Kinematic Fast Dynamo Problem

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The universe enjoys its elegance being noticed.
J. Green

A world around us

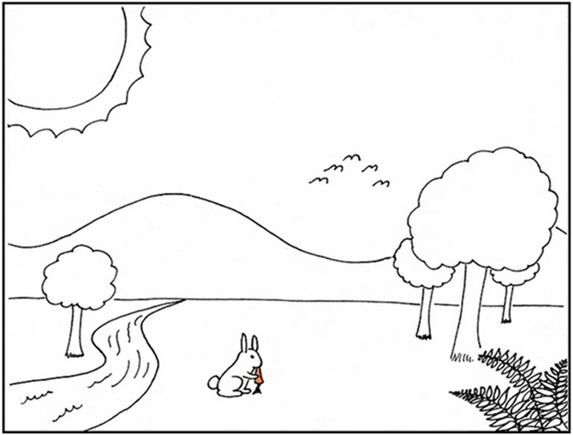
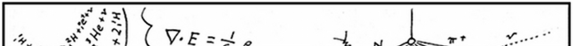


Figure: A walk to the University of Warwick on a sunny day in May.



Dynamo equations

Ignoring the Lorenz force, the system of magnetohydrodynamics may be reduced to a Navier-Stokes type equation.

The kinematic dynamo equations

$$\begin{cases} \dfrac{\partial B}{\partial t} = (B \cdot \nabla)v - (v \cdot \nabla)B + \varepsilon \Delta B; \\ \nabla \cdot v = \nabla \cdot B = 0. \end{cases}$$

- v is the (known) velocity field of a fluid filling a certain compact domain M;
- B is the (unknown) magnetic field;
- ε is a dimensionless parameter reflecting the magnetic diffusion through the boundary of M.

Kinematic fast dynamo problem

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Problem (Main fast dynamo problem)

Does there exist a divergence-free velocity field v in a compact domain M tangent to the boundary, such that the energy of the magnetic field $B(t)$ grows exponentially in time for some initial field B_0 in the presence of small diffusion ($\varepsilon > 0$)?

- a velocity field v which gives a positive answer, is called *fast dynamo*
- this is a Cauchy problem
- A case of special interest are stationary velocity fields in three-dimensional domains.

Dynamo Theorems

Theorem (The case $\varepsilon = 0$ is easy)

On an arbitrary n-dimensional manifold any divergence-free vector field with a stagnation point with a unique positive eigenvalue is a non-dissipative kinematic fast dynamo.

Theorem (Antidynamo theorem)

A transitionally, helically, or axially symmetric magnetic field in $\mathbb{R}^3$ cannot be maintained by a dissipative dynamo action.

The provisional flow

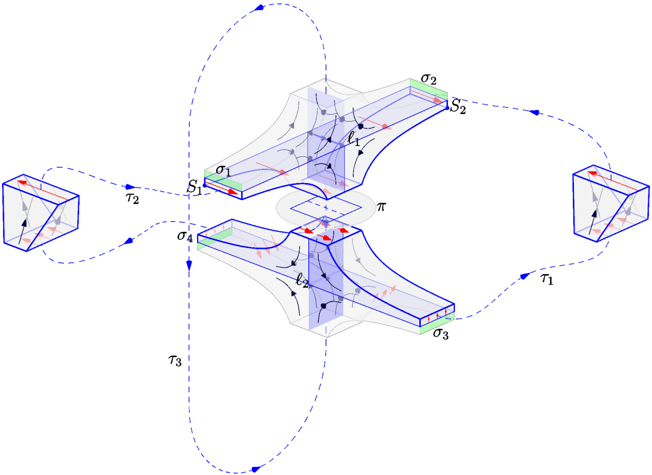

Figure: Dynamo manifold with the fluid flow (blue) and magnetic induction field (red). The labels S_1 and S_2 mark periodic saddle points. $\tau_{1,2,3,4}$ stand for manifolds equivalent to cylinders.

From flows to diffeomorphisms

Lemma

The exponent of the Laplacian operator in $\mathbb{R}^n$ is the Weierstrass transform.

$$(\exp(\varepsilon\Delta)B)(z) = (W_\varepsilon B)(z) \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \frac{1}{(\sqrt{2\pi}\varepsilon)^d} \exp\Big(-\frac{|z-t|^2}{2\varepsilon^2}\Big) B(t)\mathrm{d}t$$

The Lemma gives a natural discretization of the dynamo equation, where the action of piecewise diffeomorphisms is used instead of the transport by a flow

$$B \mapsto (W_\varepsilon g_*)B, \qquad g \text{ is a piecewise diffeomorphism.}$$

Dynamo Theorems (continued)

Theorem (Dissipative dynamos on surfaces)

Let $g\colon M \to M$ be an area-preserving diffeomorphism of the two-dimensional compact Riemannian manifold M. Then g is a dissipative fast dynamo if and only if the induced linear operator g_{*1} on the first homology group has an eigenvector ν with eigenvalue $|\lambda| > 1$. The dynamo increment is independent of ε:

$$\lim_{n\to\infty} \frac{1}{n} \ln \|B_n\| = \ln |\lambda|$$

for almost any initial vector field B_0. (Here $B_{n+1} = \exp(\varepsilon\Delta) g_* B_n$.)

The proof uses duality between vector fields and one forms and cannot be generalised to high dimensions.

Main result

Theorem (Main)

There exists a piecewise diffeomorphism $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that for some vector field B_0 in $\mathbb{R}^2$

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}\log\left\|\left(W_\varepsilon F_*\right)^n B_0\right\|_{\mathcal{L}_1} > 0.$$

The map F may be realised as the first return map of the provisional flow to the section π.

This theorem doesn't resolve the problem for the flow, because the first return time is not constant and tends to infinity near separatrices of the saddles.

Noise instead of diffusion

Definition (Small random perturbations)

Given a map $F\colon \mathbb{R}^n \to \mathbb{R}^n$ we define *a natural extension* $\widehat{F}\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by $\widehat{F}(x,y) = F(x) + y$. To any sequence $\xi \in \ell_\infty(\mathbb{R}^n)$ we can associate *a small perturbation* F^m_ξ of the map F by

$$F^m_\xi \stackrel{\text{def}}{=} \widehat{F}_{\xi(m)} \circ \widehat{F}_{\xi(m-1)} \circ \ldots \circ \widehat{F}_{\xi(1)}.$$

Lemma (Noise Lemma)

Let w_ε be the Gaussian kernel in $\mathbb{R}^k$ with isotropic variance ε. In the notations introduced above, for any vector field B in $\mathbb{R}^k$ and for any $m > 0$ and $\overline{t} = (0, t_1, t_2, \ldots, t_{m-1}) \in \mathbb{R}^{km}$

$$W_{\frac{\varepsilon}{2}}(W_\varepsilon F_*)^{m-1} W_{\frac{\varepsilon}{2}} B(z) =$$
$$= \int_{\mathbb{R}^{k(m-1)}} w_\varepsilon(t_1) w_\varepsilon(t_2) \ldots w_\varepsilon(t_{m-1}) (W_{\frac{\varepsilon}{2}} F^m_{\overline{t}*} W_{\frac{\varepsilon}{2}} B)(z) \mathrm{d}\overline{t},$$

The operator to study

- The operator $(W_\varepsilon F_*)^n$ was hard to study.
- The operator $W_{\frac{\varepsilon}{2}} F_{\xi *}^m W_{\frac{\varepsilon}{2}}$, where $\xi \in \ell_\infty(\mathbb{R}^2)$, is easier and sufficient.

Main goal

To construct an invariant cone C for the operator $W_{\frac{\varepsilon}{2}} F_{\xi *}^m W_{\frac{\varepsilon}{2}}$ for arbitrary sufficiently large $m \gg 1$, for all $\|\xi\|_\infty \leq m2^{-\alpha m}$ and $\varepsilon \leq 2^{-\alpha m}$ for some $\alpha < 1$, in the space of essentially bounded vector fields with absolutely integrable components. The cone should satisfy

$$\left\| W_{\frac{\varepsilon}{2}} F_{\xi *}^m W_{\frac{\varepsilon}{2}} \,|_C \right\| \geq 2^m \cdot \mathrm{const.}$$

The bound is justified: $\left\| W_{\frac{\varepsilon}{2}} F_*^m W_{\frac{\varepsilon}{2}} \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \chi_\square \right\| \geq 2^{m-1}$.

Main difficulty

If the diffeomorphism action causes the field to change direction rapidly; its energy cannot grow exponentially fast in the presence of diffusion.

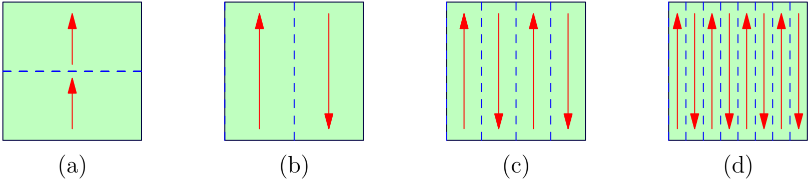

Figure: Evolution of the magnetic field (red) under iterations of the folded Baker's map. (a) initial vector field, (d) vector field after 3 iterations. Blue dashed lines mark discontinuities.

Strategy: main heros

Definition

A cone is a convex subset which is invariant with respect to multiplication by a non-negative real number. The cones we will be dealing with have a general form

$$C(x,\alpha_k) := \{dx + w \mid \|w\| < d2^{-\alpha_k}, d \in \mathbb{R}^+\}.$$

We say that the cone $C(x,\alpha_1)$ is smaller than the cone $C(y,\alpha_2)$ and write $C(x,\alpha_1) \ll C(y,\alpha_2)$, if $\alpha_1 > \alpha_2 > 0$. We do not require here that $C(x,\alpha_1) \cap C(y,\alpha_2) \neq \varnothing$.

Our Poincare map restricted to the hyperbolic set is the Baker's map,

$$F(x,y) = \begin{cases} \left(\frac{x-1}{2}, 2y+1\right), & \text{if } x \in \square,\ -1 < y < 0, \\ \left(\frac{x+1}{2}, 2y-1\right), & \text{if } x \in \square,\ 0 < y < 1, \\ Q(x,y), & \text{if } (x,y) \in \mathbb{R}^2 \setminus \square; \end{cases}$$

Strategy: key steps

1. Fix a large number $m \gg 1$ and a sequence $\|\xi\|_\infty \leq 2^{-m\alpha}$.
2. Choose a norm: maximum of the weighted $\mathcal{L}_1$-norm and weighted supremum norm; the "weights" depend on m and ξ.

$$\|f\|_{\mathcal{L}_1,\varphi} = \int |f\varphi|, \qquad \varphi \geq 0;$$
$$\|f\|_{\mathcal{L}_\infty,\beta} = \beta\|f\|_{\mathcal{L}_\infty}.$$

3. Introduce a sequence of *canonical partitions*, associated to a sequence of small perturbations ξ, a substitute for a Markov partition for m iterations.
4. Introduce a subspace of piecewise-constant vector fields $\mathfrak{X}_\Omega$, associated to a canonical partition $\Omega(m,\xi)$; and choose a *basis*.
5. Approximate the linear operator $F^{2m}_{\xi*}|_{\mathfrak{X}_{\Omega^1}}$, by a linear operator $\mathcal{A}(m,\xi)\colon \mathfrak{X}_{\Omega^1} \to \mathfrak{X}_{\Omega^2}$ (partitions Ω^1 and Ω^2 depend on ξ and m).

Strategy: key steps (continued)

1. Construct a pair of open cones $C_1 \subset \mathfrak{X}_{\Omega^1}$ and $C_2 \subset \mathfrak{X}_{\Omega^2}$ such that $\mathcal{A}\colon \overline{C_1} \to C_2 \ll C_1$. (Both cones depend on ξ and m).
2. Get rid of the dependence on ξ: show that an image of the Weierstrass transform $W_{\frac{\varepsilon}{2}} v$ may be very well approximated by a piecewise-constant vector field, associated to a canonical partition Ω. This is due to $\varepsilon \gg \sup \mathrm{diam}(\Omega_j)$.
3. Construct an invariant cone for the operator $W_{\frac{\varepsilon}{2}} F_{\xi *}^{2m} W_{\frac{\varepsilon}{2}}$ in the space $\mathfrak{X}$ of bounded vector fields with absolutely integrable components.

Canonical partitions

Fix a large number $m \gg 1$ and a sequence $\|\xi\|_\infty \leq 2^{-m\alpha}$.

Definition (Canonical partition)

To a small perturbation F_ξ^{2m} of the map F we associate a partition $\Omega(m,\xi)$ of $\mathbb{R}^2$ that satisfies the following conditions

1. The unit square $\square$ contains at most 2^{2m} and at least $2^{2(m-1)}$ elements of the partition. Interiors of the elements do not intersect the boundary of the square.
2. For any element Ω_{ij} of the partition Ω there exist two rectangles $Rec\left(\frac{1}{m}2^{-m}, \frac{1}{m}2^{-m}\right) \subseteq \Omega_{ij} \subseteq Rec\left(2^{1-m}, 2^{1-m}\right)$.
3. Any rectangle $R \subset \square$ such that $F_\xi^k(R) \subset \square$ for all $0 \leq k \leq 2m$ is contained in a single element of the partition.

Theorem

Canonical partition exists for any sequence $\|\xi\|_\infty \leq m2^{-\alpha m}$.

Further development

1. The same machinery works well for expanding piecewise differentiable one-dimensional maps with small distortion: $\log \frac{|\max f'|}{|\min f'|} \ll 1$;
2. A similar technique is applicable to the piecewise-diffeomorphisms of the real plane with a Cantor hyperbolic set.
3. We believe that this approach can lead to a positive answer to the *kinematic fast dynamo problem* in its original form.

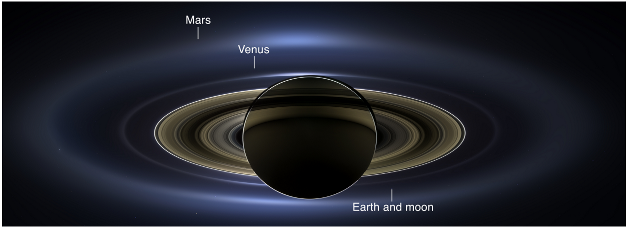

Figure: The day the Earth smiled (July 19, 2013).

References

- Arnold, V. I. and Khesin, B. A. Topological methods in hydrodynamics. Applied Mathematical Sciences, v. 125, Springer-Verlag, 1998.
- Vainshtein, S. I. and Zeldovich, Ya. B. Origin of magnetic fields in astrophysics. Soviet Phys. Usp. 15, (1972), 159–172.
- Photo: NASA Cassini spacecraft project.

Thank you!