Giant Components in Random Temporal Graphs

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Joint works with Ruben Becker, Arnaud Casteigts, Pierluigi Crescenzi, Bojana Kodric, Malte Renken, Viktor Zamaraev,

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September 2023

RANDOM 2023

Temporal graphs: context

Consider use of a UAV for aerial photography...

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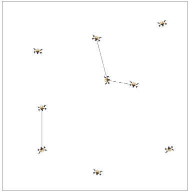

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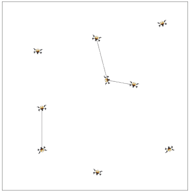

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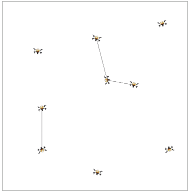

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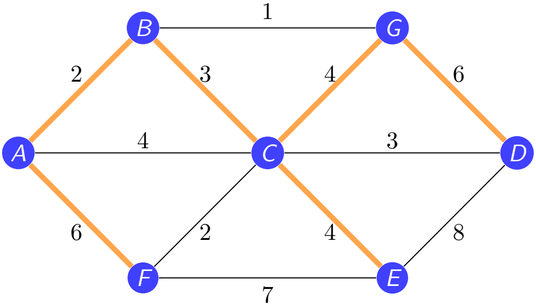
With UAVs as nodes, add temporary edges when connections are useable
We study sequences of edge-E-at-time-T for relayed data to take

Temporal graphs and paths

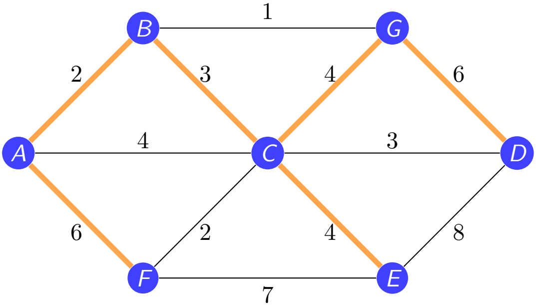


Definition: Temporal graph

Temporal graph is a graph with edge presence times

Temporal path: path with edges crossed at increasing presence times

Temporal graphs and paths



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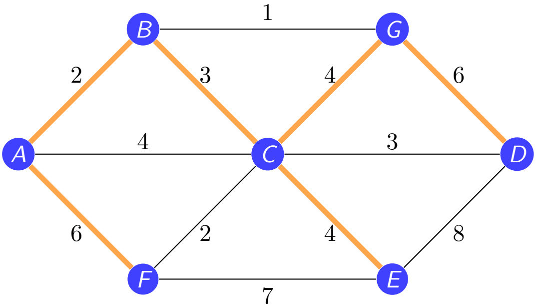

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Temporal path: path with edges crossed at increasing presence times

$A - B - C - G - D$: temporal path, times: $2 < 3 < 4 < 6$
$A - C - D$: not temporal path, times: $4 > 3$

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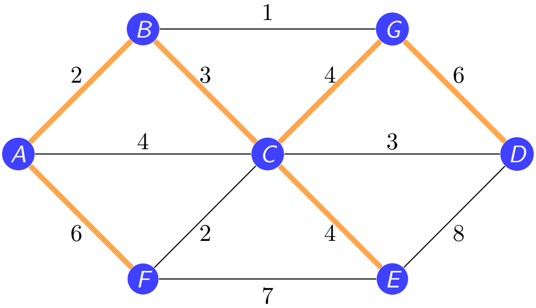

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$A - B - C - G - D$: temporal path, times: $2 < 3 < 4 < 6$
$A - C - D$: not temporal path, times: $4 > 3$

$A - C - E - D$: definition-dependent! times: $4 = 4 < 8$

Temporal graph reachability

$u \rightsquigarrow v$: there is a temporal path from u to v

- Unlike undirected graphs, $\rightsquigarrow$ is **not symmetric**
  $A \overset{1}{—} B \overset{2}{—} C$: $A \rightsquigarrow C$, but $C \not\rightsquigarrow A$
- Unlike static (non-temporal) graphs, $\rightsquigarrow$ is **not transitive**
  $C \rightsquigarrow B$, $B \rightsquigarrow A$, but $C \not\rightsquigarrow A$
- Many obvious facts about static graph do not translate
  If we have a source, a sink, and a path from the sink to the source…
  still **not always temporally connected**!
- A journey with multiple changes

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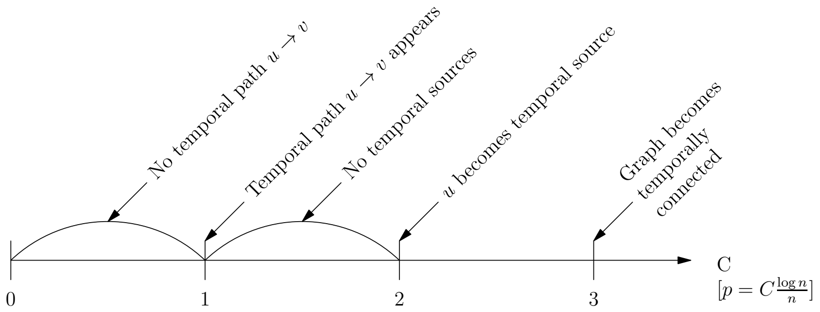
Our focus

Definition: RSTG $\mathcal{F}_{n,p}$

Random Simple Temporal Graph (RSTG) is an Erdős-Rényi random graph with uniformly random (strict) edge order
Notation: $\mathcal{F}_{n,p}$



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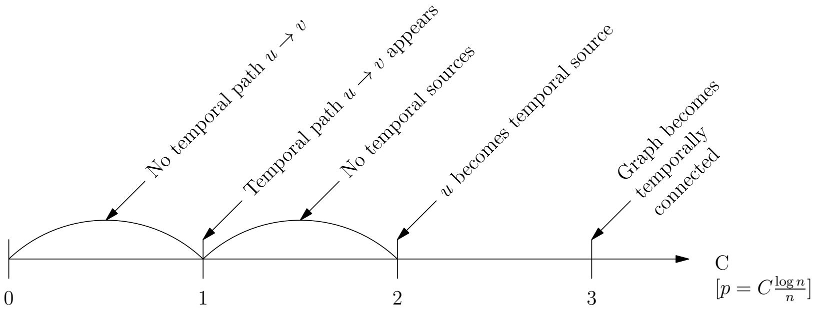
RSTG $\mathcal{F}_{n,p}$: Erdős-Réniy random graph with random edge order



We study temporal connected components

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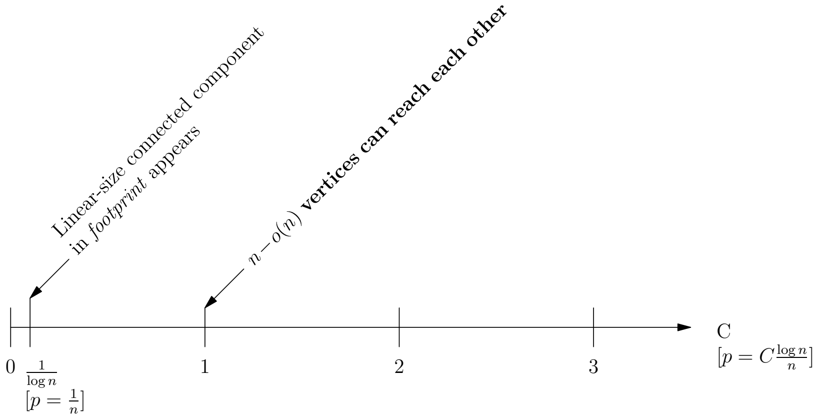

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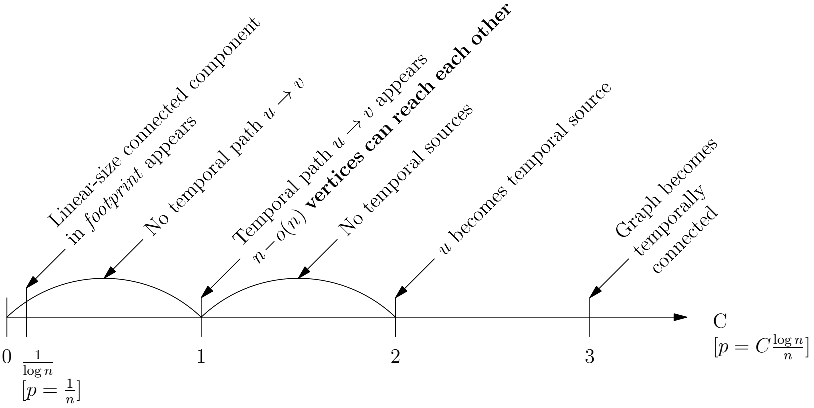

We study temporal connected components

Connected components

What is a (strongly) connected component?
«Everyone reaches everyone»

Inside or outside the component?
Normally irrelevant: A reaches B via C, then C reaches B and *everyone reached by B*

No transitivity for temporal reachability!

- *Open* connected components:
  every node in the component reaches every other node
  ... using paths that *may* leave the component and re-enter it
- *Closed* connected components:
  every node in the component reaches every other node
  via paths *inside* the component

In our random setting both happens roughly at the same time
Proofs simpler for the open case

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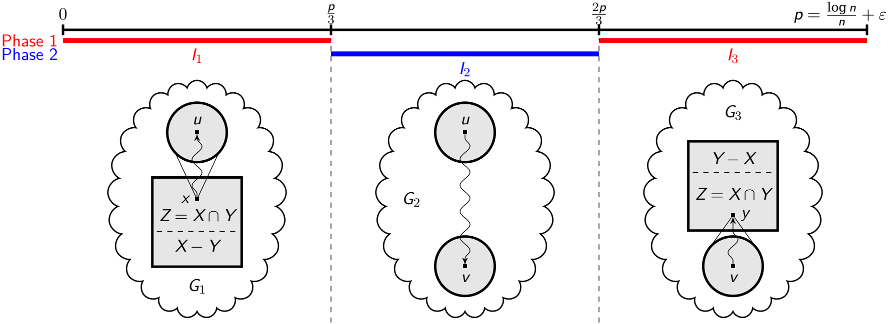
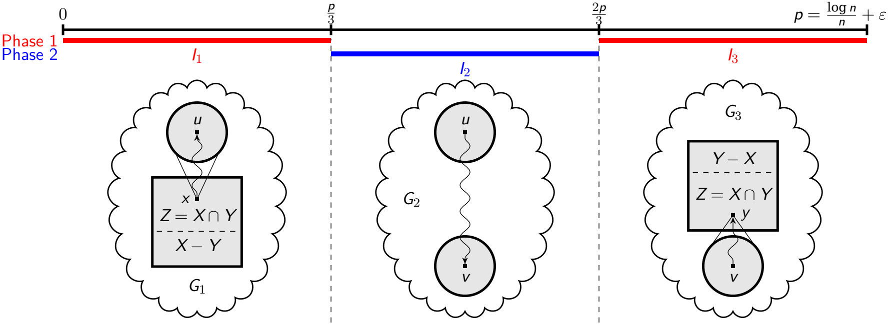


Construction: Open component

First phase: a typical vertex reaches $\omega(n^{1/3})$ vertices

Last phase: a typical vertex is reached by $\omega(n^{1/3})$ vertices

Middle phase: any two large sets of vertices are connected —
despite loss of the edges used for first/last phase



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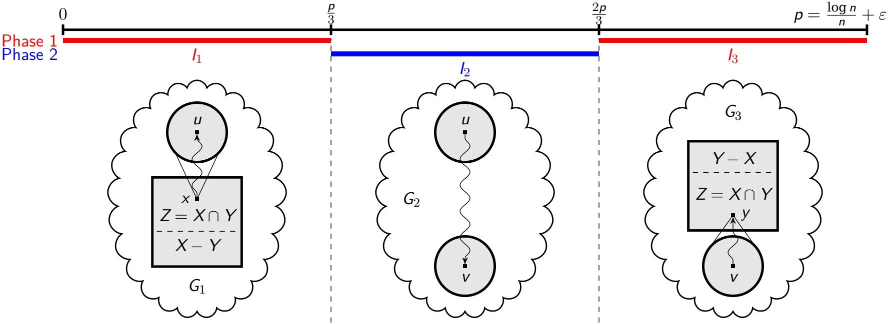

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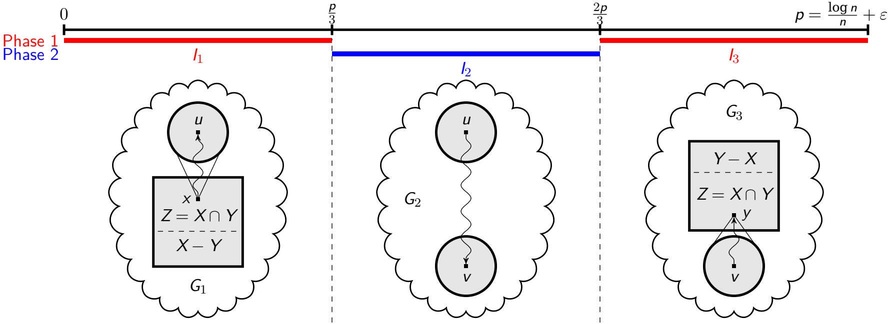

Choice of the vertices

- Normally takes $\frac{\log n}{n}$ time and $O(\log n)$ hops for $u \rightsquigarrow v$
- But a vertex can «sleep» with no edges for some time
- Longest observed sleeping time: $\frac{\log n}{n}$
- Sleep determines extra $\frac{\log n}{n}$ per level of generality!
  typical pair $\rightarrow$ source/sink $\rightarrow$ full connectivity
- Most vertices are not sleepers, yield a closed connected component

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Construction: Foremost tree

Building block for larger construction

Tree of temporal paths

- Start with a single vertex

    advanced version: and some starting time

- Add earliest later edge that adds a new vertex to tree
- Repeat

Same can be done in reverse

RSTG — Reachability-Equivalent Definitions

- $\mathcal{G}_{n,p}$ (Erdős–Rényi random graph) with random edge order
- $\mathcal{G}_{n,p}$ with random edge labels from $[0;1]$
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- K_n (clique) random edge labels from $[0;1]$, edges after p removed

When things happen as p grows for fixed labelling of K_n?

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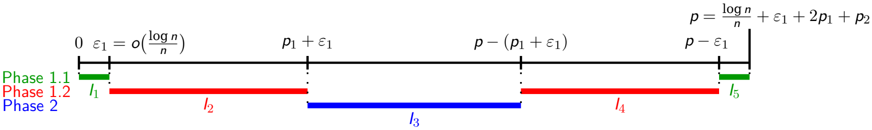
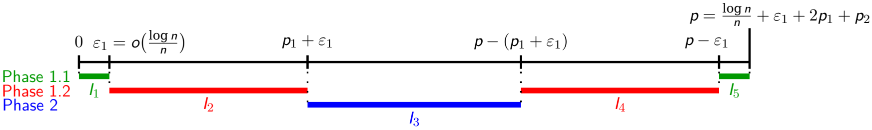


Closed components

- Similar to the open case
- Extra sub-phases added in the beginning/in the end
  - Chosen vertices quickly reach $(\log n)^{\Theta(1)}$ vertices without going through the excluded ones
  - Concentration of growth speed afterwards — starting set large enough



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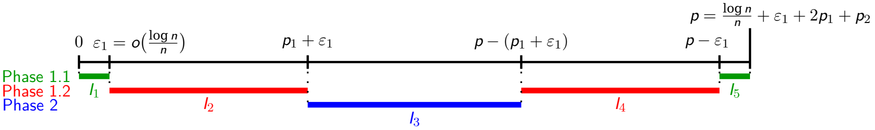

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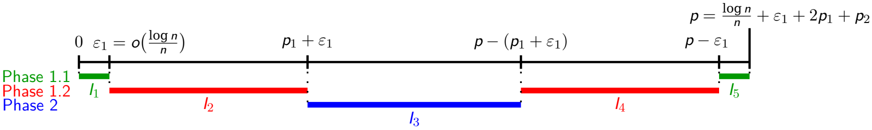

Conclusion

- Giant connected component arises once vertices are «typically» connected
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- Techniques developed for multi-phase analysis of temporal reachability

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Thanks for your attention!

Questions?

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