Illusions: curves of zeros of Selberg zeta functions

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On one property of one analytic function

Selberg Zeta Function

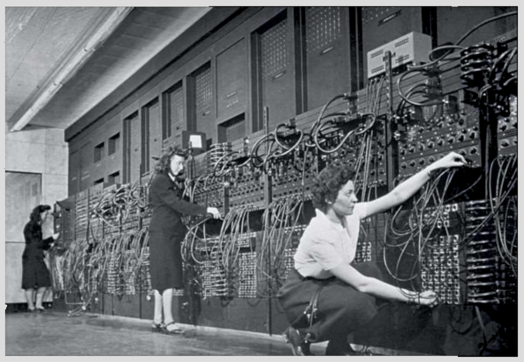
Let X be a compact surface of constant negative sectional curvature $\kappa = -1$. Define

$$Z_X(s) = \prod_{n=0}^{\infty} \prod_{\substack{\gamma = \text{primitive} \\ \text{closed geodesic}}} \left(1 - e^{-(s+n)\ell(\gamma)}\right),$$

Theorem (Selberg, 1956)

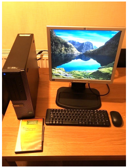
Let X be a compact Riemann surface. Then the infinite product converges to an analytic non-zero function on $\Re(s) > 1$ and extends as an analytic function to $\mathbb{C}$. The function Z_X has a simple zero at $s = 1$ and for any zero s in the critical strip $0 < \Re(s) < 1$ we have that either $s \in [0, 1]$ is real, or $\Re(s) = \frac{1}{2}$.

Years Past...



ENIAC and its first programmers, c.1950

$\Longrightarrow$



Dell Mini, 2017

Numerical Experiments Revealed

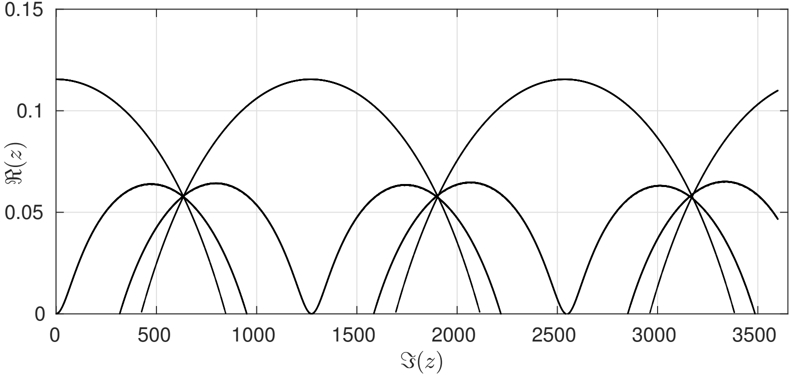


Figure: 29504 Zeros of *an approximation* to the Selberg zeta function associated to a pair of pants. D. Borthwick, 2014

This Plot Raised Many Questions

1. What exactly the approximation is? (An infinite product can't be evaluated numerically, unless it can be reduced to a finite one.)
2. If we consider another approximation to the same function, will the plot be different?
3. Are these zeros any close to the zeros of ζ?
4. Why do we see the curves?
5. If we consider another surface, how the plot will change?

Another Example

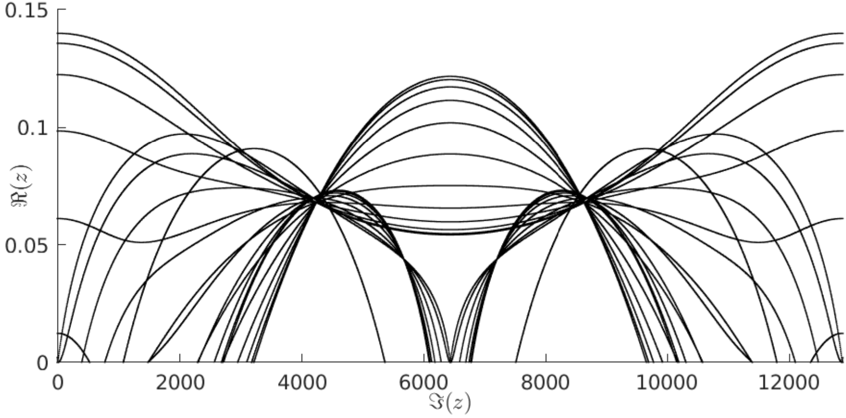


Figure: 107164 Zeros of the Selberg zeta function associated to a one-holed torus. P.V., 2018.

Why Do We See the Curves?

It is a feature (or a bug) of the outlook we have, like the photo below.

Figure: P.V. holding the Hunter's moon on the 24th of October.

Disappearance of the Curves

Take an affine transform for a closer look ...

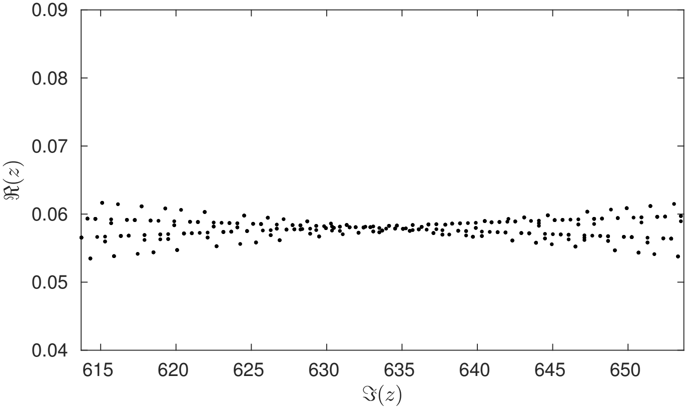
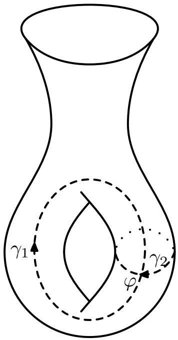


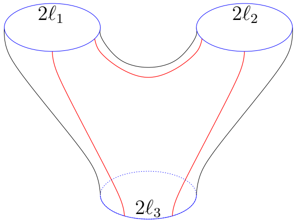
Figure: A zoom-in of the plot of the zero set of the Selberg's zeta for a pair of pants.

A One-Holed Torus



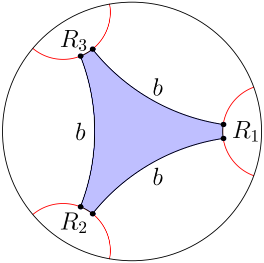
- Topologically one-holed torus T is a punctured sphere with a handle;
- It is a surface of constant negative curvature -1 and cannot be embedded into $\mathbb{R}^3$ by Efimov's theorem;
- As a metric space, it is uniquelly defined by the lengths of two geodesics and the angle inbetween $T = T(\ell_1, \ell_2, \varphi)$;
- It possess countably many closed geodesics $\{\gamma_n\}$ of lengths $0 < \ell(\gamma_1) < \ell(\gamma_2) < \ldots < \ell(\gamma_n) \ldots \to \infty$
- Symmetric torus means $\ell_1 = \ell_2$, $\varphi = \frac{\pi}{2}$.

A Pair of Pants



- Topologically pair of pants X is a 3-punctured sphere;
- It is a surface of constant negative curvature -1 and cannot be embedded into $\mathbb{R}^3$ by Efimov's theorem;
- As a metric space, it is uniquelly defined by the lengths of the three boundary geodesics: $X = X(\ell_1, \ell_2, \ell_3)$;
- It possess countably many closed geodesics $\{\gamma_n\}$ of lengths $0 < \ell(\gamma_1) < \ell(\gamma_2) < \ldots < \ell(\gamma_n) \ldots \to \infty$
- Symmetric pair of pants means $\ell_1 = \ell_2 = \ell_3 =: b$.

The Hyperbolic Action



- Cutting the pair of pants along the red geodesics, we obtain a pair of hexagons;
- The hexangons can be immersed into $\mathbb{H}^2$ as right-angled hexagons;
- The Fuchsian group $\Gamma = \langle R_1, R_2, R_3 \rangle$, generated by reflections with respect to the "cuts", gives a pair of pants as a double cover of the factor space $X(b) = \mathbb{H}^2/\Gamma$;
- To any geodesic X corresponds a geodesic in $\mathbb{H}$; for any closed geodesic γ there exists $R_\gamma \in \Gamma$ preserving γ.
- The action $\Gamma \curvearrowright \mathbb{H}^2$ is hyperbolic.

Properties of Selberg Zeta Functions

1. In 1992, Guillopé established that in the case of geometrically finite hyperbolic surfaces of infinite area, the infinite product Z_X converges for $\Re(s)$ sufficiently large and has a meromorphic extension to $\mathbb{C}$.
2. Zeros of the Selberg zeta function correspond to the poles of the Ruelle zeta function given by

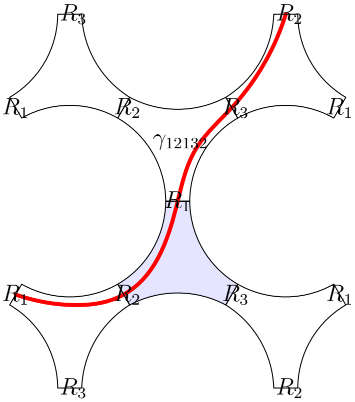
$$\zeta(s) := \frac{Z_X(s+1)}{Z_X(s)} = \prod_{\substack{\gamma = \text{primitive} \\ \text{closed geodesic}}} \left(1 - e^{-s\ell(\gamma)}\right)^{-1}$$

3. There exists the largest real zero δ, which is equal to the Hausdorff dimension of the limit set of Γ (a subset of the unit circle).
4. There is no other zeros with $\Re(s) = \delta$

Properties of Selberg Zeta Functions (continued)

5. δ is the growth rate of the number of primitive closed geodesics $\delta = \lim_{t\to\infty} \frac{1}{t} \log \#\{\gamma \colon \ell(\gamma) \leq t\}$. Moreover, $\#\{\gamma \colon \ell(\gamma) \leq t\} \sim \frac{e^{\delta t}}{\delta t}$.
6. For a symmetric pair of pants $\delta = \delta(b) \sim \frac{1}{b}$ (McMullen)
7. There exists $\varepsilon > 0$ such that there is only finite number of zeros satisfying $\Re(s) > \delta - \varepsilon$ (Jakobson–Naud)
8. Complex zeros are related to the eigenvalues of the Laplacian operator acting on L_2 functions and are a subject of intensive research (Nonnenmacher, Patterson, Perry, Zworski ...). These are defined as the poles of the resolvent and are referred to as *resonances* of X.

Closed Geodesics



To every closed geodesic γ on $X(b)$ corresponds

- a cutting sequence of period $2n$

$$\cdots j_{2n-1} j_{2n} j_{2n+1} \cdots ,$$

  where $j_k \in \{1, 2, 3\}$, $j_k \neq j_{k+1}$ for $1 \leq k \leq 2n$ and $j_{2n} \neq j_1$.

- a periodic orbit of the subshift σ of finite type on the space of 3 symbols $\Sigma = \{1, 2, 3\}^{\mathbb{Z}}$ with transition matrix

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Transition Matrices

Let's fix n and define $r_n\colon \Sigma \to \mathbb{R}$, $r_n(\xi) = \ell(\gamma_{[\xi_{[n/2]},\xi_{[n/2]+1}]})$, where γ is chosen such that

$$\ell(\gamma) = \min_{\gamma'}\{\ell(\gamma') \mid \gamma' \text{ intersects } \xi_1, \ldots \xi_n\}$$

Let $\xi^1, \ldots, \xi^N$ be all subsequences of the sequences in Σ of the length n. We define an $N \times N$ transition matrix

$$A^n_{i,j} = \begin{cases} 1, & \text{if } \xi^i_{k+1} = \xi^j_k;\ \text{for } k = 1, \ldots, n-1 \\ 0, & \text{otherwise.} \end{cases}$$

and a complex matrix function

$$A\colon \mathbb{C} \to Mat(N,N) \qquad A_{i,j}(s) = \exp(-sr_n(\xi)) \cdot A^n_{i,j},$$

where $\xi = \xi^i_1 \ldots \xi^i_n \xi^j_n$.

Key Lemma

Lemma

$$\prod_{\substack{\gamma = \text{ primitive} \\ \text{closed geodesic}}} \left(1 - e^{-s\ell(\gamma)}\right)^2 = \lim_{n\to\infty} \det\left(I_N - A^2(s)\right);$$

where I_N is the $N \times N$ identity matrix.

Choosing $n = 2$ above we get $r_2 \equiv b$

$$\det(Id - e^{-2sb}A^2) = (1 - 4e^{-2bs})(1 - e^{-2bs})^2$$

For a first approximation...

- The zero set belongs to a pair of straight lines
- The distance between consequetive zeros is $\frac{\pi}{b}$.

Curves of Zeros — I

Using $n = 3$ in the approximation of geodesics length

$$r_3(\xi) = b + c(\xi)e^{-b} + O(e^{-2b}),$$

we obtain a 6×6 matrix which determinant has the zero set on the curves

$$\mathcal{C}_1 = \left\{ \frac{1}{2b} \ln |2 - 2\cos(t)| + ie^b t \mid t \in \mathbb{R} \right\};$$

$$\mathcal{C}_2 = \left\{ \frac{1}{2b} \ln |2 + \cos(2t)| + ie^b t \mid t \in \mathbb{R} \right\};$$

$$\mathcal{C}_3 = \left\{ \frac{1}{2b} \ln \left| 1 - \frac{1}{2}e^{2it} - \frac{1}{2}e^{it}\sqrt{4 - 3e^{2it}} \right| + ie^b t \mid t \in \mathbb{R} \right\};$$

$$\mathcal{C}_4 = \left\{ \frac{1}{2b} \ln \left| 1 - \frac{1}{2}e^{2it} + \frac{1}{2}e^{it}\sqrt{4 - 3e^{2it}} \right| + ie^b t \mid t \in \mathbb{R} \right\}.$$

Curves of Zeros — II

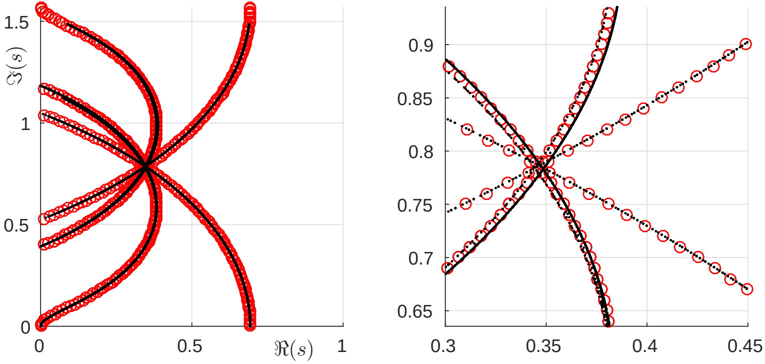


Figure: The zero sets of $\zeta_X\left(\frac{\sigma}{b} + ite^b\right)$ and renormalized curves $\mathcal{C}_k$, for $b = 6$; and a zoomed neighbourhood of $\left(\frac{\ln 2}{2}, \frac{\pi}{4}\right)$.

Comments on Geometric Approximation

1. Increasing n we do not see a change in the zero set for $\Im(z) < e^{3b}$;
2. There is no good estimates on error term (or rate of convergence).

We need to estimate the approximation error.

Transfer Operators Technique

Given a hyperbolic action, we introduce:

1. A proper Banach space of analytic functions;
2. A nuclear transfer operator acting on the Banach space;
3. The determinant of the transfer operator, which is an analytic function;
4. Ruelle–Pollicott dynamical zeta function;
5. The Ruelle zeta function turns to be an analytic function, which is closely related to the determinant (of the transfer operator);
6. The zeta function can be computed very efficiently using periodic orbits data (of the hyperbolic system) and its zeros provide quontitative information about the system.

The Banach Space

The space $\mathcal{B}$ of analytic functions on the union of disjoint disks $\sqcup_{k=1}^{3} U_k$, chosen so that $R_i(U_j \cup U_k) \subset U_i$ for any three distinct $i, j, k \in \{1, 2, 3\}$.

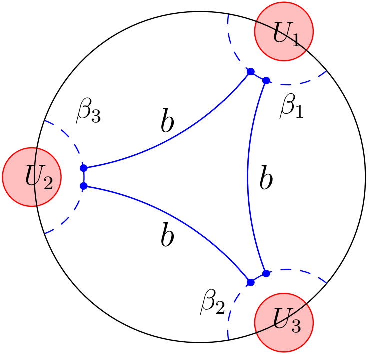


Figure: The domain of analytic functions forming the Banach space (in pale red).

Transfer Operator

We define a transfer operator $\mathcal{L}_s$ on the space $\mathcal{B}$ by

$$(\mathcal{L}_s f)\,|_{U_1}\,(z_1) = |R_1'(z_2)|^s f(z_2) + |R_1'(z_3)|^s f(z_3),$$

where z_2, z_3 are preimages of $z_1 \in U_1$ with respect to reflection with respect to the geodesic β_1.

Lemma (Grothendieck–Ruelle)

The operator $\mathcal{L}_s$ is nuclear.

We may write *the determinant* of the transfer operator as

$$\zeta(z, s) \stackrel{\text{def}}{=} \exp\Big(-\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{Tr}\mathcal{L}_s^n\Big) = \det(I - z\mathcal{L}_s).$$

Zeta Function Magic

Lemma (Grothendieck–Ruelle)

The trace of the transfer operator may be explicitly computed in terms of the closed geodesics.

$$\mathrm{Tr}\mathcal{L}_s^n = \sum_{|\gamma|=n} \frac{\exp(-s\ell(\gamma))}{1-\exp(-\ell(\gamma))}$$

Theorem (Ruelle)

There exists a constant δ such that the determinant is an analytic function in both variables in a strip $0 < s < \delta$, and

$$\zeta(1,s) = \zeta(s) = \exp\Big(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{|\gamma|=n} \frac{\exp(-s\ell(\gamma))}{1-\exp(-\ell(\gamma))}\Big)$$

Computing the Zeta Function

Using Ruelle's Theorem,

$$\zeta(s) = \sum_{n=0}^{\infty} z^n a_n(s)\Big|_{z=1} = \lim_{N\to\infty} \sum_{n=0}^{N} a_n(s),$$

where a_n are explicitly defined in terms of closed geodesics of the word length not more than $|\gamma| \leq 2n$, and are analytic in s:

$$a_n(s) = -\frac{1}{n} \sum_{j=0}^{n-2} a_j(s) \cdot \mathrm{Tr}\mathcal{L}_s^{n-j}$$

Lemma (after Grothendieck–Ruelle)

The terms $a_n(s)$ are decreasing superexponentially: $|a_n(s)| < \lambda(s)^{n^2}$, where $\lambda(s) < 1$ depend only on $\mathcal{L}_s$, but the estimate is not uniform in s.

Algorithm

Choosing truncation $\zeta_N(s) = \sum_{n=0}^{N} a_n(s)$, we can

1. find the largest real zero = the width of the critical strip,
2. consider a dense lattice in the strip,
3. compute the residue over each square,
4. find a zero using Newton method starting from a point of the lattice.

Numerical Output: Symmetric Pants

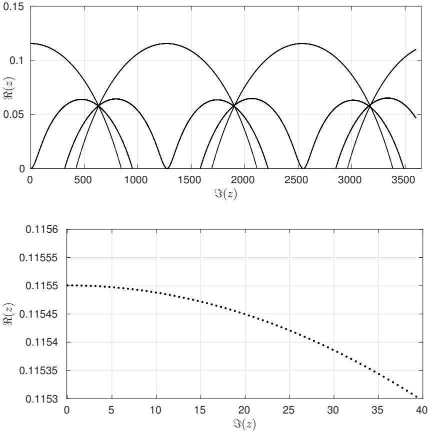


Figure: Zeros of the zeta function associated to a symmetric pair of pants and a more careful look for $b = 12$, $N = 14$.

Another Viewpoint: Exponential Sums

The function $\zeta_N(s)$ is a finite exponential sum

$$\zeta_N(s) = \sum_{j=k}^{n} \alpha_k \exp(\mu_k s),$$

where the multipliers μ_k are the lengths of closed geodesics with word length up to $2N$.

1. Zeros form a point-periodic set and belong to a finite strip, parallel to the imaginary axis
2. Their imaginary parts satisfy relation

$$\Im(s_k) = \frac{\pi}{\max \mu_k - \min \mu_k} + \varphi(k),$$

for an almost periodic function φ.

Main Approximation Result

$$\mathcal{R}(T) = \{s \in \mathbb{C} \mid 0 \leq |\Re(s)| \leq \delta \text{ and } |\Im(s)| \leq T\}.$$

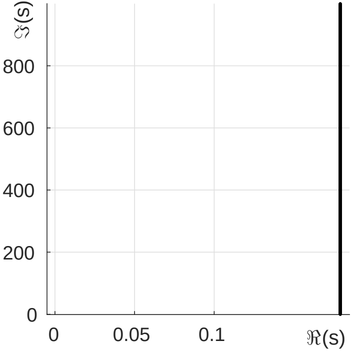
Theorem (M. Pollicott-P. V.)

Let X be a symmetric pair of pants with boundary geodesics of the length $\ell(\gamma_0) = 2b$. We may approximate ζ on the domain $\mathcal{R}(T)$ by a complex trigonometric polynomial ζ_n so that $\sup_{\mathcal{R}(T)} |\zeta - \zeta_n| \leq \eta(b, n, T)$, where $T(b) = e^{k_0 b}$ for some constant $1 < k_0 < 2$ independent of b and n, such that

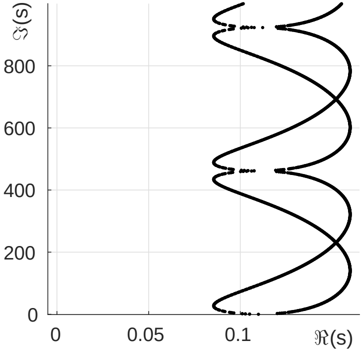
1. *for any $n \geq 14$ we have $\eta(b, n, T(b)) \leq O\left(\frac{1}{\sqrt{b}}\right)$ as $b \to \infty$*
2. *for any $b \geq 20$ we have $\eta(b, n, T(b)) \leq O\left(e^{-bk_1 n^2}\right)$ as $n \to \infty$.*

for some $k_1 > 0$ which is independent on b and n.

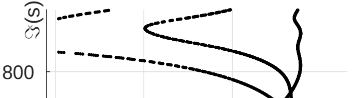
Subsequent Approximations



(a) $Z_2(s)$



(b) $Z_4(s)$



Final Approximation

Lemma (after M. Pollicott-P.V.)

There exists an explicit 6-by-6 matrix $B(s)$ *such that the real analytic function* $\zeta_{12}\left(\frac{\sigma}{b} + ite^b\right)$ *converges uniformly to* $\det(I - e^{-2\sigma - 2itbe^b} B(e^{it}))$*, and more precisely,*

$$\left| Z_{12}\left(\frac{\sigma}{b} + ite^b\right) - \det\left(I - e^{-2\sigma - 2itbe^b} B(e^{it})\right)\right| = O\left(e^{-b}\right)$$

as $b \to +\infty$*.*

- The matrix B can be constructed using a transition matrix of a subshift of finite type on the space $\{1, 2, 3\}^{\mathbb{N}}$.
- The curves $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$ computed using the formula

$$|e^{2\sigma}| = eig(B(e^{it}))$$

References

- A. Grothendieck, Produits tensoriels topologiques et espaces nucleaires, *Mem. Amer. Math. Soc.*, 16 (1955), 1–140.
- L. Guillopé, Fonctions zeta de Selberg et surfaces de géométrie finie, Adv. Stud. Pure Math., vol. 21, Kinokuniya, Tokyo, 1992, pp. 33–70.
- Parry, W. and Pollicott, M. Zeta functions and the periodic orbit structure of hyperbolic dynamics. Astérisque No. 187–188 (1990), 268 pp.
- D. Ruelle, Zeta-functions for expanding maps and Anosov flows, *Invent. Math.*, 34 (1976), 231–242.

Thank you!