Zeros of the Selberg zeta function for non-compact surfaces

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Have nothing in your houses that you do not know to be useful, or believe to be beautiful.
W. Morris
Riemann Zeta Function

Let \( \mathcal{P} \) be a set of prime numbers. Define

\[
\zeta(s) := \prod_{p \in \mathcal{P}} (1 - p^{-s})^{-1}.
\]

**Theorem (Riemann)**

The zeta function \( \zeta \) converges to a non-zero analytic function for \( \Re(s) > 1 \). Moreover, the function \( \zeta \)

1. has a unique pole at \( s = 1 \),
2. has an analytic extension to \( \mathbb{C}^* \).

**Riemann Hypothesis**

The zeta function \( \zeta \) has zeros only at negative even integers or on the critical line \( \Re(s) = \frac{1}{2} \).
Selberg Zeta Function

Let $\Gamma$ be the set of prime closed geodesics on a compact surface $X$ of constant negative sectional curvature $\kappa = -1$. Define

$$Z_X(s) = \prod_{\gamma \in \Gamma} (1 - e^{-s\ell(\gamma)}) .$$

**Theorem (Selberg)**

*Let $X$ be a compact Riemann surface. Then the function $Z_X$ has a simple zero at $s = 1$ and for any zero $s$ in the critical strip $0 < \Re(s) < 1$ we have that either $s \in [0, 1]$ is real, or $\Re(s) = \frac{1}{2}$.*

In 1992, Guillopé established that in the case of geometrically finite hyperbolic surfaces of infinite area, the function $Z_X$ has a meromorphic extension to $\mathbb{C}$.
First Attempt on Location of Zeros

We may expand (after Grothendieck–Ruelle–Pollicott)

\[ Z_{12}(s) = 1 + a_2(s) + a_4(s) + \ldots + a_{12}(s), \]

where \( a_{2j} \) are explicitly defined in terms of closed geodesics of the length not more than \( n_X j \), so that

- the constant \( n_X \) depends on the surface only,
- each \( a_{2j}(s) \) is an analytic function in \( s \).

Then we can

1. find the largest real zero = the width of the critical strip,
2. consider a dense lattice in the strip,
3. compute the residue over each square,
4. find a zero using Newton method starting from a point of the lattice.
Numerical Output: Symmetric Pants

Figure: Zeros of the zeta function associated to a symmetric pair of pants and a more careful look (after D. Borthwick).
Q&A

1. Is the zero set of $Z_{12}$ close to the zero set of $Z_X$? → Yes!
2. How can we prove this? → Use transfer operators
3. What are characteristic properties of the set of zeros of $Z_X$?

Qualitative observations

Let the length of boundary geodesics be $2b$. Then

- The vertical spacing of zeros is approximately $\frac{\pi}{b}$.
- The pattern of zeros appears to lie on four distinct curves, which seem to have a common point at $\frac{\delta}{2} + i \frac{\pi}{2} e^b$.
- The vertical apparent periodicity of the pattern of zeros is approximately $\pi e^b$.

4. How can we explain them? → Study the very beginning of the geodesics length spectrum
Transfer Operators Technique

Given a hyperbolic action, we introduce:

1. A proper Banach space of analytic functions;
2. A nuclear transfer operator acting on the Banach space;
3. The determinant of the transfer operator, which is an analytic function;
4. Ruelle–Pollicott dynamical zeta function;
5. The Selberg zeta function turns to be an analytic function, which is closely related to the determinant (of the transfer operator);
6. The zeta function can be computed very efficiently using periodic orbits data (of the hyperbolic system) and its zeros provide quantitative information about the system.
A Pair of Pants

- Topologically pair of pants $X$ is a 3-punctured sphere;
- It is a surface of constant negative curvature $-1$ and cannot be embedded into $\mathbb{R}^3$ by Efimov’s theorem;
- As a metric space, it is uniquely defined by the lengths of the three boundary geodesics: $X = X(\ell_1, \ell_2, \ell_3)$;
- It possess countably many of closed geodesics $\{\gamma_n\}$ of the lengths $0 < \ell(\gamma_1) < \ell(\gamma_2) < \ldots < \ell(\gamma_n) \ldots \to \infty$;
- Symmetric pair of pants means $\ell_1 = \ell_2 = \ell_3 =: b$. 

\[
\begin{tikzpicture}
  \draw [thick,blue] (0,0) to [out=90,in=180] (1,1) to [out=0,in=90] (2,0);
  \draw [thick,blue] (2,0) to [out=-90,in=0] (1,-1) to [out=180,in=-90] (0,0);
  \draw [thick,red] (1,1) to (1,-1);
  \draw [thick,red] (0,0) to [out=90,in=180] (1,1);
  \draw [thick,red] (2,0) to [out=-90,in=0] (1,-1);
  \node at (0,0) {$2\ell_1$};
  \node at (2,0) {$2\ell_2$};
  \node at (1,-1) {$2\ell_3$};
\end{tikzpicture}
\]
The Hyperbolic Action

- Cutting the pair of pants along the red geodesics, we obtain a pair of hexagons;
- The hexagons can be immersed into $\mathbb{H}^2$ as right-angled hexagons;
- The Fuchsian group $\Gamma = \langle R_1, R_2, R_3 \rangle$, generated by reflections with respect to the “cuts”, gives a pair of pants as the factor space $X(b) = \mathbb{H}^2 / \Gamma$.
- Any closed geodesic is uniquely defined by a periodic cutting sequence, and can be associated to an element of the Fuchsian group: $\cdots R_{k_1} R_{k_2} R_{k_3} \cdots \leftrightarrow \gamma_{\cdots k_1, k_2, k_3}$.
- The action $\Gamma \acts \mathbb{H}^2$ is hyperbolic;
The Banach Space

The space $\mathcal{B}$ of analytic functions on the union of disjoint disks $\bigcup_{k=1}^{3} U_k$, chosen so that $R_i(U_j \cup U_k) \subset U_i$ for any three distinct $i, j, k \in \{1, 2, 3\}$.

Figure: The domain of analytic functions forming the Banach space (in pale red).
Transfer Operator

We define a transfer operator $\mathcal{L}_s$ on the space $\mathcal{B}$ by

$$(\mathcal{L}_sf) |_{U_1} (z_1) = |R'_1(z_2)|^s f(z_2) + |R'_1(z_3)|^s f(z_3),$$

where $z_2, z_3$ are preimages of $z_1 \in U_1$ with respect to reflection with respect to the geodesic $\beta_1$.

Lemma (Grothendieck–Ruelle)

The operator $\mathcal{L}_s$ is nuclear.

We may write the determinant of the transfer operator as

$$\zeta(z, s) \overset{\text{def}}{=} \exp \left( - \sum_{n=1}^{\infty} \frac{z^n}{n} \text{Tr} \mathcal{L}_s^n \right).$$
Lemma (Grothendieck–Ruelle)

The trace of the transfer operator may be explicitly computed in terms of the closed geodesics.

\[ \text{Tr} \mathcal{L}^n_s = \sum_{|\gamma|=n} \frac{\exp(-s \ell(\gamma))}{1 - \exp(-\ell(\gamma))} \]

Theorem (Ruelle)

There exists a constant \( \delta \) such that the determinant is an analytic function in both variables in a strip \( 0 < s < \delta \), and

\[ \zeta(1, s) = Z_X(s) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{|\gamma|=n} \frac{\exp(-s \ell(\gamma))}{1 - \exp(-\ell(\gamma))}\right) \]
Estimating Approximation Error

Lemma (after M. Pollicott–P. V.)

Let $X$ be a pair of pants with boundary geodesics of the length $2b$. Then we may expand the determinant in a Taylor series in $z$:

$$
\zeta(z, s) = 1 + \sum_{n=1}^{\infty} a_n(s) z^s
$$

with the coefficients $a_n$ bounded by $|a_n(s)| \leq C^n(s) \lambda \frac{n(n+1)}{2}$, where

1. $0 < \lambda \leq (4e^{-b} + O(e^{-2b}))^{\frac{1}{3}}$ as $b \to \infty$.
2. $|C(\sigma + it)| \leq 16e^{-2b\sigma} + e^{2\pi t}$.
Main Approximation Result

\[ \mathcal{R}(T) = \{ s \in \mathbb{C} \mid 0 \leq |\Re(s)| \leq \delta \text{ and } |\Im(s)| \leq T \}. \]

**Theorem (M. Pollicott-P. V.)**

Let \( X \) be a symmetric pair of pants with boundary geodesics of the length \( \ell(\gamma_0) = 2b \). We may approximate \( Z_X \) on the domain \( \mathcal{R}(T) \) by the complex trigonometric polynomial \( Z_n \) so that
\[
\sup_{\mathcal{R}(T)} |Z_X - Z_n| \leq \eta(b, n, T),
\]
where \( T(b) = e^{k_0 b} \) for some constant \( 1 < k_0 < 2 \) independent of \( b \) and \( n \), such that

1. for any \( n > 3 \) we have \( \eta(b, n, T(b)) \leq O\left(\frac{1}{\sqrt{b}}\right) \) as \( b \to \infty \)
2. for any \( b \geq 8 \) we have \( \eta(b, n, T(b)) \leq O\left(e^{-b k_1 n^2}\right) \) as \( n \to \infty \).

for some \( k_1 > 0 \) which is independent on \( b \) and \( n \).
Subsequent Approximations — I

Figure: Plots of the zero set of $Z_{2n}(s)$, for $b = 5$
Subsequent Approximations — II

Figure: Plots of the zero set of $Z_{2n}(s)$, for $b = 5$
Subsequent Approximations — III

Figure: Plots of the zero set of $Z_{2n}(s)$, for $b = 5$
Subsequent Approximations — IV

Figure: Plots of the zero set of $Z_{2n}(s)$, for $b = 5$
Curves of Zeros — I

Figure: The zero sets of $Z_X\left(\frac{\sigma}{b} + ite^b\right)$ (red) and the curves $C_k$, (black) for $b = 5$; and a zoomed neighbourhood of $\left(\frac{\ln 2}{2}, \frac{\pi}{4}\right)$. 
The curves

\[ C_1 = \{ \ln |e^{2it} + 1| + it \mid t \in \mathbb{R} \} ; \]
\[ C_2 = \{ \ln |e^{2it} - 1| + it \mid t \in \mathbb{R} \} ; \]
\[ C_3 = \{ \ln \left| 2 - e^{4it} - e^{2it} \sqrt{4e^{2it} - 3e^{4it}} \right| - \ln 2 + it \mid t \in \mathbb{R} \} ; \]
\[ C_4 = \{ \ln \left| 2 - e^{4it} + e^{2it} \sqrt{4e^{2it} - 3e^{4it}} \right| - \ln 2 + it \mid t \in \mathbb{R} \} . \]

contain the zero set of a real analytic function

\[ g_X(\sigma + it) = \det (I - \exp(-2\sigma + 2ite^b)B(e^{2it})) , \]

where \( B(z) \) is a transition matrix for SSFT on the space of cutting sequences, encoding data about closed geodesics.
Asymptotic Result for Large $b$

$$B(z) = \begin{pmatrix} 1 & z & 0 & 0 & z^2 & z \\ z & 1 & z^2 & z & 0 & 0 \\ 0 & 0 & 1 & z & z & z^2 \\ z^2 & z & z & 1 & 0 & 0 \\ 0 & 0 & z & z^2 & 1 & z \\ z & z^2 & 0 & 0 & z & 1 \end{pmatrix}$$

$$\mathcal{R}(T) = \{ s \in \mathbb{C} \mid 0 \leq |\Re(s)| \leq \ln 2 \text{ and } |\Im(s)| \leq T \}.$$ 

**Theorem (M. Pollicott–P. V.)**

For any $1 < k < 2$ a real analytic function $Z_X\left(\frac{\sigma}{b} + it e^b\right)$ converges uniformly to $g_X(\sigma + it)$, more precisely, as $b \to \infty$,

$$\sup_{s \in \mathcal{R}(e^{kb})} \left| Z_X \left(\frac{\sigma}{b} + it e^b\right) - g_X(\sigma + it) \right| = O \left( \frac{1}{\sqrt{b}} \right).$$
Consider a closed geodesic $\gamma$ on $X(b)$ corresponding to a cutting sequence of period $2n$

$$\cdots j_{2n-1}j_{2n}j_{2n+1} \cdots,$$

where $j_k \in \{1, 2, 3\}$, $j_k \neq j_{k+1}$ for $1 \leq k \leq 2n$ and $j_{2n} \neq j_1$. Then

$$\ell(\gamma) = 2nb + c(\gamma)e^{-b} + O(e^{-2b}),$$

where

$$c(\gamma) = \# \{1 \leq k \leq 2n : j_k \neq j_{k+2} \mod 2n\}.$$
References


