PD}_A(v)$ add up to a reduced double signed pipe dream of shape w.

Denote by $\mathrm{DPD}^k_{\mathcal{F}_n}(w)$ the set of all reduced double signed pipe dreams of type B, C or D of shape $w \in \mathcal{F}_n$. Then the formula from Corollary 3.17 can be rewritten as follows:

Corollary 3.18. *If $w \in \mathcal{F}_n$ and $k \geq 0$, the k-truncation of a double Schubert polynomial is obtained as the sum of monomials over all reduced double signed pipe dreams of a given shape:*

$$\mathfrak{F}^{[k]}_w(\mathbf{z}, \mathbf{x}, \mathbf{t}) = \sum_{D \in \mathrm{DPD}^k_{\mathcal{F}_n}(w)} \mathbf{x}^{\alpha(D)} \mathbf{z}^{\beta(D)} (-\mathbf{t})^{\beta'(D)}.$$

Remark 3.19. We can consider the base for double signed pipe dreams with countably many Γ-blocks in the middle. This allows us to obtain double Schubert polynomials $\mathfrak{F}_w(\mathbf{z}, \mathbf{x}, \mathbf{t})$ as sums of monomials over infinite reduced double signed pipe dreams.

3.7. Examples

Example 3.20. Let $w = s_0 \in \mathcal{BC}_\infty$. All pipe dreams $D \in \mathrm{PD}_{\mathcal{B}_\infty}(s_0)$ have exactly one significant element: a faucet in a corner of any Γ-block. Hence

$$\mathfrak{B}_{s_0}(\mathbf{z}, \mathbf{x}) = \sum x_i = p_1(\mathbf{x});$$
$$\mathfrak{C}_{s_0}(\mathbf{z}, \mathbf{x}) = 2 \sum x_i = 2p_1(\mathbf{x}).$$

Example 3.21. Let $w = s_i \in \mathcal{BC}_\infty$ and $i \geq 1$, or $w = s_i \in \mathcal{D}_\infty$ and $i \geq 2$. Each pipe dream (of any type) of this shape has a unique significant element: a cross in a box labeled by i. It can be located either in one of the first i rows of the staircase block, or in any of the two parts of any Γ-block. Hence

$$\mathfrak{B}_{s_i}(\mathbf{z}, \mathbf{x}) = \mathfrak{C}_{s_i}(\mathbf{z}, \mathbf{x}) = \mathfrak{D}_{s_i}(\mathbf{z}, \mathbf{x}) = z_1 + z_2 + \cdots + z_i + 2p_1(\mathbf{x}).$$

Now consider double signed pipe dreams of shape s_i. The cross can be also located in one of the first i columns of the upper staircase block, so

$$\mathfrak{B}_{s_i}(\mathbf{z},\mathbf{x},\mathbf{t}) = \mathfrak{C}_{s_i}(\mathbf{z},\mathbf{x},\mathbf{t}) = \mathfrak{D}_{s_i}(\mathbf{z},\mathbf{x},\mathbf{t}) = (z_1 - t_1) + (z_2 - t_2) + \cdots + (z_i - t_i) + 2p_1(\mathbf{x}).$$

Example 3.22. In a reduced d-signed pipe dream of shape $s_1 \in \mathcal{D}_\infty$ the unique cross can be located either in the corner of the staircase block, or in the corner of any Γ-block. In the case of d-double signed pipe dreams the cross can also be situated in the corner of the upper staircase block. This means that

$$\mathfrak{D}_{s_1}(\mathbf{z},\mathbf{x}) = z_1 + p_1(\mathbf{x});$$
$$\mathfrak{D}_{s_1}(\mathbf{z},\mathbf{x},\mathbf{t}) = (z_1 - t_1) + p_1(\mathbf{x}).$$

Now consider a reduced d-signed pipe dream of shape $s_{\hat{1}} \in \mathcal{D}_n$. They contain one cross with a faucet, which can be situated only in the corner of a Γ-blocks, hence

$$\mathfrak{D}_{s_{\hat{1}}}(\mathbf{z},\mathbf{x}) = \mathfrak{D}_{s_{\hat{1}}}(\mathbf{z},\mathbf{x},\mathbf{t}) = p_1(\mathbf{x}).$$

Example 3.23. Let $w = s_1 s_{\hat{1}} \in \mathcal{D}_\infty$. If D is a reduced d-signed pipe dream of shape w, the following cases may occur:

- There is a cross in the corner of the staircase block and a cross with a faucet in the corner of the ith Γ-block. For each i we obtain a summand $z_1 x_i$.
- In the corner of the ith Γ-block there is a cross, and in the jth Γ-block there is a cross with a faucet. For each $i \neq j$ this gives us a summand $x_i x_j$.
- In the corner of the ith Γ-block there is an elbow joint with two faucets. For each i we obtain a summand x_i^2.

So we have

$$\mathcal{D}_{s_1 s_{\hat{1}}}(\mathbf{z},\mathbf{x}) = z_1 \sum_i x_i + \sum_i x_i^2 + \sum_{i \neq j} x_i x_j = z_1 p_1(\mathbf{x}) + p_1^2(\mathbf{x}).$$

If D is a reduced d-double signed pipe dream, another possibility occurs: a cross can be located in the corner of the upper staircase block, while a cross with a faucet is in the corner of the ith Γ-block. These cases provide summands of the form $-t_1 x_i$, hence

$$\mathcal{D}_{s_1 s_{\hat{1}}}(\mathbf{z},\mathbf{x},\mathbf{t}) = (z_1 - t_1) p_1(\mathbf{x}) + p_1^2(\mathbf{x}).$$

Example 3.24. Let $w = s_1 s_0 s_1 = 1\bar{2} \in \mathcal{BC}_\infty$. If $D \in \mathrm{PD}_{\mathcal{B}_2}(w)$, the pipe dream D has two crosses in the boxes indexed by 1 and a faucet between them. The following cases may occur:

- The lower cross is in the corner of the staircase block, the faucet and the upper cross are in the same (ith) Γ-block. For each i we obtain $z_1 x_i^2$.
- The lower cross is in the corner of the staircase block, the faucet is in the ith Γ-block, the upper cross is in any of the two parts of the jth Γ-block. For each $i < j$ we obtain a summand $2z_1 x_i x_j$.
- All three elements are situated in the ith Γ-block. For each i we obtain a summand x_i^3.
- The lower cross is in any of the two parts of the ith Γ-block, the upper cross and the faucet are in the jth Γ-block. For each $i < j$ we get a summand $2x_i x_j^2$.
- The lower cross and the faucet are in the ith Γ-block, and the upper cross is in any of the two parts of the jth Γ-block. For each $i < j$ we have $2x_i^2 x_j$.
- The lower cross is in any of the two parts of the ith Γ-block, the faucet is in the jth Γ-block, the upper cross is in any of the two parts of the kth Γ-block. For each $i < j < k$ we have a summand $4x_i x_j x_k$.

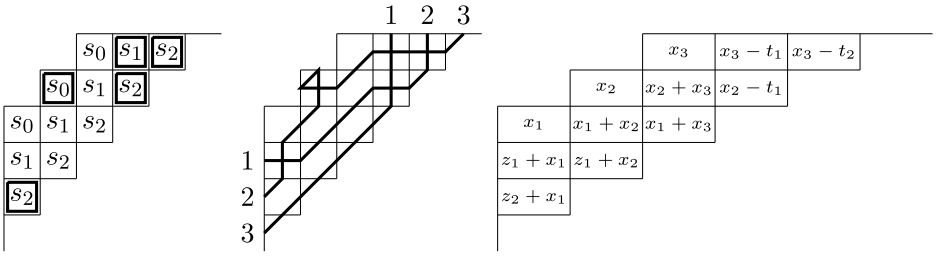


Fig. 8. An extended excited Young diagram.

Summarizing, we obtain

$$\mathfrak{B}_{s_1 s_0 s_1}(\mathbf{z}, \mathbf{x}) = z_1 \sum_i x_i^2 + 2z_1 \sum_{i<j} x_i x_j + \sum_i x_i^3 +$$
$$+ 2\sum_{i<j} x_i^2 x_j + 2\sum_{i<j} x_i x_j^2 + 4\sum_{i<j<k} x_i x_j x_k = z_1 p_1^2(\mathbf{x}) + \frac{2}{3} p_1^3(\mathbf{x}) + \frac{1}{3} p_3(\mathbf{x}).$$

4. Relation to extended excited Young diagrams

4.1. The construction by Kirillov and Naruse

In this section we compare our construction to the construction of pipe dreams described by An. Kirillov and H. Naruse in [12] by means of "extended excited Young diagrams". We refer the reader to [12, Sec. 8.2].

Let us recall their construction for type B. They start with a trapezoidal skew Young diagram[1] consisting of a staircase block $(n-1, n-2, \ldots, 2, 1)$ with n rows of length n above it, with each upper row shifted by one to the right with respect to the row below it, see Fig. 8. This diagram is called an extended excited Young diagram (extended EYD for short). A pipe dream is a subset of boxes of this diagram, marked by crosses.

Each box of an extended EYD corresponds to the sum of two variables: $z_i + x_j$, $x_i + x_j$, or $x_i - t_j$ (our variables z_i and $-t_j$ are denoted in [12] by a_i and b_j, respectively). The *weight* of a pipe dream is the product of all such binomials corresponding to the boxes marked by crosses.

Moreover, each box of the extended EYD corresponds to a simple reflection in the corresponding Weyl group $s_0, \ldots, s_{n-1}$ for the types B and C, $s_1, s_{\hat{1}}, s_2, \ldots, s_{n-1}$ for the type D. The *permutation* corresponding to a pipe dream is the product of all simple reflections corresponding to the marked boxes, read from right to left and from top to bottom.

Thus, the (truncated) Schubert polynomial $\mathcal{F}_w^{[n]}(\mathbf{z}, \mathbf{x}, \mathbf{t})$ is equal to the sum of the weights for all reduced pipe dreams with the permutation w (see [12, Thm 5, Thm 6]).

Here we reproduce Example 8 from the aforementioned paper.

Example 4.1. Consider the signed permutation $w = 2\bar{3}1 = s_2 s_1 s_2 s_0 s_1 \in \mathcal{BC}_3$. Here is an example of pipe dream corresponding to this permutation. Its weight is equal to $(x_3 - t_2)(x_3 - t_1)(x_2 - t_1)x_2(z_2 + x_1)$.

The boxes of each diagonal of this skew Young diagram are indexed by the corresponding simple reflection; this allows us to assign to an extended EYD an element of $\mathcal{F}_n$. There is another

[1] We turn the figures from [12] 90 degrees clockwise; this allows us to draw Young diagrams in the English notation, as opposed to the French one, used in the cited paper.

indexing of its boxes, shown on the same figure on the right: each box, except those on the topmost antidiagonal, corresponds to a *binomial* (as opposed to a monomial in our case), i.e. the sum of variables x_i, z_i, or $-t_i$. Each extended EYD with d crosses thus produces a polynomial with 2^d monomials. In our setting, it corresponds to 2^d different pipe dreams. In the next subsection we describe the procedure producing these pipe dreams from the extended EYD.

Remark 4.2. These two constructions of pipe dreams for are somewhat parallel to the constructions of pipe dreams of double Schubert polynomials due to Bergeron–Billey and Fomin–Kirillov, respectively. In the former paper, the authors assign a pipe dream to each monomial of double Schubert polynomial; this is similar to our construction presented in Section 3. Corollary 3.18 is thus a direct generalization of [6, (4.1)]. Fomin and Kirillov, on the other hand, assign to each pipe dream a product of several binomials (see [5, Thm 6.2]), just like in Theorem 5 of the paper by Kirillov and Naruse [12] discussed in this section. The latter presentations are thus more efficient (or, put it differently, coarser): each diagram corresponds to several monomials, as opposed to one. Algebraically, the relation between these two constructions is nothing but the Cauchy expansion formula (see Theorem 3.15).[2]

4.2. Producing pipe dreams from an extended EYD

Starting from an extended excited Young diagram, one can produce several pipe dreams corresponding to the same permutation, such that the sum of the monomials corresponding to the pipe dreams is equal to the weight of the extended EYD.

Let us illustrate the bijection of our pipe dreams with those from the paper [12] with an example of the type B. For the types C and D, this bijection is constructed similarly.

Informally, a pipe dream from [12] can be obtained from our pipe dream by pushing the bottom staircase block, n copies of Γ-blocks, and the top staircase block one into another. Our aim is to "pull these blocks apart" in such a way that the shape of the pipe dream and the number of significant elements in each of the blocks remain unchanged.

To do this, for each binomial $x_i - t_j$, $x_i + x_j$, or $z_i + x_j$ corresponding to a cross in the extended EYD let us select one of the two terms. We want to construct a pipe dream with the same permutation and the monomial equal to the product of the selected terms. That means that the crosses in the lower/upper staircase blocks of this pipe dream should correspond to the crosses in the extended EYD marked by z_i and $-t_j$ respectively, and each cross in the extended EYD marked by x_i should produce a cross or a faucet in the ith Γ-block of the resulting pipe dream. This will be done as follows.

The diagram can be represented as the union of several Γ-blocks (an example with two of them is shown in Fig. 9). They will be indexed, counting from top to bottom, by the following variables: $-t_{n-1}, \dots, -t_1, x_n, \dots, x_1$. On the figure we write the corresponding variable under each Γ-block. The Γ-blocks indexed by the $-t_i$'s contain significant elements only in their vertical parts: their horizontal parts are located "outside" the pipe dream.

Let us pull out the Γ-blocks one by one in the northeast direction, starting from the top one. After shifting the first block far enough, we proceed with the next Γ-block, counted from the top, and so on. We shift the first $n-1$ blocks, corresponding to the $-t_i$'s, by the same number of positions northeast, in such a way that the shifted blocks would again form an upper staircase of our pipe dream, but it does not intersect any lower blocks. Then we shift all the subsequent Γ-blocks, corresponding to x_i, also starting with the top one, in such a way that finally they are separate (and thus form a pipe dream) (see Fig. 10).

Further we describe one step of this procedure, i.e., how to shift the topmost, i.e. the nth, Γ-block one box up and right.

[2] We are grateful to the referee for this remark.

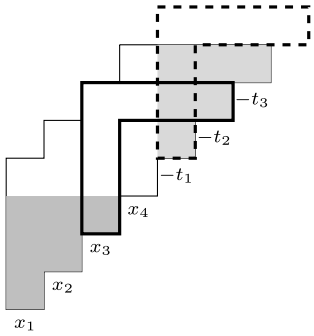


Fig. 9. An extended EYD. The regions corresponding to lower (resp. upper) staircase blocks are shaded in dark (resp. light) gray; the Γ-blocks corresponding to t_1 and x_3 are highlighted.

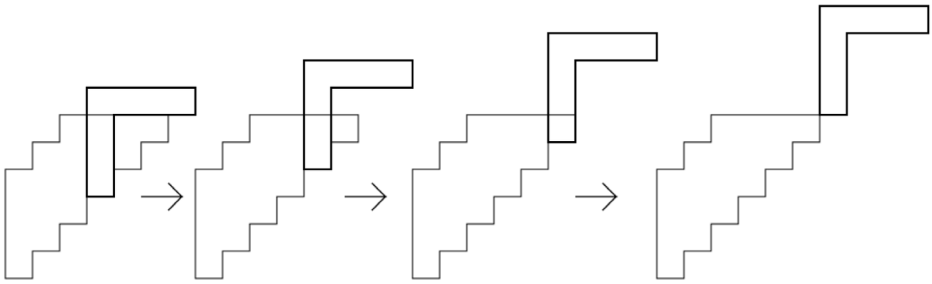
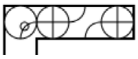


Fig. 10. Shifting of the topmost Γ-block northeast.

Let us draw circles around the significant elements corresponding to the variable x_n, i.e. the elements which belong to the block we are shifting.

All the significant elements in the topmost row necessarily belong to the topmost Γ-block; we shift them one box northeast.

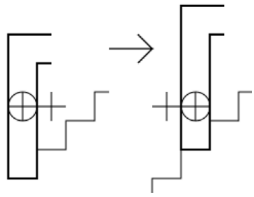


Now consider the crosses in the vertical part of the Γ-block that we are shifting. Denote the column containing this vertical part and the column immediately right of it by V and $V+1$, respectively. Note that all nonencircled crosses in the column V correspond to the horizontal parts of their Γ-blocks (or belong to the rows of the staircase block). So we need to shift some crosses in such a way that:

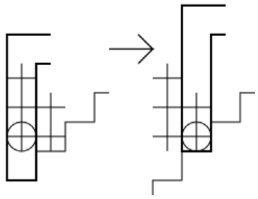
- the shape of the pipe dream remains the same;
- the number of encircled crosses in the column $V+1$ after the shift is equal to the number of encircled crosses in the column V before the shift;
- the number of nonencircled crosses in each row remains the same.

Let us move the encircled crosses one by one, starting from the top. We distinguish between the two cases:

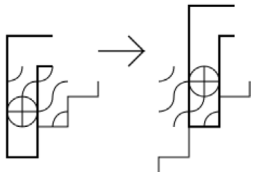
- There is a cross to the right of the encircled cross. This means that the encircled cross does not belong to the bottom line of the Γ-block. In this case, we swap the encircled cross with the nonencircled one (so the encircled cross moves one box down with respect to the Γ-block, see the figure below).



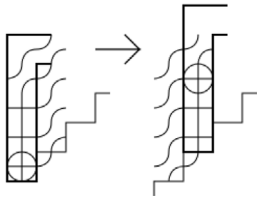
This operation does not involve any elements to the top of the encircled cross.



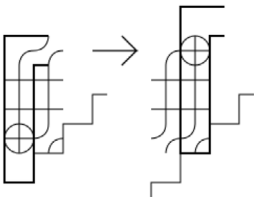
- There is an elbow to the right of the encircled cross. We need to distinguish between the following subcases.
  - If above  there are two elbows  , the encircled cross is shifted northeast.



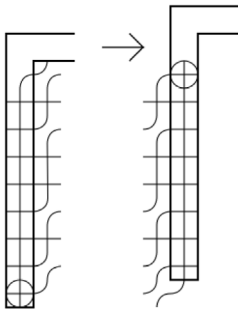
  - If above  there are several rows with a cross followed by an elbow:  , the whole column of crosses is shifted northeast, and we put a circle around the topmost of them.



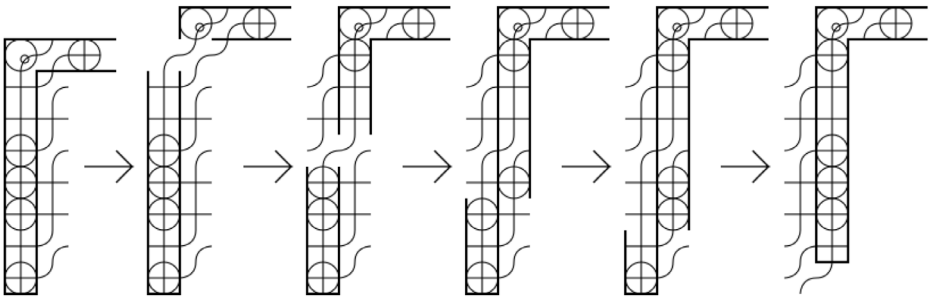
– If above  there are several lines with double crosses:  , the encircled cross is shifted by a ladder move.



– In the general case we proceed as follows. Consider the columns V and $V+1$. Above  we have some sequence of pairs  and  , and on the top of this sequence there are two elbows  (since our pipe dream is reduced, the configuration  cannot occur below ). Starting from the top, we pull the crosses from the configurations  through  using ladder moves and encircle the topmost of the shifted crosses.



The figure below shows the procedure of shifting a Γ-block in case of several encircled crosses.



We shift the encircled crosses one by one, starting from the top. However, as we proceed, the order of the encircled crosses may change.

Finally, is clear that the operation of "pulling apart" of Γ-blocks is invertible: we can bring the blocks back together, starting from the bottom ones and keeping in mind which significant element belongs to which block. This would give us an excited Young diagram together with one of the monomials occuring in its weight.

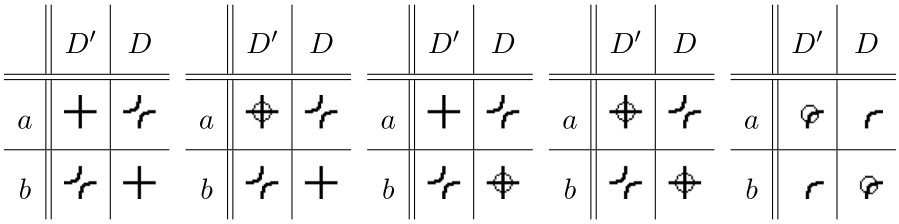
The procedure for the types C and D is similar, so we do not describe it here.

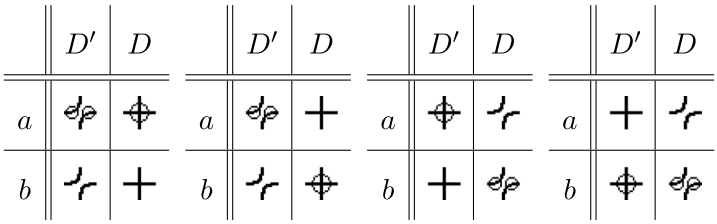
Remark 4.3. T. Ikeda and H. Naruse [22] proposed a description of P- and Q-Schur functions in terms of the so-called *excited Young diagrams*. These functions are particular cases of Schubert polynomials of types B, C and D (see Section 6), corresponding to Grassmannian permutations. The procedure described above is a generalization of the *separation of variables* described in [22, Sec. 10].

5. Bottom pipe dreams

5.1. Admissible moves and extended Lehmer codes of signed permutations

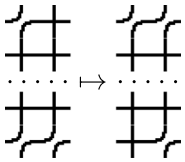
Let $w \in \mathcal{S}_n$ or $w \in \mathcal{F}_n$. Consider the set of pipe dreams of a given shape: $\mathrm{PD}_A(w)$ or $\mathrm{PD}_{\mathcal{F}_n}(w)$. We introduce a partial order on this set as follows. Let D and D' be two pipe dreams of the same shape that coincide everywhere except two boxes a and b, and the row containing a is above the row containing b. We shall say that D is obtained from D' by an *admissible move* (notation: $D \prec D'$) if the elements in a and b are located in one of the following nine ways:





(For the entries in the first line, D is obtained from D' by shifting one significant element down.)

Example 5.1. Let D, D' be pipe dreams of the type A. In this case, only the first of these nine moves is allowed. As a particular case of this move, we have a *ladder move* defined in [6]:



(here the dots represent a sequence of crosses).

Extending the relation $\prec$ by transitivity, we obtain a partial order on the set of pipe dreams. This is indeed a partial order. To show this, let us introduce an order on the set of variables as follows: $z_{n-1} < z_{n-2} < \cdots < z_1 < x_1 < x_2 < \cdots$. Consider the corresponding lexicographic order $>$ on the set of monomials. It is clear that for $D \prec D'$ we have $\mathbf{x}^{\alpha(D)}\mathbf{z}^{\beta(D)} < \mathbf{x}^{\alpha(D')}\mathbf{z}^{\beta(D')}$.

Our next goal is to show that for each w the set $\mathrm{PD}_A(w)$ (resp. $\mathrm{PD}_{\mathcal{F}_n}(w)$) has a unique minimal element with respect to the partial order $\prec$. Such a pipe dream is called *the bottom pipe dream* of w.

Recall the definition of the Lehmer code of a permutation. We also need to generalize the notion of the Lehmer code for a signed permutation.

Definition 5.2. Let $w \in \mathcal{S}_n$. The *Lehmer code* $L(w)$ of this permutation is a sequence $(L_1(w), \dots, L_n(w))$, where

$$L_i(w) = \#\{j > i \mid w(i) > w(j)\}.$$

Definition 5.3. Let $w \in \mathcal{BC}_n$. Consider w as a bijection from $\{-n, \dots, -1, 1, \dots, n\}$ to itself. The *Lehmer code* of w is a sequence $L(w) = (L_1(w), \dots, L_n(w))$ defined as

$$L_i(w) = \#\{j > i \mid w(i) > w(j)\}.$$

The *signed Lehmer code* of w is the pair $(L(w), N(w))$, where

$$N(w) = \left(N_1(w) > N_2(w) > \cdots > N_{s(w)}(w) \mid w^{-1}(N_i(w)) < 0 \text{ for all } i\right)$$

is the decreasing sequence of positive entries with negative preimages.

It is well-known (cf., for instance, [4, p. 9]) that a permutation is uniquely defined by its Lehmer code. This immediately implies that a signed permutation is uniquely defined by its signed Lehmer code.

Remark 5.4. There seems to be no prevalent notion of the Lehmer code for signed permutations in the literature. For example, the definitions of Lehmer codes for signed permutations given in the papers [23,24] are different from ours, and from each other as well.

5.2. Type A pipe dreams

Here we recall the proof of existence of bottom pipe dream for type A permutations, due to S. Billey and N. Bergeron [6] (cf. also [8]).

Theorem 5.5. *Let $w \in \mathcal{S}_n$. There exists a unique* bottom *pipe dream $D_b \in \mathrm{PD}_A(w)$ with all crosses adjusted to the left (i.e., not containing fragments of the form*  *). It can be constructed as follows: for all $i = 1, \dots, n$ the ith row of D_b contains exactly $L_i(w)$ left-adjusted crosses (here $L(w) = (L_1(w), \dots, L_n(w))$ is the Lehmer code of w.)*

Each pipe dream $D \in \mathrm{PD}_A(w)$ can be brought to D_b by a sequence of ladder moves.

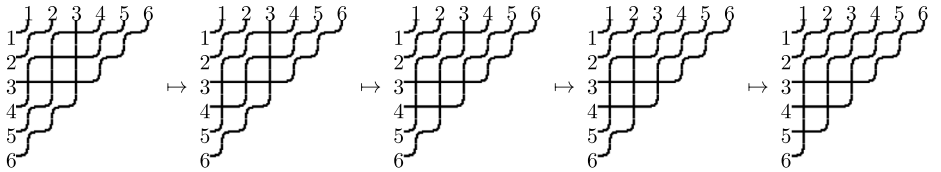
Proof. First observe that every pipe dream without fragments of the form  is reduced.

It is easy to check that if for each $i = 1, \dots, n$ the ith row of the pipe dream D_b has $L_i(w)$ crosses adjusted to the left, then D_b has indeed the shape $w \in \mathcal{S}_n$. Since a permutation is uniquely determined by its Lehmer code, D_b is the unique bottom pipe dream in $\mathrm{PD}_A(w)$.

Consider a pipe dream $D \in \mathrm{PD}_A(w)$. Since it is reduced, it cannot contain a fragment of the form  . If D is not the bottom pipe dream, it contains fragments of the form  . Let us take the 

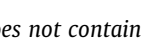
lowest row containing such fragments; among them, take the leftmost occurrence. The cross in it can be shifted downstairs by means of a ladder move. Repeating such an operation, we will obtain the bottom pipe dream for w. □

Example 5.6. Let $w = 126543 \in \mathcal{S}_6$. The Lehmer code of this permutation equals $L(w) = (0, 0, 3, 2, 1, 0)$. Let us show how to bring one of the pipe dreams of shape w to the bottom one.



5.3. B-signed pipe dreams

Theorem 5.7. *Let $w \in \mathcal{B}_n$. There exists a unique* bottom *b-signed pipe dream $D_b \in \mathrm{PD}_{\mathcal{B}_n}(w)$ that satisfies the following conditions:*

- *D_b does not contain fragments of the form*  *and* *;*
- *The numbers of significant elements in its Γ-blocks form a strictly decreasing sequence (ending by zeros).*

This pipe dream D_b can be constructed as follows. Let $(L(w), N(w))$ be the signed Lehmer code of w. For each $i = 1, \ldots, n$ the ith row of the staircase block in D_b contains $L_i(w)$ left-adjusted crosses. And for each $j = 1, \ldots, s(w)$ the horizontal part of the jth Γ-block contains $N_j(w)$ left-adjusted significant elements (i.e. an elbow with a faucet and $N_i(w) - 1$ crosses).

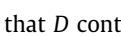
Moreover, each b-signed pipe dream $D \in \mathrm{PD}_{\mathcal{B}_n}(w)$ can be brought to D_b by a sequence of admissible moves.

Proof. By definition, the bottom *b*-signed pipe dream D_b satisfies the following conditions:

- All the crosses in the staircase block are left-adjusted.
- All the significant elements in each Γ-block are located in its horizontal part and are left-adjusted. Hence each nonempty Γ-block has a faucet in its corner. Denote the number of significant elements in the ith Γ-block by μ_i.
- The total number of faucets equals $s(w)$, which is the number of nonempty Γ-blocks. Since the numbers of significant elements in Γ-blocks decrease, we have $n \geq \mu_1 > \mu_2 > \cdots > \mu_{s(w)} > 0$.

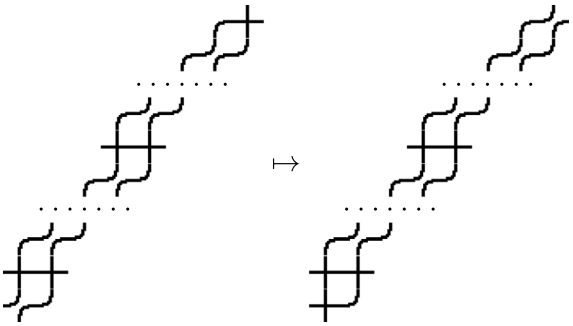
It is easy to observe that such b-signed pipe dream D_b with $L_i(w)$ crosses in ith row of the staircase block and $N_j(w)$ significant elements in jth Γ-block is reduced and has the shape w. Since a permutation $w \in \mathcal{BC}_n$ is uniquely determined by its signed Lehmer code $(L(w), N(w))$, D_b is the unique bottom b-signed pipe dream in $\mathrm{PD}_B(w)$.

It remains to show that each non-bottom *b*-signed pipe dream $D \in \mathrm{PD}_{\mathcal{B}_n}(w)$ can be brought to the bottom one by shifting significant elements down.

Suppose that D contains a fragment  or . Consider the lowest of them; let a be its box containing a cross. Let us go down along the strands crossing at a, and consider the next pairs of adjacent boxes containing these strands. They can look as follows:

-  or . Let b be the left box of the first such pair. Then the strands cross at a and "nearly meet" in b. Note that if one of these strands has a faucet on it, it is located below b. Thus we can move the cross from a into b without changing the shape of our b-signed pipe dream.
-  : in this case the two strands continue passing next to each other.
-  cannot occur, because a was the lowest box with such a condition.
-  only can occur if one of the strands had a faucet on it (otherwise D is nonreduced). But before the faucet the corresponding strand should turn left, hence there is a fragment of the form  before it, and this case was already considered.

Summarizing, we see that the two strands crossing in a follow next to each other and do not contain faucets before passing through a fragment of the form  or . Since each strand contains at least one elbow in the staircase block, we necessarily obtain such a fragment. So the cross from a can be moved down:

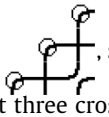


This allows us to bring D to the following form: all significant elements in the Γ-blocks are located in their horizontal parts and are left-adjusted.

Now suppose that the number of significant elements in Γ-blocks does not form a strictly decreasing sequence. Let these blocks have $\mu_1, \mu_2, \ldots$ significant elements and let i be the minimal number satisfying $\mu_{i+1} \geq \mu_i > 0$ or $\mu_{i+1} > 0, \mu_i = 0$.

If $\mu_i = 0$ and $\mu_{i+1} > 0$, we can move all the elements from the $(i+1)$-st block into the ith one; obviously, this does not change the shape of our pipe dream. Thus we can reduce the situation to the case $\mu_1 > 0, \mu_2 > 0, \ldots, \mu_{s(w)} > 0$, with all the following blocks being empty.

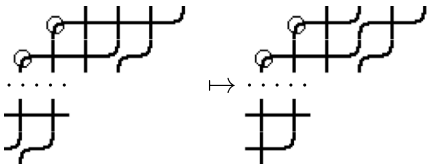
In the next examples we will draw the horizontal parts of the Γ-blocks adjacent to each other, ignoring the elbows between them.

A reduced pipe dream cannot contain a fragment of the form  , so the horizontal parts of all non-empty Γ-blocks except the top one must contain at least two crosses. Also, reduced pipe cannot contain  , so the horizontal parts of all non-empty Γ-blocks except the top two must contain at least three crosses. Proceeding in this way further, we obtain $\mu_j \geq s(w) - j + 1$.

Now consider the box in the $(i+1)$th Γ-block on the μ_ith position, counted from the left; denote this box by a. Since $\mu_{i+1} \geq \mu_i$, this box contains a cross. Consider the pair of strands crossing in a and follow then down.

It is clear that both of these strands have faucets; moreover, they enter the staircase block at adjacent positions. Similarly to the previous case we show that they pass through a box b of the staircase block containing an elbow  . Since both strands have a faucet between a and b, we can

move the cross from a to b without changing the shape of our b-signed pipe dream, as shown on the diagram below:



Repeating these operations, we will obtain the bottom b-signed pipe dream D_b. □

5.4. C-signed pipe dreams

The case of c-signed pipe dreams is treated similarly to the case of b-signed pipe dreams. For $w \in \mathcal{C}_n$ the bottom pipe dream $D_b \in \mathrm{PD}_{\mathcal{C}_n}(w)$ looks like the bottom pipe dream $D'_b \in \mathrm{PD}_{\mathcal{B}_n}(w)$, with one exception: all the elbows with faucets are located in the vertical parts of the Γ-blocks. Let us state an analogue of Theorem 5.7:

Theorem 5.8. *Let $w \in \mathcal{BC}_n$. There exists a unique* bottom *c-signed pipe dream $D_b \in \mathrm{PD}_{\mathcal{C}_n}(w)$ with the following properties:*

- *all the crosses in the staircase block are left-adjusted;*
- *every nonempty Γ-block contains an elbow with a faucet in the vertical part of the block;*
- *all the crosses in the Γ-blocks are located in the horizontal parts and left-adjusted: the box indexed by* 0 *contains an elbow*  *and is followed by a sequence of crosses;*
- *the numbers of significant elements in Γ-blocks form a strictly decreasing sequence.*

This pipe dream D_b has $L_i(w)$ crosses in the ith row of the staircase block and $N_j(w)$ significant elements in the jth Γ-block (here $(L(w), N(w))$ is the signed Lehmer code of $w \in \mathcal{BC}_n$).

Any c-signed pipe dream $D \in \mathrm{PD}_{\mathcal{C}_n}(w)$ can be brought to D_b by a sequence of admissible moves.

5.5. D-signed pipe dreams

Theorem 5.9. *Let $w \in \mathcal{D}_n$. There exists a unique* bottom *d-signed pipe dream $D_b \in \mathrm{PD}_{\mathcal{D}_n}(w)$ satisfying the following properties:*

- *D_b does not contain fragments of the form*  *(however,*  *may occur);*
- *the numbers of significant elements in the Γ-blocks form a strictly decreasing sequence (ending by zeros);*
- *The $1'$-boxes of nonempty Γ-blocks contain crosses*  *and crosses with faucets*  *. These elements alternate: the blocks with odd numbers contain crosses with faucets* *, while those with even numbers contain ordinary crosses* *.*

This pipe dream D_b can be constructed as follows. Let $(L(w), N(w))$ be the signed Lehmer code for $w \in \mathcal{D}_n$ and let

$$\mu(w) = \begin{cases} \big(N_1(w) - 1, N_2(w) - 1, \ldots, N_{s(w)}(w) - 1\big), & \text{if } N_{s(w)}(w) > 1; \\ \big(N_1(w) - 1, N_2(w) - 1, \ldots, N_{s(w)-1}(w) - 1\big), & \text{if } N_{s(w)}(w) = 1. \end{cases}$$

For each $i = 1, \ldots, n$ the ith row of the staircase block contains $L_i(w)$ left-adjusted crosses. For each $j = 1, \ldots, \ell(\mu)$ the horizontal part of jth Γ-block contains $\mu_j(w)$ left-adjusted crosses. If j is odd, then there is a faucet on the cross in the corner box.

Moreover, each d-signed pipe dream $D \in \mathrm{PD}_{\mathcal{D}_n}(w)$ can be brought to D_b by a sequence of admissible moves.

Proof. The proof is similar to the proof of Theorem 5.7.

The bottom d-signed pipe dream D_b constructed as in the statement of the theorem is reduced and has the shape $w \in \mathcal{D}_n$. Since the number of sign changes in $w \in \mathcal{D}_n$ is even, the partition $\mu(w)$ uniquely determines $N(w)$ and each permutation $w \in \mathcal{D}_n$ is uniquely determined by the pair $(L(w), \mu(w))$. Thus, D_b is the unique bottom d-signed pipe dream in $\mathrm{PD}_D(w)$.

Let us show how to reduce a d-signed pipe dream $D \in \mathrm{PD}_{\mathcal{D}_n}(w)$ to D_b.

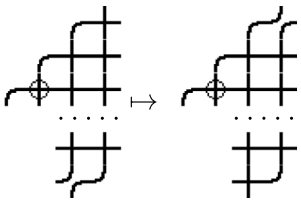
The cross from the lowest fragment  can be moved down: to do this, we follow the two crossing strands down; since they are not the leftmost, these two strands cannot contain the fragment of the form  . Then we proceed similarly to the proof of Theorem 5.7.

These operations can bring our pipe dream to the following form: the significant elements are only in the horizontal parts of the Γ-blocks; in each row they are left-adjusted. Further in the examples we draw the horizontal parts of the blocks one under another, ignoring the elbows between them.

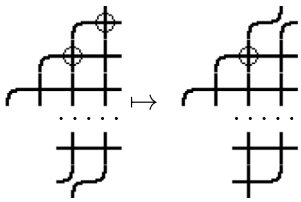
If certain Γ-block is empty, we can move to it all the significant elements of the block above it. So we can suppose that all the empty blocks are located above the nonempty ones.

Now look at the elements in the boxes of the Γ-blocks indexed by $1'$. In the odd blocks we should have crosses with faucets  , and in the even ones crosses  . Suppose this is not true; take the lowest (ith) block where this does not hold, and distinguish between the following possibilities:

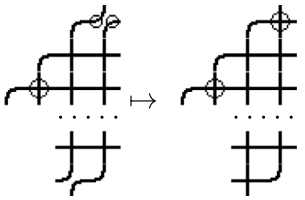
- The number i is odd, and there is a cross  in the corner of the ith block. In the corner of the $(i-1)$-st block there is also a cross  . Since D is reduced, the $(i-1)$-st block also contains at least one more cross. Then we can move the cross from the corner of the ith block down similarly to Theorem 5.7 (since both strands do not have faucets on them):



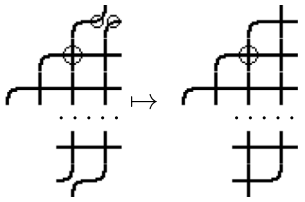
- The number i is even, and there is a cross with a faucet in the corner of the ith block (denote this box by a). The corner of the $(i-1)$-st block also contains an element  . Since D is reduced, the $(i-1)$-st block should contain at least one more cross. Similarly to the previous cases we can show that the strands crossing at a also "nearly meet" at some other box b containing an elbow  . Then we can replace the cross with a faucet at a by an elbow joint and replace the elbow at b by a cross; this would preserve the shape of the permutation:



- The number i is odd, there is an elbow with two faucets in the corner of the ith block:

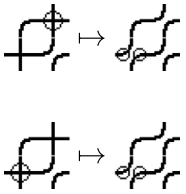


- The number i is even, there is an elbow with two faucets in the corner of the ith block:

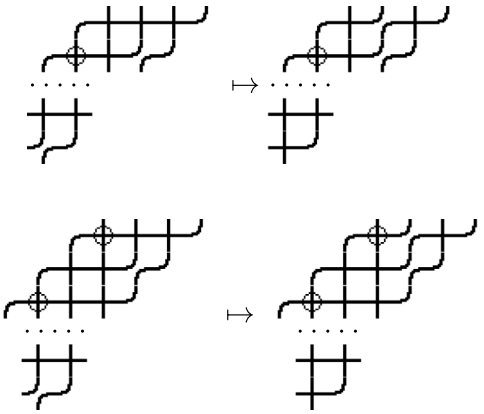


Now suppose that the numbers of significant elements in the Γ-blocks do not form a decreasing sequence. Let these blocks contain $\mu_1, \mu_2, \dots$ significant elements, and let i be the least number such that $\mu_{i+1} \geq \mu_i > 0$.

If $\mu_i = 1$, the crosses and from the two neighboring corners can be shifted into one elbow joint :



Otherwise consider the box a in the $(i+1)$-st Γ-block at the μ_ith position counted from the left. Similarly to Theorem 5.7, the cross at this box can be moved down:



Let us summarize the previous discussion. Consider a d-signed pipe dream $D \in \mathrm{PD}_{\mathcal{D}_n}(w)$. If it is not the bottom pipe dream, we can bring one of the significant elements down:

- if D has a fragment  , we can move the cross from the lowest of such fragments;
- if D has no  , but certain nonempty block is above an empty one, we can shift all the elements from this block;
- if D has no  , all the empty blocks are above the nonempty ones, but at some of the boxes $1'$ the alternation of crosses  and  is violated, we can move down the element from this box;
- if D has no  , all the empty blocks are above the nonempty ones, the crosses and  at the boxes $1'$ are alternating, but the number of significant elements is not decreasing, we can move the lowest cross that violates this rule. □

The bottom pipe dream of $w \in \mathcal{F}_\infty$ corresponds to the lexicographically highest monomial of $\mathfrak{F}_w$. This immediately implies the result by Billey and Haiman that Schubert polynomials form a basis (cf. [9, Thm 3, Thm 4]).

Theorem 5.10. *Each family of Schubert polynomials* $\{\mathfrak{B}_w \mid w \in \mathcal{BC}_\infty\}$, $\{\mathfrak{C}_w \mid w \in \mathcal{BC}_\infty\}$, $\{\mathfrak{D}_w \mid w \in \mathcal{D}_\infty\}$ *forms a* $\mathbb{Q}$*-basis in* $\mathbb{Q}[\mathbf{z}, p_1(\mathbf{x}), p_3(\mathbf{x}), \ldots]$.

Proof. As before, we consider the order $\ldots > z_2 > z_1 > x_1 > x_2 > \cdots$ on the set of variables and the corresponding lexicographic order $\prec$ on the set of monomials. The highest (with respect to this order) monomial $\mathrm{in}_\prec(\mathfrak{F}_w)$ in a Schubert polynomial $\mathfrak{F}_w$, where $w \in \mathcal{F}_n \subset \mathcal{F}_\infty$, corresponds to the bottom pipe dream $D_b \in \mathrm{PD}_{\mathcal{F}_n}(w)$.

From the description of the bottom pipe dreams we see that:

- for different $w \in \mathcal{F}_\infty$ the highest terms of the Schubert polynomials $\mathfrak{F}_w$ are distinct, hence the Schubert polynomials of each given type (B, C or D) are linearly independent;
- for Schubert polynomials of a given type (B, C or D) and each $\beta = (\beta_1, \beta_2, \ldots, 0, 0, \ldots)$ and $\alpha = (\alpha_1 > \alpha_2 > \cdots)$ there exists an element $w \in \mathcal{F}_\infty$ such that the highest term of $\mathfrak{F}_w$ equals $\mathbf{x}^\alpha \mathbf{z}^\beta$.

Now let us show that the highest term of each monomial $f \in \mathbb{Q}[\mathbf{z}, p_1(\mathbf{x}), p_3(\mathbf{x}), \ldots]$ satisfies the following property: the powers of x_i occuring in it strictly decrease.

Let $f \in \mathbb{Q}[p_1(\mathbf{x}), p_3(\mathbf{x}), \ldots]$, and let the highest term of f be equal to $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_s^{\alpha s}$. Since f is a symmetric function, we have $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_s$. Suppose that $\alpha_1 = \alpha_2 = \alpha$.

Let $x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3} \ldots$ be another monomial occuring in f. Then f also contains the monomial $x_1^{\beta_2} x_2^{\beta_1} x_3^{\beta_3} \ldots$, so $\beta_1 \leq \alpha$, $\beta_2 \leq \alpha$ and we have either $\beta_1 = \beta_2 = \alpha$, or $\beta_1 + \beta_2 < \alpha_1 + \alpha_2 = 2\alpha$. This means that

$$f(x_1, x_2, x_3, \ldots) = x_1^\alpha x_2^\alpha g(x_3, x_4, \ldots) + h(x_1, x_2, x_3 \ldots).$$

Here g is a nonzero symmetric function, and h is a symmetric function with the following property: for each of its monomials the sums of powers of x_1 and x_2 is less than 2α. The function f is *supersymmetric*:

$$f(y, -y, x_1, x_2, \ldots) = f(x_1, x_2, \ldots),$$

since every odd power sum $p_{2k-1}(\mathbf{x})$ satisfies this property. On the other hand,

$$f(y, -y, x_1, x_2, \ldots) = \pm y^{2\alpha} g(x_1, x_2, x_3, \ldots) + h(y, -y, x_1, \ldots),$$

where the power of h with respect to y does not exceed $2\alpha - 1$. So $f(y, -y, x_1, x_2, \ldots)$ depends on y and hence cannot be equal to $f(x_1, x_2, \ldots)$. This contradiction shows that $\alpha_1 > \alpha_2$.

Now let us express $f(\mathbf{x})$ as

$$f(\mathbf{x}) = x^{\alpha_1} g(x_2, x_3, \ldots) + h(x_1, x_2, \ldots),$$

where $g \in \mathbb{Q}[p_1(x_2, x_3, \ldots), p_3(x_2, x_3, \ldots), \ldots]$, and the power of h with respect to x_1 does not exceed $\alpha_1 - 1$. The highest term of $g(x_2, x_3, \ldots)$ is equal to $x_2^{\alpha_2} x_3^{\alpha_3} \ldots x_s^{\alpha_s}$. A similar reasoning applied to g shows that $\alpha_2 > \alpha_3$. Proceeding in this way further, we obtain the desired inequalities $\alpha_1 > \alpha_2 > \cdots > \alpha_s$. □

Remark 5.11. Note that the Schubert polynomials, viewed as elements of the ring $\mathbb{Q}[\mathbf{z}, p_1(\mathbf{x}), p_3(\mathbf{x}), \ldots]$, do not necessarily have integer coefficients. For instance, if $w = s_0 s_1 s_0$, the corresponding Schubert polynomial of type B equals

$$\mathfrak{B}_{s_0 s_1 s_0} = \sum_{i<j} (x_i^2 x_j + x_i x_j^2) + 2 \sum_{i<j<k} x_i x_j x_k = \frac{1}{3} p_1^3(\mathbf{x}) - \frac{1}{3} p_3(\mathbf{x}).$$

More examples can be found in Appendix.

However, if we consider them as elements of the ring $\mathbb{Z}[\mathbf{z}] \otimes \Omega[\mathbf{x}]$, where $\Omega[\mathbf{x}]$ is the ring of supersymmetric functions in $\mathbf{x}$, then the Schubert polynomials of types B and D form bases of this ring. The argument for proving this is similar to the one in type A: every monomial $\mathbf{x}^\alpha \mathbf{z}^\beta$ with $\alpha_1 > \alpha_2 > \ldots$ occurs as the highest term in some Schubert polynomial with coefficient 1.

6. Schubert polynomials of Grassmannian permutations

6.1. Schubert polynomials of Grassmannian permutations are Schur polynomials

It is well known that the Schur polynomials appear as the Schubert polynomials of permutations with a unique descent (such permutations are called Grassmannian ones). This fact can be proved combinatorially (cf., for instance, the notes [8] by A. Knutson). The Schur polynomial $s_\lambda(z_1, \ldots, z_n)$, where λ is a partition, can be obtained as the sum of monomials z^T over Young tableaux T of shape λ filled by the integers not exceeding n; one can construct a bijection between the Young tableaux of a given shape and the pipe dreams of the Grassmannian permutation corresponding to λ. For the reader's convenience, we recall this result as Theorem 6.4 and provide its proof.

It is well-known that this result can be generalized to the case of other classical groups. In this case the ordinary Schur polynomials are replaced by the P- and Q-Schur polynomials: P for the types B and D, while Q corresponds to the type C. This observation is essentially due to P. Pragacz [10]; in [9, Theorem 3], this fact was shown using a modification of the Edelman–Greene correspondence. The main goal of this section is to give a proof of this fact without referring to the Edelman–Greene correspondence; instead, we mimic the type A proof and construct the bijection between the pipe dreams of a Grassmannian permutation and circled shifted Young tableaux, which index monomials in P- and Q-Schur functions.

Definition 6.1. Let $2 \le k \le n-1$. A permutation $w \in \mathcal{S}_n$ is said to be *k-Grassmannian*, if $w(1) < w(2) < \cdots < w(k) > w(k+1) < w(k+2) < \cdots < w(n)$. A permutation is said to be Grassmannian, if it is k-Grassmannian for some k. In other words, a Grassmannian permutation has a unique descent at k.

A k-Grassmannian permutation $w \in \mathcal{S}_n$ bijectively corresponds to the partition $\lambda(w) = (\lambda_1, \ldots, \lambda_k)$ of the number $\ell(w)$ into at most k parts: it is given by $\lambda_i = w(i) - i$.

Definition 6.2. Let $\mathcal{F}_n = \mathcal{BC}_n$ or $\mathcal{D}_n$. A permutation $w \in \mathcal{F}_n$ is said to be *Grassmannian* if $w(1) < w(2) < \cdots < w(n)$.

Let k be a number such that $w(k) < 0 < w(k+1)$. A permutation thus defines a *strict* partition, i.e. a partition of strictly decreasing integers, $-w(1) > -w(2) > \cdots > -w(k) > 0$. This provides a bijection between strict partitions and Grassmannian permutations in $\mathcal{BC}_\infty$.

In the one-line notation, saying that a permutation $w \in \mathcal{BC}_\infty$ is Grassmannian is equivalent to saying that $w = \bar{i}_1 \ldots \bar{i}_k i_{k+1} \ldots i_n$, with all $i_j > 0$ distinct, $i_1 > i_2 > \cdots > i_k$, and $i_{k+1} < i_{k+2} < \cdots < i_n$: all the entries with bars precede those without bars, the absolute values of the barred entries

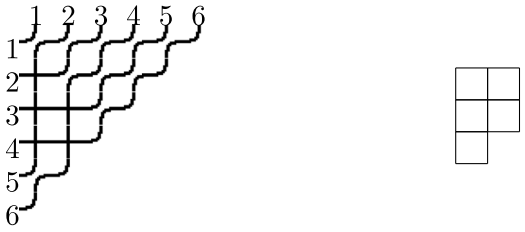


Fig. 11. The bottom pipe dream for $w = 135624$ and the Young diagram $\lambda(w)$.

decrease, and the values of the entries without bars increase. The corresponding strict partition is $(i_1, i_2, \dots, i_k)$.

In this section we prove that the Schubert polynomials of Grassmannian permutations for $\mathcal{F}_n = \mathcal{BC}_n$ and $\mathcal{D}_n$ are equal to P- (in types B and D) and Q-Schur functions (in type C), respectively, by constructing an explicit bijection between the pipe dreams and the circled Young tableaux.

We start with recalling the combinatorial description of (usual) Schur polynomials.

Definition 6.3. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0)$ be a partition and $k \geq 0$. Consider the *Young diagram* (in the English notation) with rows of length $\lambda_1, \dots, \lambda_l$. A *(semistandard) Young tableau of shape* λ is a labeling of its boxes by the numbers $1, \dots, k$ weakly increasing along the rows and strictly increasing along the columns. The set of all semistandard Young tableaux of shape λ will be denoted by $\mathrm{SSYT}_k(\lambda)$.

If $T \in \mathrm{SSYT}_k(\lambda)$, denote by $\mathbf{z}^T$ the monomial obtained as the product of z_i over all occurrences of i in the tableau T. A *Schur polynomial* is defined as the sum of monomials $\mathbf{z}^T$ over all semistandard Young tableaux of a given shape:

$$s_\lambda(z_1, \dots, z_k) = \sum_{T \in \mathrm{SSYT}_k(\lambda)} \mathbf{z}^T.$$

It is well-known (cf., for example, [25]) that s_λ is a symmetric polynomial in $z_1, \dots, z_k$.

The following theorem is a fundamental property of Schubert polynomials (see, for instance, [4, (4.8)]). Its bijective proof is probably folklore; it can be found, for example, in [8]. For the reader's convenience, we give this proof here. It will be generalized for the cases B, C, and D in the next subsection.

Theorem 6.4. *Let $w \in \mathcal{S}_n$ be a k-Grassmannian permutation. Let*

$$\lambda(w) = (\lambda_1(w), \dots, \lambda_l(w)) = (w(k) - k, \dots, w(2) - 2, w(1) - 1).$$

Then we have

$$\mathfrak{S}_w(\mathbf{z}) = s_{\lambda(w)}(z_1, z_2, \dots, z_k).$$

Proof. We first observe that $L_i(w) = w(i) - i$ for $i \leq k$ and $L_i(w) = 0$ for $i > k$. This means that for the bottom pipe dream $D_b \in \mathrm{PD}_A(w)$ its crosses are located in the boxes forming the Young diagram λ flipped upside down: (see Fig. 11).

Now let $D \in \mathrm{PD}_A(w)$ be an arbitrary pipe dream of shape w. Let us assign to D a Young tableau $T \in \mathrm{SSYT}_k(\lambda(w))$ as follows. First let us bring it to D_b by moving some crosses down. Now we replace each cross in D_b by the number $k - i + 1$, where i is the number of the row which initially contained this cross.

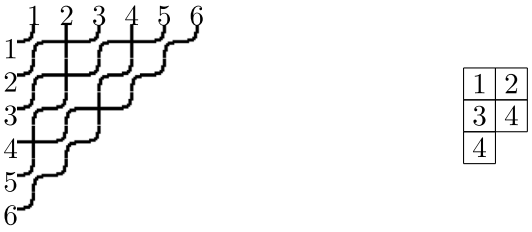


Fig. 12. Pipe dream and the corresponding Young tableau.

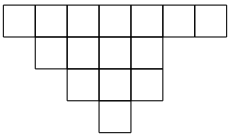
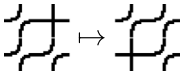


Fig. 13. Shifted Young diagram $\mu = (7, 4, 3, 1)$.

One can check that all the crosses were only shifted southwest:



This means that the order of crosses along each strand remains unchanged. Hence the numbers along the rows of T weakly increase, and the numbers along the columns strictly increase (see Fig. 12).

This correspondence between $\mathrm{PD}_A(w)$ and $\mathrm{SSYT}_k(\lambda(w))$ is a bijection. Indeed, let us start with a tableau T; we will read the crosses in D_b from right to left, from top to bottom, and lift each of the crosses into the row prescribed by the value inside the corresponding box. Since the numbers in T weakly increase along the rows and strictly increase along the columns, each of the crosses will not "bump" into the crosses lifted before, so all of them will be lifted to the prescribed positions.

Let $\widetilde{\mathbf{z}} = (z_k, z_{k-1}, \dots, z_1)$. If $T \in \mathrm{SSYT}_k(\lambda(w))$ is the tableau corresponding to $D \in \mathrm{PD}_A(w)$, the monomial $\mathbf{z}^{\beta(D)}$ is equal to $\widetilde{\mathbf{z}}^T$.

Hence

$$s_\lambda(\widetilde{\mathbf{z}}) = \sum_{T \in \mathrm{SSYT}(\lambda(w))} \widetilde{\mathbf{z}}^T = \sum_{D \in \mathrm{PD}_A(w)} \mathbf{z}^{\beta(D)} = \mathfrak{S}_w(\mathbf{z}).$$

Since the Schur polynomials are symmetric, we have $s_\lambda(\mathbf{z}) = \mathfrak{S}_w(z_1, \dots, z_k)$. □

6.2. Schubert polynomials of types B, C and D are P- and Q-Schur functions

Let $\mu = (\mu_1, \dots, \mu_\ell)$, where $\mu_1 > \mu_2 > \cdots > \mu_\ell$ be a partition into distinct parts. Consider a *shifted Young diagram*, as shown on the figure below: its ith row consists of μ_i boxes, and the rows are aligned along the NW-SE diagonal. The partition $\mu = (\mu_1, \mu_2, \dots, \mu_\ell)$ is called the *shape* of this diagram (see Fig. 13).

Let us fill the boxes of a shifted Young diagram by the numbers $1^\circ < 1 < 2^\circ < 2 < \cdots$ according to the following rules:

- the numbers weakly increase along the rows and the columns;
- a row cannot contain two equal numbers with circles;
- a column cannot contain two equal numbers without circles.

Such an object is called a *circled Young tableau* of shape μ. The set of all such tableaux of shape μ will be denoted by $\mathrm{CYT}(\mu)$.

Let $\mathrm{CYT}'(\mu) \subset \mathrm{CYT}(\mu)$ be the subset of circled Young tableaux such that the leftmost number of each row has no circle.

To each tableau $T \in \mathrm{CYT}(\mu)$ we can assign a monomial $\mathbf{x}^T$ obtained as the product of x_i over all the occurrences of i° and i in T.

Definition 6.5. The *P- and Q-Schur functions* are defined as the sums of monomials over all circled Young tableaux:

$$Q_\mu(\mathbf{x}) = \sum_{T \in \mathrm{CYT}(\mu)} \mathbf{x}^T;$$
$$P_\mu(\mathbf{x}) = \sum_{T \in \mathrm{CYT}'(\mu)} \mathbf{x}^T.$$

Remark 6.6. Let $T \in \mathrm{CYT}(\mu)$ be a circled Young tableau. Note that we can add or remove a circle at the number of the leftmost box of each row, still obtaining a valid circled tableau. This means that

$$P_\mu(\mathbf{x}) = 2^{-\ell(\mu)} Q_\mu(\mathbf{x}),$$

where $\ell(\mu)$ is the number of parts in the partition μ.

The following theorem was proved in [9, Prop. 3.13, 3.14] using a modification of the Edelman–Greene correspondence. Here we prove it with a different (and easier) bijective argument.

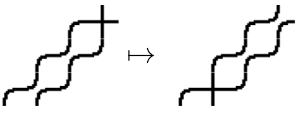
Theorem 6.7. *Let $w = \overline{w}_1 \overline{w}_2 \dots \overline{w}_{s(w)} w_{s(w)+1} \dots w_n \in \mathcal{BC}_n$ be a Grassmannian permutation with the decreasing segment $w_1, \dots, w_{s(w)}$ and the increasing segment $w_{s(w)+1}, \dots, w_n$. Let $\mu = (w_1, \dots, w_{s(w)})$. Then $\mathfrak{B}_w = P_\mu$ and $\mathfrak{C}_w = Q_\mu$.*

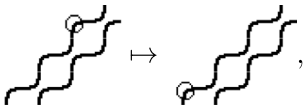
Proof. Consider the bottom pipe dream $D_b \in \mathrm{PD}_{\mathcal{B}_n}(w)$. The positions of its crosses can be deduced from Theorem 5.7: since $-w_1 < -w_2 < \dots < -w_{s(w)} < w_{s(w)+1} < \dots < w_n$, its Lehmer code equals $L(w) = (0, 0, \dots, 0)$, hence the staircase block of D_b does not contain any crosses. On the other hand, the ith Γ-block for $i \le s(w)$ contains w_i left-adjusted significant elements.

Let us establish a correspondence between the significant elements of D_b and the boxes in the shifted Young diagram in a natural way: the jth significant element (counted from the left) of the ith Γ-block corresponds to the jth box of the ith row in the shifted Young diagram. If two boxes are located in the same column of the Young diagram, there is a strand passing vertically through both corresponding significant elements. Meanwhile, two boxes in the same row of the Young diagram correspond to a horizontal segment of a strand.

Since each $D \in \mathrm{PD}_{\mathcal{B}_n}(w)$ can be reduced to D_b by shifting down significant elements, the staircase block of D also does not contain crosses. We assign to D a circled Young tableau $T \in \mathrm{CYT}'(\mu)$ as follows. For each significant element from D_b let us look at its initial position in D. If it was in the vertical part (excluding the corner) of the ith Γ-block, we put i° into the corresponding box of our tableau. If it was in the horizontal part (including the corner), we put i.

One can show that for Grassmannian permutations the significant elements will only move along the NE–SW diagonal:





so the order of the elements corresponding to each strand will remain unchanged. This implies the non-decreasing of the numbers in T along the rows and the columns. Also, if a strand passes through two significant elements horizontally (resp. vertically), these elements cannot be located in the same vertical (resp. horizontal) part of the same Γ-block. This implies that the circled integers are not repeating along the rows of T, and the non-circled ones are not repeating along the columns.

This correspondence between $\mathrm{PD}_{\mathcal{B}_n}(w)$ and $\mathrm{CYT}'(\mu)$ is a bijection. Indeed, consider a tableau $T \in \mathrm{CYT}'(\mu)$. Let us read the significant elements of D_b one by one, right to left top to bottom, and lift each of them into the vertical or horizontal part of the Γ-block prescribed by the entry in the corresponding box. It is easy to see that the elements will not "bump" into each other, so all of them will be lifted to the prescribed positions.

To conclude, note that if a b-signed pipe dream D corresponds to the tableau T, then $\mathbf{x}^T = \mathbf{x}^{\alpha(D)} = \mathbf{x}^{\alpha(D)}\mathbf{z}^{\beta(D)}$. This means that

$$\mathfrak{B}_w(\mathbf{x}, \mathbf{z}) = \sum_{D \in \mathrm{PD}_{\mathcal{B}_n}(w)} \mathbf{x}^{\alpha(D)}\mathbf{z}^{\beta(D)} = \sum_{T \in \mathrm{CYT}'(\mu)} \mathbf{x}^T = P_\mu(\mathbf{x}).$$

In the case of c-signed pipe dreams a similar reasoning gives us the following relation:

$$\mathfrak{C}_w(\mathbf{x}, \mathbf{z}) = \sum_{D \in \mathrm{PD}_{\mathcal{C}_n}(w)} \mathbf{x}^{\alpha(D)}\mathbf{z}^{\beta(D)} = \sum_{T \in \mathrm{CYT}(\mu)} \mathbf{x}^T = Q_\mu(\mathbf{x}). \quad \square$$

We conclude by a similar statement in type D.

Theorem 6.8. *Let $w = \overline{w}_1\overline{w}_2 \dots \overline{w}_{s(w)}w_{s(w)+1} \dots w_n \in \mathcal{D}_n$ be a Grassmannian permutation, with the decreasing segment $w_1, \dots, w_{s(w)}$ and the increasing segment $w_{s(w)+1}, \dots, w_n$. Let $\mu' = (w_1 - 1, w_2 - 1, \dots, w_{s(w)} - 1)$ (if the last entry of the sequence is zero, we omit it). Then $\mathfrak{D}_w = P_{\mu'}$.*

Proof. Similar to the proof of Theorem 6.7. $\square$

Acknowledgments

We are grateful to the anonymous referees for their useful comments and remarks, in particular, for bringing the paper [12] to our attention. This research was supported by the HSE University Basic Research Program and the Theoretical Physics and Mathematics Advancement Foundation "BASIS", Russia. E.S. was also partially supported by the RFBR, Russia grant 20-01-00091-a and the Simons-IUM Fellowship.

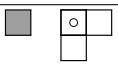
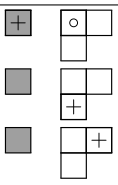
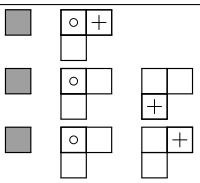
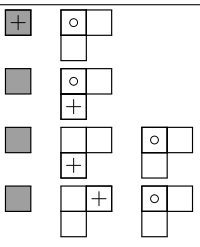
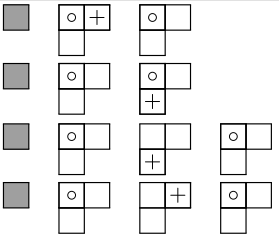
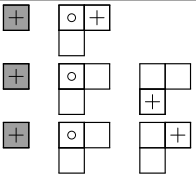
Appendix. More examples

In the appendix we present an alternative, more compact, notation for pipe dreams. Then we list all the elements, except the identity, for the group $\mathcal{BC}_2$, as well as some elements of the group $\mathcal{D}_3$. For each element we list all the signed pipe dreams of types B and D respectively, with the corresponding monomials.

While working with usual pipe dreams, it is convenient to omit elbows and to draw only crosses. Thus, a pipe dream for a permutation $w \in S_n$ is represented by a staircase Young diagram $(n-1, \dots, 2, 1)$ with some boxes filled by crosses.

For the pipe dreams of other classical types, we will do the same: a pipe dream of type B_n will be represented by a sequence of Young diagrams consisting of a staircase $(n-1, \dots, 2, 1)$ and several hooks $(n, 1, \dots, 1)$, with some boxes filled by crosses and eventually by the sign $\circ$. The latter sign can only be located in the upper-left corner of a hook; it represents a faucet. Type C is treated similarly.

Table 1
B-signed pipe dreams for $w \in \mathcal{BC}_2$.

w	Pipe dreams	Weight
$\overline{1}2 = s_0$		x_i
$\mathfrak{B}_{s_0} = \sum_i x_i = p_1(\mathbf{x})$		
$21 = s_1$		z_1 x_i x_i
$\mathfrak{B}_{s_1} = z_1 + 2\sum_i x_i = z_1 + 2p_1(\mathbf{x})$		
$\overline{2}1 = s_1 s_0$		x_i^2 $x_i x_j$ $x_i x_j$
$\mathfrak{B}_{s_1 s_0} = \sum_i x_i^2 + 2\sum_{i<j} x_i x_j = p_1^2(\mathbf{x})$		
$2\overline{1} = s_0 s_1$		$z_1 x_i$ x_i^2 $x_i x_j$ $x_i x_j$
$\mathfrak{B}_{s_0 s_1} = z_1 \sum_i x_i + \sum_i x_i^2 + 2\sum_{i<j} x_i x_j = z_1 p_1(\mathbf{x}) + p_1^2(\mathbf{x})$		
$\overline{2\overline{1}} = s_0 s_1 s_0$		$x_i^2 x_j$ $x_i x_j^2$ $x_i x_j x_k$ $x_i x_j x_k$
$\mathfrak{B}_{s_0 s_1 s_0} = \sum_{i<j}(x_i^2 x_j + x_i x_j^2) + 2\sum_{i<j<k} x_i x_j x_k = \frac{1}{3}p_1^3(\mathbf{x}) - \frac{1}{3}p_3(\mathbf{x})$		
$1\overline{2} = s_1 s_0 s_1$		$z_1 x_i^2$ $z_1 x_i x_j$ $z_1 x_i x_j$

(*continued on next page*)

Table 1 (*continued*).

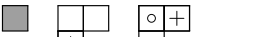
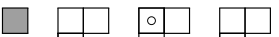
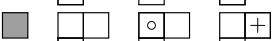
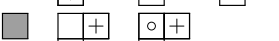
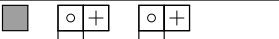
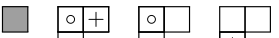
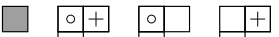
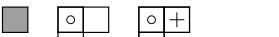
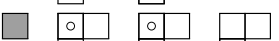
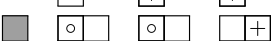
w	Pipe dreams	Weight
		x_i^3
		$x_i^2x_j$
		$x_i^2x_j$
		$x_ix_j^2$
		$x_ix_jx_k$
		$x_ix_jx_k$
		$x_ix_j^2$
		$x_ix_jx_k$
		$x_ix_jx_k$
$\mathfrak{B}_{s_1s_0s_1} = z_1\sum_i x_i^2 + 2z_1\sum_{i<j} x_ix_j + 2\sum_{i<j}(x_i^2x_j + x_ix_j^2)+$ $+4\sum_{i<j<k} x_ix_jx_k = z_1p_1^2(\mathbf{x}) + \frac{2}{3}p_1^3(\mathbf{x}) + \frac{1}{3}p_3(\mathbf{x})$		
$\overline{12} = s_0s_1s_0s_1$		$x_i^2x_j^2$
$= s_1s_0s_1s_0$		$x_i^2x_jx_k$
		$x_i^2x_jx_k$
		$x_i^2x_j^3$
		$x_ix_j^2x_k$
		$x_ix_j^2x_k$
		$x_ix_jx_k^2$
		$x_ix_jx_kx_m$
		$x_ix_jx_kx_m$
		$x_ix_jx_k^2$
		$x_ix_jx_kx_m$

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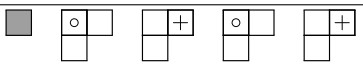
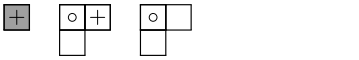
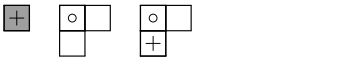
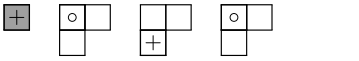
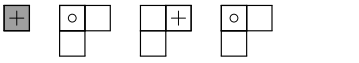
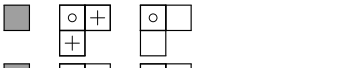
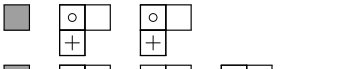
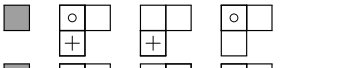
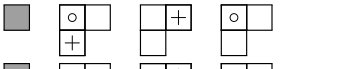
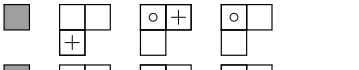
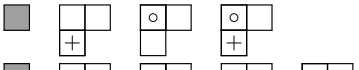
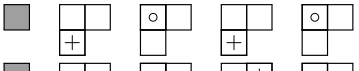
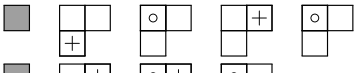
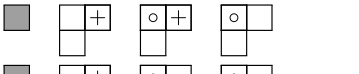
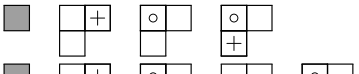
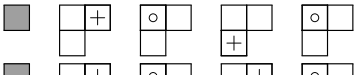
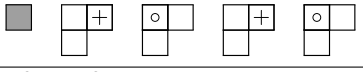
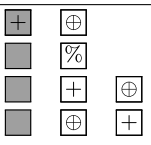
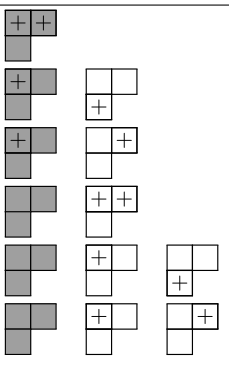
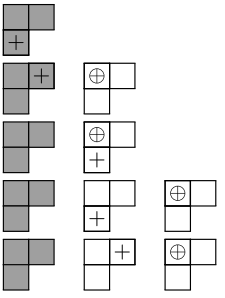
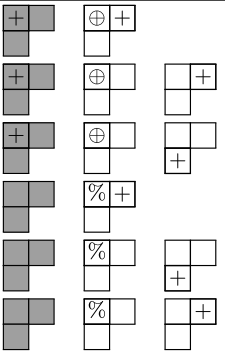
Table 1 (*continued*).

w	Pipe dreams	Weight
		$x_ix_jx_kx_m$
		$z_1x_i^2x_j$
		$z_1x_ix_j^2$
		$z_1x_ix_jx_k$
		$z_1x_ix_jx_k$
		$x_i^3x_j$
		$x_i^2x_j^2$
		$x_i^2x_jx_k$
		$x_i^2x_jx_k$
		$x_ix_j^2x_k$
		$x_ix_jx_k^2$
		$x_ix_jx_kx_m$
		$x_ix_jx_kx_m$
		$x_ix_j^2x_k$
		$x_ix_jx_k^2$
		$x_ix_jx_kx_m$
		$x_ix_jx_kx_m$
	$\mathfrak{B}_{s_1s_0s_1s_0} = z_1\sum_{i<j}(x_i^2x_j + x_ix_j^2) + 2z_1\sum_{i<j<k} x_ix_jx_k +$ $+\sum_{i<j}(2x_i^2x_j^2 + x_ix_j^3 + x_i^3x_j) + 4\sum_{i<j<k}(x_i^2x_jx_k + x_ix_j^2x_k + x_ix_jx_k^2) +$ $+8\sum_{i<j<k<m} x_ix_jx_kx_m =$ $\frac{1}{3}z_1p_1^3(\mathbf{x}) - \frac{1}{3}z_1p_3(\mathbf{x}) + \frac{1}{3}p_1^4(\mathbf{x}) - \frac{1}{3}p_1(\mathbf{x})p_3(\mathbf{x})$	

Likewise, for type D the upper-left corner of each hook can be either empty or contain one of the signs: $+$, $\oplus$, or %, representing a cross, a cross with a faucet, or an elbow with two faucets, respectively.

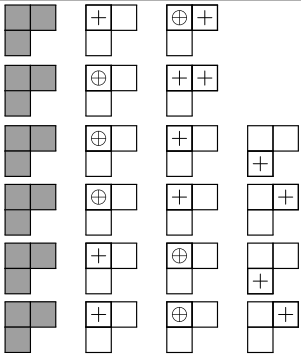
In the tables below we omit the empty Γ-blocks. The total weight of all such pipe dreams is a polynomial in z_i times a quasisymmetric function in x_i, given in the right column. The first (staircase) Young diagram is shaded gray (see Table 1, Table 2). .

Table 2
D-signed pipe dreams for some $w \in \mathcal{D}_3$.

w	Pipe dreams	Weight
$\overline{12} = s_1 s_{\bar{1}}$		$z_1 x_i$ x_i^2 $x_i x_j$ $x_i x_j$
$\mathfrak{D}_{s_1 s_{\bar{1}}} = z_1 \sum_i x_i + \sum_i x_i^2 + 2\sum_{i<j} x_i x_j = z_1 p_1(\mathbf{x}) + p_1^2(\mathbf{x})$		
$312 = s_2 s_1$		z_1^2 $z_1 x_i$ $z_1 x_i$ x_i^2 $x_i x_j$ $x_i x_j$
$\mathfrak{D}_{s_2 s_1} = z_1^2 + 2z_1 \sum_i x_i + \sum_i x_i^2 + 2\sum_{i<j} x_i x_j = z_1^2 + 2z_1 p_1(\mathbf{x}) + p_1^2(\mathbf{x})$		
$\overline{2}3\overline{1} = s_{\bar{1}} s_2$		$z_2 x_1$ $z_1 x_i$ x_i^2 $x_i x_j$ $x_i x_j$
$\mathfrak{D}_{s_{\bar{1}} s_2} = z_1 \sum_i x_i + z_2 \sum_i x_i + \sum_i x_i^2 + 2\sum_{i<j} x_i x_j =$ $(z_1 + z_2) p_1(\mathbf{x}) + p_1^2(\mathbf{x})$		
$\overline{13}2 = s_2 s_1 s_{\bar{1}}$		$z_1 x_i^2$ $z_1 x_i x_j$ $z_1 x_i x_j$ x_i^3 $x_i^2 x_j$ $x_i^2 x_j$

(*continued on next page*)

Table 2 (*continued*).

w	Pipe dreams	Weight
		$x_i x_j^2$ $x_i x_j^2$ $x_i x_j x_k$ $x_i x_j x_k$ $x_i x_j x_k$ $x_i x_j x_k$
$\mathfrak{D}_{s_2 s_1 s_2^{\vee}} = z_1 \sum_i x_i^2 + 2z_1 \sum_{i<j} x_i x_j + 2 \sum_{i<j} (x_i^2 x_j + x_i x_j^2) +$ $+4 \sum_{i<j<k} x_i x_j x_k = z_1 p_1^2(\mathbf{x}) + \frac{2}{3} p_1^3(\mathbf{x}) + \frac{1}{3} p_3(\mathbf{x})$		

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