

From a pair of pants to the Hausdorff dimension of the generator's limit set

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A pair of pants and the limit set of a Fuchsian group

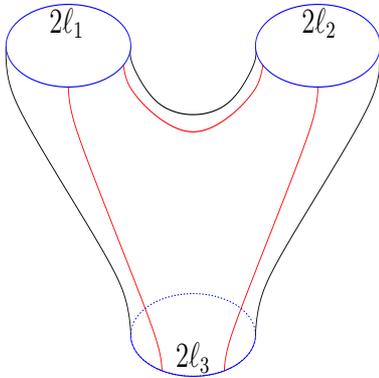


Figure 1: A pair of pants is a three-punctured sphere. It is uniquely defined by the lengths of boundary geodesics: $2l_1$, $2l_2$, and $2l_3$. Cutting the pants along the red geodesics, we obtain a right-angled hyperbolic hexagon.

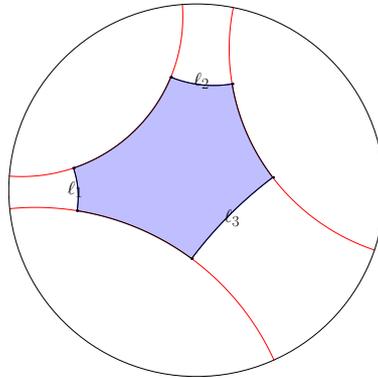


Figure 2: The pair of pants appears then as a factor-space \mathbb{H}/Γ , where the group Γ is generated by reflections with respect to red geodesics. The group Γ is uniquely defined by the pairwise distances l_1 , l_2 , and l_3 .

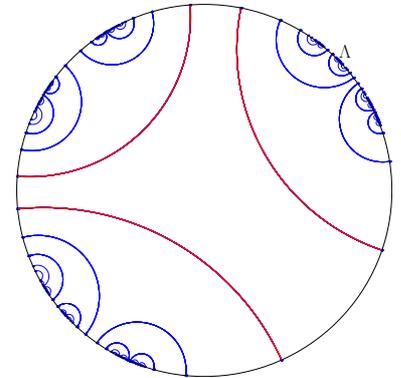


Figure 3: The group of reflections Γ is acting hyperbolically on the hyperbolic plane \mathbb{H} . We are interested in the Hausdorff dimension of the limit set as a function of length $\dim_{\mathbb{H}}(\Lambda) = \dim(l_1, l_2, l_3)$.

Dynamical Zeta Function Toolbox

The Selberg zeta function for a hyperbolic surface is a complex analytic function defined in terms of lengths of closed geodesics $\lambda(\gamma)$:

$$Z(s) = \prod_n \prod_{\gamma} (1 - e^{-(s+n)\lambda(\gamma)}).$$

Any closed geodesics is uniquely defined by a sequence of reflections with respect to the red cuts. This allows one to write a dynamical zeta function

$$\zeta(z, s) = \exp \left(- \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{|\gamma|=m} \frac{e^{-s\lambda(\gamma)}}{1 - e^{-\lambda(\gamma)}} \right)$$

The power series coefficients converge to zero superexponentially, and this gives us an efficient way of computing zeta function numerically, as $\zeta(1, s) = Z(s)$.

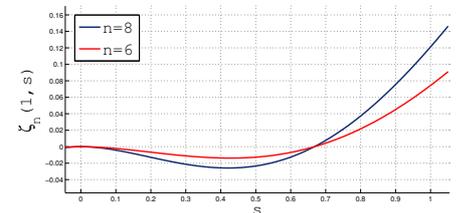


Figure 4: Typical graphs of a zeta function, computed using closed geodesics with $n = 6$ and $n = 8$ reflections.

Results on Hausdorff Dimension

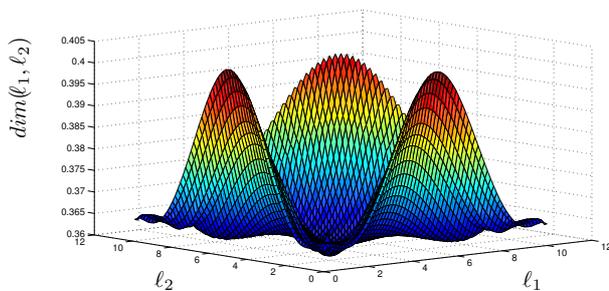


Figure 5: Dimension of a limit set of a pair of pants, as a function of l_1, l_2 under additional assumption $l_1 + l_2 + l_3 =: l = 11$.

- The Hausdorff dimension is the largest real zero of the zeta function and can be efficiently computed numerically.
- The dimension function has, at least, 4 points of global minima and 3 saddle points.
- The dimension of the limit set in the middle $\dim(\frac{l}{3}, \frac{l}{3}, \frac{l}{3}) \approx \frac{1}{l} \rightarrow 0$ as $l \rightarrow \infty$.
- The dimension is a continuous function up to the boundary.
- It is known that when the original surface has cusps, the Hausdorff dimension is at least $\frac{1}{2}$. However, the numerical method we use is not applicable to this case.

Acknowledgements, Credits, References

The conference “*Fractals and Related Fields – III*” poster is based on the following works mathematical works

1. For background in Fuchsian groups actions, see A. F. Beardon, The geometry of discrete groups. Graduate Texts in Mathematics, 91. Springer-Verlag, New York, 1983. xii+337 pp.
2. Selberg zeta function has been introduced by Atle Selberg, in “Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series”, J. Indian Math. Soc. (N.S.) 20 (1956), 47–87.
3. For relation between Selberg Zeta Function and Dynamical Zeta Function, see D. Ruelle, Zeta-functions for expanding maps and Anosov flows, *Invent. Math.*, 34 (1976), 231–242.
4. For error term estimates in Taylor series of the Dynamical Zeta Function, see A. Grothendieck, Produits tensoriels topologiques et espaces nucleaires, *Mem. Amer. Math. Soc.*, 16 (1955), 1–140.
5. The problem has been introduced to us by N. Peyrimhoff in private conversation, also see K. Gittins, N. Peyerimhoff, M. Stoiciu, D. Wirosoetisno, Some spectral applications of McMullen’s Hausdorff dimension algorithm. *Conform. Geom. Dyn.* 16 (2012), 184–203.
6. The lower bound for dimension on the boundary of the simplex, is due to A. F. Beardon, Inequalities for certain Fuchsian groups. *Acta Math.* 127 (1971) 221–258.

Software used

1. The poster prepared with \LaTeX , using [beamerposter](#) package.
2. The hyperbolic plane graphics is prepared with [Asymptote](#), using [hyperbolic geometry](#) module by Raoul Bourquin.
3. Numerical data for dimension obtained with a help of **Fortran**.
4. The plots of dimension and zeta function prepared with **Matlab**®.