FAST DYNAMO ON THE REAL PLANE

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ABSTRACT. In this paper we show that the Baker map of a square, extended to the real plane by a non-expanding map satisfying mild extra conditions does induce a fast dynamo action on the vector fields on the plane in the sense that there exist a $\mathcal{L}_1$ vector field whose norm grows exponentially under the induced action. This is the second step towards a solution of the kinematic fast dynamo problem.

1. INTRODUCTION

The present work contains the third chapter of my PhD Thesis, where the fast dynamo theorem is proved for piecewise diffeomorphisms of the real plane $\mathbb{R}^2$.

Theorem 2. There exists a volume preserving piecewise diffeomorphism $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that for some vector field B_0 in $\mathbb{R}^2$

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}\ln\left\|\left(\exp(\varepsilon\Delta)F_*\right)^n B_0\right\|_{\mathcal{L}_1} > 0.$$

The map F may be realised as a Poincaré map of an incompressible fluid flow filling a compact domain in $\mathbb{R}^3$ (an immersed 3-dimensional manifold with a boundary).

2. PREFACE

The classical kinematic fast dynamo problem dates back to 1970s and concerns the evolution of a magnetic field in a conducting fluid flow in the presence of small diffusion.

The kinematic dynamo equations read [4], [2]

$$\begin{cases} \dfrac{\partial B}{\partial t} = (B\cdot\nabla)v - (v\cdot\nabla)B + \varepsilon\Delta B \\ \nabla\cdot v = \nabla\cdot B = 0, \end{cases} \tag{1}$$

where v is the known velocity field of the conducting fluid filling a certain compact domain M, tangent to the boundary ∂M; B is the magnetic field, and ε is a parameter corresponding to the speed of diffusion through the boundary ∂M. The case of slow diffusion corresponds to an almost perfectly conducting fluid.

As usual, ∇ is the divergence and $\Delta = \nabla^2$ stands for the Laplacian operator.

Problem 1 ([5],[1])**.** Whether or not there exist a divergence-free velocity field v with a compact support $\operatorname{supp} v = M$ such that the energy $E(t) = \|B(t)\|^2_{\mathcal{L}^1(M)}$ of the magnetic field $B(t)$ grows exponentially with time for some initial condition $B(0) = B_0$ with $\operatorname{supp} B_0 = M$, and for arbitrary small diffusivity ε?

The exponential growth of the magnetic energy is equivalent to

$$\lim_{\varepsilon\to 0}\lim_{t\to\infty}\frac{1}{t}\ln\int_{\mathbb{R}^d}|B(z,t,\varepsilon)|\mathrm{d}z > 0 \tag{2}$$

This is a Cauchy problem for a Navier-Stokes type equation. The main interest is related to stationary velocity fields v in 2– and 3-dimensional domains M.

A scheme (a drawing) of a possible 3-dimensional flow has been suggested by my supervisor, Dr. O. Kozlovski. The goal of my Thesis work was to complete the details and to find an analytic argument that will verify the construction.

The provisional flow resembled a hyperbolic flow in places. This suggests the following approach: to choose a Poincaré section such that the Poincaré map possess a hyperbolic set and is easy to analyse, and to prove an analogue of the inequality (2) for the Poincaré map, replacing the flow action by the diffeomorphism action composed with the exponent of the Laplacian. In other words, let g be the Poincaré map, and let consider the operator

$$B \to \exp(\varepsilon\Delta)g_* B.$$

Then a discrete analogue of the inequality (2) is [1]

$$\lim_{\varepsilon\to 0}\lim_{t\to\infty}\frac{1}{t}\ln\int_{\mathbb{R}^d}\big|(\exp(\varepsilon\Delta)g_*)B_0(z)\big|\mathrm{d}z > 0 \tag{3}$$

The hyperbolic two-dimensional Poincaré map can be reduced even further by considering induced transformation on a suitably chosen unstable manifold; which would be a one-dimensional non-invertible piecewise smooth map with a hyperbolic set. The general theory for these maps is very well developed.

Following this course, in the second chapter of the Thesis, we study one-dimensional case, develop an approach, and establish the following fast dynamo theorem in dimension one.

Theorem 5. Let w_ε be the Gaussian kernel on $\mathbb{R}$ with variance ε. There exist a piecewise diffeomorphism $g\colon \mathbb{R}\to\mathbb{R}$ and a function $v\colon \mathbb{R}\to\mathbb{R}$ such that[1]

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}\ln\big\|(w_\varepsilon * g_*)^n v\big\| > 0,$$

[1] It is generally known [6] that $\exp(\varepsilon\Delta)v = w_\varepsilon * v$.

where $*$ stands for convolution and g_* is a transfer operator induced by g according to

$$(g_*\phi)(x) \colon = \sum_{y \in g^{-1}(x)} \operatorname{sgn} \mathrm{d}g(y)\phi(y)$$

The third chapter, presented here, is independent of the second chapter, apart from one construction, one Theorem, and a Lemma, that we present here for coherence.
Small random perturbations. We construct a random dynamical system using skew-products. Let X be a real manifold and let $f \colon X \to X$ be a transformation. We consider its extension

$$\widehat{f} \colon X \times \mathbb{R}^n \to X \qquad \widehat{f}(x, \xi) \stackrel{\text{def}}{=} f(x) + \xi(1). \tag{4}$$

Let $\Sigma \subset \ell_\infty(\mathbb{R}^n)$ be a shift-invariant subset of two-sided bounded sequences of vectors in $\mathbb{R}^n$. We introduce a skew product over the Bernoulli shift

$$\sigma \times \widehat{f} \colon \Sigma \times X \to \Sigma \times X \qquad (\sigma \times \widehat{f})(\xi, z) \stackrel{\text{def}}{=} (\sigma(\xi), \widehat{f}(z, \xi(1))). \tag{5}$$

The induced transformation on fibers we denote by

$$f_\xi \colon X \to X, \qquad f_\xi(z) \stackrel{\text{def}}{=} \widehat{f}(z, \xi(1)). \tag{6}$$

Its iterations are given by

$$f_\xi^k(z) \stackrel{\text{def}}{=} \widehat{f}(f_\xi^{k-1}(z), \xi(k)). \tag{7}$$

Definition 1. We call the map f_ξ *a random perturbation* of the map f associated to the sequence $\xi \in \Sigma$.

Canonical partitions. If the map f is Markov, its perturbation, depending on the sequence ξ may or may not be Markov. To study the latter case, we introduce the notion of a canonical partition associated to a sequence ξ, a substitute for the Markov partition.

We are particularly interested in the following class of maps. Let $s_2 \le 2 \le s_1$, be two real numbers such that $\log \frac{s_1}{s_2} = \varkappa \ll 1$. Let $m \gg 1$ be a large integer and let $\delta = 2^{-m\alpha}$ be a small real number with $\frac{15}{16} \le \alpha < 1$. Consider a map $f \colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} s_1 x + s_1 - 1, & \text{if } -1 < x < \frac{2}{s_1} - 1; \\ s_2 x + 1 - s_2, & \text{if } \frac{2}{s_1} - 1 < x < 1; \\ -x, & \text{otherwise.} \end{cases} \tag{8}$$

and define its extension $\widehat{f}\colon \mathbb{R}^2 \to \mathbb{R}$ by $\widehat{f}(x,y) = f(x) + y$. We associate a small perturbation f_ξ to any sequence $\xi \in \ell_\infty(\mathbb{R})$ and $\|\xi\|_\infty \le \delta$.

It may seem at first sight that the examples chosen are too simple since they are linear. However, they appear to be sufficiently complicated to analyse and the same approach will work for non-trivial perturbations, since most estimates are based on distortion estimates and the distortion is easy to control for perturbations of hyperbolic maps.

Definition 2. Let $I \subset [-1,1]$ be an interval of continuity of the map f_ξ^n. We call a branch $f_\xi^n(I)$ of the map f_ξ^n *main*, if for any $0 < k < n$ we have that $f_\xi^k(I) \subset [-1,1]$.

Theorem 1. *For any sequence $\xi \in \ell_\infty(\mathbb{R})$ with $\|\xi\|_\infty \le \delta$ there exist a partition $\Omega = \bigsqcup_{j\in\mathbb{Z}} \Omega_j$ of $\mathbb{R}$ such that*

(1) *The interval $[-1,1]$ contains at least 2^{m-1} and at most 2^m intervals of the partition, and $\{\pm 1\}$ are the end points of some intervals of the partition.*

(2) *The length of intervals Ω_j is bounded away from zero and from infinity*
$$\frac{1}{m s_1^m} \le |\Omega_j| \le 2\Big(\frac{1}{s_1^m} + \frac{1}{s_2^m}\Big).$$

(3) *Any interval $I \subset \mathbb{R}$ of the length $|I| = \delta$ contains not more than $N_\delta = 2^{m+1}\delta^{\log_{s_1} 2}$ intervals of the partition.*

(4) *Any interval of the partition $\Omega_j \subset \mathbb{R} \setminus [-1-m\delta; 1+m\delta]$ has length $|\Omega_j| = 2^{-m}$.*

(5) *Any main branch belongs to a single element of Ω and any element of Ω contains not more than 2 main branches.*

We call the partition Ω a canonical partition for the map f_ξ^m associated to the perturbation ξ.

Lemma 2.1. *We call a main branch $f_\xi^k(I)$ of the map f_ξ^k* long, *if $\big|f_\xi^k(I) \cap [-1,1]\big| > \frac{2}{s_2}$. The map f_ξ^k for any $1 \le k \le m\alpha \log_{s_1} 2$ has exactly 2^k long branches.*

3. Notation

The following notations will be used throughout.

We denote the unit square in the plane $\mathbb{R}^2$ by $\square \stackrel{\text{def}}{=} [-1,1]^2$.

The Jacobian of a function F we denote by $\mathrm{d}F$, and by $|\mathrm{d}F|$ we denote its determinant. For a function of two variables, by ∂_x we denote the derivative in the first variable and by ∂_y we denote its derivative in the second variable. Similarly, for any point $z \in \mathbb{R}^2$ we denote by z_x and z_y its first and second coordinates.

The indicator function of a set X we denote by χ_X. In particular, $\chi_\square$ is the indicator function of the square $[-1,1]^2$. Given a subset $X \subset \mathbb{R}^2$ and a partition $\Omega = \{\Omega_{ij}\}_{(i,j)\in\mathbb{Z}^2}$ of the plane $\mathbb{R}^2$ we abuse notations and write $(i,j) \in X$ for $\Omega_{ij} \subset X$. We denote by π_x and π_y the natural orthogonal projections

$$\pi_x \colon \mathbb{R}^2 \to \mathbb{R} \qquad \pi_x(z_x, z_y) = z_x, \tag{9}$$

$$\pi_y \colon \mathbb{R}^2 \to \mathbb{R} \qquad \pi_y(z_x, z_y) = z_y. \tag{10}$$

The length of a vector v we denote by $|v|$ and the n-dimensional Lebesgue measure of a subset $A \in \mathbb{R}^n$ we denote by $|A|$. For any sequence of vectors $\xi \in \ell_\infty(\mathbb{R}^2)$ we denote by $\xi_x \in \ell_\infty(\mathbb{R})$ and $\xi_y \in \ell_\infty(\mathbb{R})$ two sequences of x- and y-coordinates of elements of ξ, respectively. We denote by Σ_δ the subset of sequences with $\|\xi\|_\infty \le \delta$.

The two dimensional Gaussian kernel w_δ is specified by

$$w_\delta(x,y) \overset{\text{def}}{=} \frac{1}{2\pi\delta^2} e^{-\frac{x^2+y^2}{2\delta^2}}. \tag{11}$$

The Weierstrass transform is a convolution operator with the Gaussian kernel. For any absolutely integrable function f it is given by

$$W_\delta f(z) \overset{\text{def}}{=} w_\delta * f(z) = \int_{\mathbb{R}^2} w_\delta(z-t) f(t) \mathrm{d}t. \tag{12}$$

For a vector field $v = (v_x, v_y)$ with absolutely Lebesgue-integrable components v_x and v_y the Weierstrass transform is defined by $W_\delta v = (w_\delta * v_x, w_\delta * v_y)$.

The space of essentially bounded vector field in $\mathbb{R}^2$ with absolutely integrable coordinates we denote by $\mathfrak{X}$.

The supremum norm of a matrix A is supremum of absolute values of its elements, we denote it by $\|A\|_\infty \overset{\text{def}}{=} \sup_{ij} |A_{ij}|$. The matrices we are dealing with will be bi-infinite.

The following letters are reserved for real constants: M, M_1, μ_1, μ_2, α, $\gamma_{1,2,3,4} > 0$. Suitable intervals of values will be specified later.

4. The dynamical system

Here we introduce the dynamical system we will be studying. It consists of the phase space $\mathfrak{X}$; the norm, which is the maximum of weighted $\mathcal{L}_1$ and $\mathcal{L}_\infty$ norms; and the transformation of the phase space, which is an action, induced by a piecewise diffeomorphism of $\mathbb{R}^2$. To define the piecewise diffeomorphism we use a tower construction.

4.1. Action on vector fields.

A tower of M floors. Let $M > 1$ be a large natural number; and let $0 < \mu_1 < 0.1$, $0 < \mu_2 \ll 1$ be two small real numbers.

Let F_0 be the Baker's map on the unit square

$$F_0(z_x, z_y) \stackrel{\text{def}}{=} \begin{cases} \left(\frac{1}{2}(z_x - 1); 2z_y + 1\right), & \text{if } z_y < 0; \\ \left(\frac{1}{2}(z_x + 1); 2z_y - 1\right), & \text{if } z_y > 0. \end{cases}$$

Consider $M - 1$ maps $F_1, \ldots, F_{M-1} \colon \mathbb{R}^2 \setminus \square \to \mathbb{R}^2 \setminus \square$ with the following properties

(1) each F_k is a smooth map;
(2) each F_k is area-preserving: $|\mathrm{d}F_k| = 1$;
(3) the Euclidean norm of the differential is uniformly bounded $\|\mathrm{d}F_k\| \leq 1 + \mu_1$;
(4) the Hessian is small $\|\mathrm{d}^2 F_k\| \leq \mu_2$.
(5) all F_k are polynomials, most are linear, some are not; the product of degrees of all of them is bounded by a small number d, which is independent of M. In particular, $d^{\frac{2}{M}} \leq 2^{\frac{1}{500}}$. This condition holds true, for example if $F_k \equiv F_j$, for all $1 \leq k \leq j \leq M - 1$. We use this a strict assumption only to claim that for any point $z \in \mathbb{R}^2$ $\#\{\pi_x^{-1}(F_1 \circ \ldots \circ F_M(z))\} \leq d$ and $\#\{\pi_y^{-1}(F_1 \circ \ldots \circ F_M(z))\} \leq d$. This bound is required in Proposition 6.2 only.

We build a tower $X \subset \mathbb{R}^3$ defined by

$$X \stackrel{\text{def}}{=} \left(\mathbb{R}^2 \times \{0\}\right) \bigcup \left((\mathbb{R}^2 \setminus \square) \times \{1, 2, \ldots, M - 1\}\right)$$

with coordinates (z, n), where $z = (z_x, z_y) \in \mathbb{R}^2$ and $n \in \{0, 1, \ldots, M - 1\}$. We will abuse notations and identify $\square \times \{0\} \subset X$ with $\square$.

The choice of piecewise diffeomorphism. We are ready to introduce a map $F \colon X \to X$ defined by

$$F(z, n) \stackrel{\text{def}}{=} \begin{cases} (F_0(z), 0), & \text{if } n = 0 \text{ and } z \in \square; \\ (F_{n+1}(z), (n + 1) \bmod (M - 1)), & \text{otherwise.} \end{cases} \tag{13}$$

Consider an extension $\widehat{F} \colon X \times \mathbb{R}^2 \to X$

$$\widehat{F}((z, n), w) \stackrel{\text{def}}{=} \begin{cases} (F_0(z) + w, 0), & \text{if } n = 0 \text{ and } z \in \square; \\ (F_{M-1}(z) + w, 0), & \text{if } n = M - 1; \\ (F_{n+1}(z), (n + 1)), & \text{otherwise.} \end{cases} \tag{14}$$

Given a sequence $\xi \in \Sigma \subset \ell_\infty(\mathbb{R}^2)$, we define a small random perturbation F_ξ of the map F, as described in Subsection 2. Then the zero floor $\mathbb{R}^2 \times \{0\}$ is invariant with respect to $F_{\underline{\xi}}^M$ and we may consider the M'th iteration as a map $F_{\underline{\xi}}^M \colon \mathbb{R}^2 \to \mathbb{R}^2$. We denote by $F_{\underline{0}} \colon X \to X$ the map corresponding to the zero sequence $\xi \equiv 0$.

Remark 1. The inverse map $F_{\xi^k}^{-1}$ is given by

$$F_{\xi^k}^{-1}(z,n) = \begin{cases} (F_0^{-1}(z-\xi^k), 0), & \text{if } z \in \square + \xi^k \text{ and } n = 0; \\ (F_{M-1}^{-1}(z-\xi^k), M-1), & \text{if } z \notin \square + \xi^k \text{ and } n = 0; i \\ (F_n^{-1}(z), n-1), & \text{otherwise.} \end{cases} \tag{15}$$

Also observe that the inverse Baker's map is given by

$$F_0^{-1}(z-\xi^k) = \begin{cases} \left(2z_x + 1 - 2\xi_x^k; \tfrac{1}{2}(z_y - 1) - \tfrac{1}{2}\xi_y^k\right), & \text{if } z_x < \xi_x^k, \text{ and } z \in (\square + \xi^k); \\ \left(2z_x - 1 - 2\xi_x^k; \tfrac{1}{2}(z_y + 1) - \tfrac{1}{2}\xi_y^k\right), & \text{if } z_x > \xi_x^k, \text{ and } z \in (\square + \xi^k). \end{cases} \tag{16}$$

Let $m_0 \gg 1$ be a large natural number. We set $m = 4Mm_0$ and choose a small real number $\delta = 2^{-m\alpha}$ with $\frac{15}{16} < \alpha \le 1$. The subset of sequences in $\ell_\infty(\mathbb{R}^2)$ with $\|\xi\|_\infty \le \delta$ we denote by Σ_δ. Given a sequence $\xi \in \Sigma_\delta$ we may define a map

$$P_\xi \colon \mathbb{R}^2 \to \mathbb{R}^2 \qquad P_\xi(z) \stackrel{\text{def}}{=} F_\xi^m(z, 0). \tag{17}$$

The map P_ξ defines induced action on the space $\mathfrak{X}$ according to

$$(P_{\xi *} v)(z) \stackrel{\text{def}}{=} \mathrm{d}P_\xi(P_\xi^{-1} z) v(P_\xi^{-1} z). \tag{18}$$

The number of iterations m remains fixed throught the manuscript. We assume it to be sufficiently large so that all inequalities hold true.

4.2. **The choice of the norm in $\mathfrak{X}$.** In this Subsection we introduce a norm in the space of vector fields in $\mathbb{R}^2$. We also give a general definition of a cone in $\mathfrak{X}$.

Given a partition Ω of $\mathbb{R}^2$, we define an associated weighted $(\Omega, \mathcal{L}_1)$-norm of a vector field v on the plane by

$$\|v\|_{\Omega,\mathcal{L}_1} \stackrel{\text{def}}{=} \sum_{ij} \frac{2^{-m}}{|\pi_y(\Omega_{ij})|} \int_{\Omega_{ij}} |v|.$$

Observe that $\|v\|_{\Omega,\mathcal{L}_1}$ is finite if the ordinary $\mathcal{L}_1$-norm is finite and the size of elements of partition is bounded away from zero:

$$\|v\|_{\Omega,\mathcal{L}_1} = \sum_{ij} \frac{2^{-m}}{|\pi_y(\Omega_{ij})|} \int_{\Omega_{ij}} |v| \le \frac{2^{-m}}{\inf |\pi_y(\Omega_{ij})|} \int_{\mathbb{R}^2} |v| = \frac{2^{-m}}{\inf |\pi_y(\Omega_{ij})|} \|v\|_{\mathcal{L}_1}.$$

The supremum norm of a vector field v we denote by $\|v\|_\infty \stackrel{\text{def}}{=} \sup |v|$. We denote by $\mathfrak{X}$ the space of vector fields on the real plane with finite $\mathcal{L}_1$ and supremum norms.

Definition 3 (Norm)**.** We introduce a new norm in $\mathfrak{X}$, associated to the partition Ω, combining the two:

$$\|v\|_\Omega \stackrel{\text{def}}{=} \max\Big(\|v\|_{\Omega,\mathcal{L}_1}, 2^{-m/4}\sup|v|\Big). \tag{19}$$

The subspace of piecewise constant vector fields associated to the partition Ω we denote $\mathfrak{X}_\Omega$. We reserve Greek letters for piecewise constant vector fields. We shall call the basis

$$\Big\{\chi^s_{\Omega_{ij}} \stackrel{\text{def}}{=} \frac{1}{|\pi_x(\Omega_{ij})|}\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\chi_{\Omega_{ij}}; \chi^u_{\Omega_{ij}} \stackrel{\text{def}}{=} \frac{1}{|\pi_x(\Omega_{ij})|}\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\chi_{\Omega_{ij}}\Big\}_{i,j\in\mathbb{Z}}.$$

the canonical basis of the subspace $\mathfrak{X}_\Omega$.

Whenever we are dealing with several partitions Ω^1, Ω^2, and Ω^3, say, we omit Ω in the norm index and write $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_3$, respectively.

We have for the norm of a piecewise constant vector field $\nu = \sum_{ij} \nu^{ij}_s \chi^s_{\Omega_{ij}} + \nu^{ij}_u \chi^u_{\Omega_{ij}}$:

$$\|\nu\|_\Omega \geq \max\Big(2^{-m}\sum_{ij}|\nu^{ij}|, \frac{2^{-m/4}}{\sup|\pi_x(\Omega_{ij})|}\sup|\nu^{ij}|\Big),$$

in particular,

$$\|\nu\|_\Omega = 1 \quad \Longrightarrow \quad \sum|\nu_{ij}| < 2^m \quad \text{and} \quad \sup|\nu_{ij}| \leq 2^{-\frac{3}{4}m}. \tag{20}$$

Invariant cones. By analogy with one-dimensional part, cones of a special form in the spaces $\mathfrak{X}$ and $\mathfrak{X}_\Omega$ play an important role. We reserve notation for a cone of radius r with main axis $\chi_\Box$ in the subspace of piecewise constant vector fields associated to the partitions Ω^1 and Ω^2:

$$\text{Cone}\left(r, \Omega^1\right) \stackrel{\text{def}}{=} \Big\{\eta = d\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\chi_\Box + \varphi \mid \varphi \in \mathfrak{X}_{\Omega^1},\ \|\varphi\|_1 \leq dr,\ \sum_{\Box}\varphi^{ij}_u = 0\Big\}. \tag{21}$$

We extend the cone $\text{Cone}\,(r, \Omega^1)$ to include general functions from the main space:

$$\widehat{\text{Cone}}\left(r, \varepsilon, \Omega^1\right) \stackrel{\text{def}}{=} \Big\{f = \eta + v \mid \eta \in \text{Cone}\left(r, \Omega^1\right),\ \|v\|_1 \leq \varepsilon\|\eta\|_1\Big\}. \tag{22}$$

4.3. **The canonical partition.** In this subsection we introduce the notion of canonical partition of $\mathbb{R}^2$ associated to a sequence of perturbations $\xi \in \ell_\infty(\mathbb{R}^2)$ as a direct product of a pair of canonical partitions of $\mathbb{R}$ and list the main properties.

Definition 4. *The k'th escaping set* for $k \in \mathbb{Z}$ is defined by

$$E_k \stackrel{\text{def}}{=} \Big\{z \in \Box \subset X \mid \prod_{j=0}^{k}\chi_\Box(F^j_\xi(z)) = 0\Big\}. \tag{23}$$

Obviously, $E_k \subset E_{k+1}$, if $k > 0$; and $E_{k+1} \subset E_k$ if $k < 0$.

Lemma 4.1. *Let $\xi \in \Sigma_\delta \subset \ell_\infty(\mathbb{R}^2)$ be a sequence of small vectors in the plane. Define a sequence $\varsigma(\xi)$ of the length m by $\varsigma^1 = -2\xi^{2m}$, $\varsigma^2 = -2\xi^{-2m-1}$, ..., $\varsigma^m = -2\xi^{m+1}$. Let p_{ς_x} and p_{ξ_y} be two random perturbations of the doubling map p defined by (8) with $s_1 = s_2 = 2$. Then the following diagrams are commutative.*

$$\begin{array}{ccc} \square \setminus E_{-m} & \xrightarrow{P^{-1}_{\sigma^m\xi}} & \mathbb{R}^2 \\ \downarrow{\scriptstyle \pi_x} & & \downarrow{\scriptstyle \pi_x} \\ \mathbb{R} & \xrightarrow{p^m_{\varsigma_x}} & \mathbb{R} \end{array} \qquad \begin{array}{ccc} \square \setminus E_{m} & \xrightarrow{P_{\xi}} & \mathbb{R}^2 \\ \downarrow{\scriptstyle \pi_y} & & \downarrow{\scriptstyle \pi_y} \\ \mathbb{R} & \xrightarrow{p^m_{\xi_y}} & \mathbb{R} \end{array}$$

Proof. Straightforward from definition. The Baker's map preserves the horizontal and vertical foliations, so the second diagram is trivial. For the first diagram, recall that by definition (Subsection 4.1) $P_\xi^{-1} = (F_\xi^m)^{-1} = F_{\xi^1}^{-1} F_{\xi^2}^{-1} \dots F_{\xi^m}^{-1}$. Using (16) and (15), we conclude that the corresponding sequence ς for the doubling map associated to P_ξ^{-1} is as defined in supposition of the Lemma. ∎

We associate a chain $\Upsilon^1, \Upsilon^2, \dots$ of partitions of $\mathbb{R}^2$ to a sequence $\xi \in \Sigma_\delta$.

The first element Υ^1 is defined as follows. Let $\Upsilon^s = \left\{\Upsilon^s_i = \left[\frac{i}{2^m}; \frac{i+1}{2^m}\right]\right\}$, $i \in \mathbb{Z}$, be a partition of $\mathbb{R}$ into equal intervals and let $\Upsilon^u = \{\Upsilon^u_j\}_{j\in\mathbb{Z}}$ be the canonical partition of the map $p^m_{\xi_y}$. Then

$$\Upsilon^1 = \{\Upsilon_{ij}\}, \ \Upsilon_{ij} = \Upsilon^s_i \times \Upsilon^u_j.$$

To define partition Υ^k, consider a sequence

$$\varsigma^1 = -2\xi^{2km}, \varsigma^2 = -2\xi^{2km-1}, \dots, \varsigma^m = -2\xi^{(2k-1)m}.$$

Let Υ^s be the canonical partition for the perturbation $p^m_{\varsigma_x}$ of the doubling map, and let Υ^u be the canonical partition of the perturbation $p^m_{\sigma^{2mk}\xi_y}$ of the doubling map. Then Υ^k is given by

$$\Upsilon^k = \{\Upsilon_{ij}\}, \ \Upsilon_{ij} = \Upsilon^s_i \times \Upsilon^u_j.$$

Definition 5. We say that a partition Υ of the plane $\mathbb{R}^2$ is *a partition of the class* $\mathcal{G}(m,\delta)$, if there exists a sequence $\xi \in \Sigma_\delta$ such that $\Upsilon = \Upsilon^k$ for some partition Υ^k from the chain of partitions associated to ξ.

Definition 6. *A rectangle* $\left(z_x - \frac{l_x}{2}, z_x + \frac{l_x}{2}\right) \times \left(z_y - \frac{l_y}{2}, z_y + \frac{l_y}{2}\right)$ with centre at z and sides l_x and l_y we denote by $Rec_z(l_x, l_y)$. Whenever location of the centre of the rectangle is of no importance, we omit z and write $Rec(l_x, l_y)$.

Lemma 4.2. *Any partition Υ of the class $\mathcal{G}(m,\delta)$ has the following properties*

(1) *The unit square* $\square$ *contains at most* 4^m *and at least* 4^{m-1} *elements of the partition.*

(2) *For any element* Υ_{ij} *of the partition* Υ *we have two rectangles*

$$Rec\big(\frac{2^{-m}}{m}, \frac{2^{-m}}{m}\big) \subseteq \Upsilon_{ij} \subseteq Rec\big(2^{1-m}, 2^{1-m}\big).$$

(3) *Any square with a side* δ *may be covered by at most* $N_\delta = 4^{m(1-\alpha)+1}$ *elements of the partition.*

Proof. Follows from the properties of the canonical partition for perturbation ξ of the doubling map. ■

5. Fast dynamo theorem in dimension two

In this we show that the main result, the fast dynamo theorem for the Poincaré map of the provisional fluid flow, follows from Noise Lemma 5.1, that we prove immediately, and Theorem 4, which is proved in the Section 7 after the preparatory Section 6.

Consider a map $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(z) \stackrel{\text{def}}{=} F^M(z,0)$, where the map F is defined by (13). Our goal is to show that for the vector field $v = \binom{0}{1} W_{\frac{\delta}{2m}} \chi_\square$

$$\lim_{\delta\to 0} \lim_{n\to\infty} \frac{1}{n} \ln \|(W_\delta T_*)^n v\| > 0. \tag{24}$$

The argument is based on two ideas. The first idea is the Noise Lemma, which suggests to replace the operator $(W_\delta F_*)^{2m}$ with operator $W_{\frac{\delta}{2m}} F_{t*}^{2m} W_{\frac{\delta}{2m}} = W_{\frac{\delta}{2m}} P_{t*}^2 W_{\frac{\delta}{2m}}$ for large, suitably chosen $m \gg 1$ and a sequence $t \in \ell_\infty(\mathbb{R}^2)$. The second idea is to construct explicitly an invariant cone for the operator $W_{\frac{\delta}{2m}} P_{t*}^2 W_{\frac{\delta}{2m}}$, independent of the choice of t.

The proof of the existence of an invariant cone requires a new approach to operators P_{t*}^2, which is developed in Section 6. The existence of an invariant cone is established in Section 7 in the following

Theorem 4. Let Ω be a partition of $\mathbb{R}^2$ of the class $\mathcal{G}(m,\delta)$; and let $\|\xi\|_\infty \le \delta$ be a sequence of real numbers. There exists $r_1(m) \ll r_2(m)$ and $\varepsilon_1(m) \ll \varepsilon_2(m)$ such that

$$W_{\frac{\delta}{2m}} P_{\xi*}^2 W_{\frac{\delta}{2m}} \colon \overline{\operatorname{Cone}(r_1,\varepsilon_1,\Omega)} \to \operatorname{Cone}(r_2,\varepsilon_2,\Omega) \subsetneq \operatorname{Cone}(r_1,\varepsilon_1,\Omega).$$

$$\left\| W_{\frac{\delta}{2m}} P_{\xi*}^2 W_{\frac{\delta}{2m}} |_{\operatorname{Cone}(r_1,\varepsilon_1,\Omega)} \right\| \ge 2^{m-5}$$

(See p. 8 for definition of a cone in the space of vector fields).

In this Section we denote the n-dimensional Lebesgue measure on $\mathbb{R}^n$ by λ_n.

Lemma 5.1 (Noise Lemma)**.** *In the notations introduced above, for any vector field v on the tower X and for any $n > 0$ we have*

$$(W_\delta F_*)^n v(z,k) = \int_{\mathbb{R}^{2(n-1)}} w_\delta(t_1) w_\delta(t_2) \dots w_\delta(t_{n-1}) (W_\delta F^n_{\overline{0t}*} v)(z,k) \mathrm{d}\lambda_{n-1}(\bar{t}), \qquad (25)$$

where $\overline{0t} = (0, t_1, t_2, \dots, t_{n-1}) \in \mathbb{R}^{2n}$.

Proof. Observe that for any $t \in \mathbb{R}^2$

$$F_{t*}v(z,k) = \mathrm{d}F_t(F_t^{-1}(z,k)) \cdot v(F_t^{-1}(z,k)) = \mathrm{d}F(F^{-1}(z-t,k)) \cdot v(F^{-1}(z-t,k)).$$

By straightforward calculation,

$$\begin{aligned}(W_\delta F_*)^n v(z,k) &= (W_\delta F_*)^{n-1} W_\delta F_* v(z,k) = (W_\delta F_*)^{n-1} \int_{\mathbb{R}^2} w_\delta(t)(F_*v)(z-t,k)\mathrm{d}\lambda_1(\bar{t}) = \\ &= (W_\delta F_*)^{n-1} \int_{\mathbb{R}^2} w_\delta(t_1)(F_{t_1*}v)(z,k)\mathrm{d}\lambda_1(\bar{t}) = \dots = \\ &= W_\delta \int_{\mathbb{R}^{2(n-1)}} w_\delta(t_1)\dots w_\delta(t_{n-1})(F_* F_{t_1*} \dots F_{t_{n-1}*} v)(z,k)\mathrm{d}\lambda_{n-1}(\bar{t}) = \\ &= \int_{\mathbb{R}^{2(n-1)}} w_\delta(t_1)\dots w_\delta(t_{n-1})(W_\delta F^n_{\overline{0t}*} v)(z,k)\mathrm{d}\lambda_{n-1}(\bar{t}).\end{aligned}$$

■

To any sequence of vectors $t \in \ell_\infty(\mathbb{R}^2)$ we associate a sequence $\widehat{t} \in \ell_\infty(\mathbb{R}^2)$ defined by

$$\widehat{t_j} : = \begin{cases} t_k, & \text{if } j = Mk; \\ 0, & \text{otherwise.} \end{cases} \qquad (26)$$

Then for any $t \in \mathbb{R}^2$ we have $T_t(z) = T(z) + t = F^M(z,0) + t = F^M_{\widehat{t}}(z,0)$. In particular for $m = m_0 M$ we have $T_t^{m_0}(z) = F^m_{\widehat{t}}(z,0)$; and for any vector field v on $\mathbb{R}^2$ we have $T^{m_0}_{t*}v(z) = (F^m_{\widehat{t}*}v)(z,0)$.

The following Lemma is a corollary of the previous one.

Lemma 5.2. *In the notations introduced above, for any vector field v on the plane $\mathbb{R}^2$ and for any $n > 1$*

$$W_{\frac{\delta}{2m}} T_* (W_{\frac{\delta}{m}} T_*)^{n-1} v(z) = \int_{\mathbb{R}^{2(n-1)}} w_{\frac{\delta}{m}}(t_1) w_{\frac{\delta}{m}}(t_2) \dots w_{\frac{\delta}{m}}(t_{n-1}) (W_{\frac{\delta}{2m}} F^{Mn}_{\widehat{t}*} v)(z) \mathrm{d}\lambda_n(\bar{t}); \qquad (27)$$

We choose three constants $\alpha = \frac{15}{16}$, $\gamma_3 = 1 - \alpha + 0.001$, $\gamma_4 = \frac{1}{4} - \gamma_1 - 0.001$, and four cone parameters $r_1 = \frac{2^{2m}\delta^2}{4m^4 N_\delta}$, $r_2 = 2^{-m\frac{1-\alpha}{4}} = 2^{-\frac{m\alpha}{64}}$, $\varepsilon_1 = 2^{-m\frac{1-\alpha}{2}} = 2^{-\frac{m\alpha}{32}}$, and $\varepsilon_2 = 2^{-2m\frac{1-\alpha}{2}} = 2^{-\frac{m\alpha}{16}}$ such that Theorem 4 holds true (see the proof in the end of Section 7 for details).

Remark 2. One can check by straighforward calculation that for any real number δ and for any function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ we have $W_\delta W_\delta f = 2W_{\sqrt{2}\delta} f$.

Lemma 5.3. *Let $m = \frac{Mn}{2} \gg 1$ be a large number and let Ω be a partition of the class $\mathcal{G}(m,\delta)$. Then for any $f \in \widehat{\text{Cone}}\,(r_1(m), \varepsilon_1(m), \Omega)$*

$$\int_{[-\delta,\delta]^{4n-2}} w_{\frac{\delta}{m}}(t_1)\dots w_{\frac{\delta}{m}}(t_{2n-1})(W_{\frac{\delta}{2m}} T^{2n}_{t*} W_{\frac{\delta}{2m}} v) \mathrm{d}\lambda_{4n-2}(\bar{t}) \in \widehat{\text{Cone}}\,(r_2', \varepsilon_2', \Omega)\,; \tag{28}$$

where $r_2' = r_2'(m) = (1+3e^{-M})r_2(m)$ and $\varepsilon_2' = \varepsilon_2'(m) = (1+3e^{-M})\varepsilon_2(m)$.

Proof. Observe that

$$\int_{[-\delta,\delta]^{4n-2}} w_{\frac{\delta}{m}}(t_1)\dots w_{\frac{\delta}{m}}(t_{2n-1})(W_{\frac{\delta}{2m}} T^{n}_{t*} W_{\frac{\delta}{2m}} v) \mathrm{d}\lambda_{4n-2}(\bar{t}) =$$
$$= \int_{[-\delta,\delta]^{4n-2}} w_{\frac{\delta}{m}}(t_1)\dots w_{\frac{\delta}{m}}(t_{2n-1})(W_{\frac{\delta}{2m}} F^{2m}_{\hat{t}*} W_{\frac{\delta}{2m}} v) \mathrm{d}\lambda_{4n-2}(\bar{t}) =$$
$$= \int_{[-\delta,\delta]^{4n-2}} w_{\frac{\delta}{m}}(t_1)\dots w_{\frac{\delta}{m}}(t_{2n-1})(W_{\frac{\delta}{2m}} P^{2}_{\hat{t}*} W_{\frac{\delta}{2m}} v) \mathrm{d}\lambda_{4n-2}(\bar{t}). \tag{29}$$

By Theorem 4 we know that $W_{\frac{\delta}{2m}} P^2_{t^*} W_{\frac{\delta}{2m}} v = d\binom{0}{1}\chi_\square + \psi_t + g_t$ for any $t \in [-\delta,\delta]^{2m}$, where $\psi_t \in \mathfrak{X}_\Omega$, $\|\psi_t\|_\Omega \le dr_2$ and $\|g_t\|_\Omega \le d\varepsilon_2$. Observe that Ω is independent on t. Therefore, for m large enough

$$\int_{[-\delta,\delta]^{2(2n-1)}} w_{\frac{\delta}{m}}(t_1)\dots w_{\frac{\delta}{m}}(t_{2n-1})\chi_\square(z) \mathrm{d}\lambda_{4n-2}(t) = \chi_\square(z)\Big(\int_{\square_\delta} w_{\frac{\delta}{m}}(t)\mathrm{d}t\Big)^{2n-1} \ge$$
$$\ge \chi_\square(z)\big(1-e^{-m}\big)^{2n-1} \ge (1-2(2n-1)e^{-m})\chi_\square(z). \tag{30}$$

Since $\psi_t \in \mathfrak{X}_\Omega$ for any $\|\bar{t}\|_\infty < \delta$,

$$\int_{[-\delta,\delta]^{4n-2}} w_{\frac{\delta}{m}}(t_1)\dots w_{\frac{\delta}{m}}(t_{2n-1})\psi_t\lambda_{4n-2}(t) \in \mathfrak{X}_\Omega.$$

and we calculate Ω-norm.

$$\Big\|\int_{[-\delta,\delta]^{4n-2}} w_{\frac{\delta}{m}}(t_1)\dots w_{\frac{\delta}{m}}(t_{2n-1})\psi_t \mathrm{d}\lambda_{4n-2}(t)\Big\|_\Omega \le$$
$$\le \sum_{ij} \frac{2^{-m}}{|\pi_x(\Omega_{ij})|} \int_{[-\delta,\delta]^{4n-2}} w_{\frac{\delta}{m}}(t_1)\dots w_{\frac{\delta}{m}}(t_{2n-1})\Big(\int_{\Omega_{ij}} |\psi_t(z)|\mathrm{d}z\Big)\mathrm{d}\lambda_{4n-2}(t) \le$$
$$\le \sup_t \|\psi_t\|_\Omega \le dr_1.$$

Similarly,

$$\Big\|\int_{[-\delta,\delta]^{4n-2}} w_{\frac{\delta}{m}}(t_1)\dots w_{\frac{\delta}{m}}(t_{2n-1})g_t \mathrm{d}\lambda_{4n-2}(t)\Big\|_\Omega \le d\varepsilon_2$$

Observe that

$$\int_{\square} \mathring{U}U \int_{[-\delta,\delta]^{4n-2}} w_{\frac{\delta}{m}}(t_1)\dots w_{\frac{\delta}{m}}(t_{2n-1})\psi_t(z)\mathrm{d}\lambda_{4n-2}(t)\mathrm{d}z =$$

$$= \int_{[-\delta,\delta]^{4n-2}} w_{\frac{\delta}{m}}(t_1)\dots w_{\frac{\delta}{m}}(t_{2n-1})\mathrm{d}\lambda_{4n-2}(t) \cdot \int_{\square} (\mathring{U}U\psi_t)(z)\mathrm{d}z = 0$$

Summing up, for any partition Ω of the class $\mathcal{G}(m,\delta)$ and for any vector field v on the plane $\mathbb{R}^2$ from the cone $\widehat{\mathrm{Cone}}\,(r_1(m),\varepsilon_1(m),\Omega)$

$$\int_{[-\delta,\delta]^{4n-2}} w_{\frac{\delta}{m}}(t_1)\dots w_{\frac{\delta}{m}}(t_{2n-1})(W_{\frac{\delta}{2m}} T_{t*}^{2n} W_{\frac{\delta}{2m}} v)\mathrm{d}\lambda_{4n-2}(t) \in \widehat{\mathrm{Cone}}\,(r_2'(m),\varepsilon_2'(m),\Omega)\,.$$

∎

Lemma 5.4. *Let $m = Mn \gg 1$ be a large number. Let Ω be a partition of the class $\mathcal{G}(m)$. Then in the notations introduced above,*

$$(W_{\frac{\delta}{2m}} T_*)(W_{\frac{\delta}{m}} T_*)^{2n-1} W_{\frac{\delta}{2m}} : \overline{\widehat{\mathrm{Cone}}\,(r_1(m),\varepsilon_1(m),\Omega)} \to \widehat{\mathrm{Cone}}\,(r_2(m),\varepsilon_2(m),\Omega)\,;$$
$$\left\|(W_{\frac{\delta}{2m}} T_*)(W_{\frac{\delta}{m}} T_*)^{2n-1} W_{\frac{\delta}{2m}} \,|_{\widehat{\mathrm{Cone}}(r_1(m),\varepsilon_1(m),\Omega)}\right\| \ge 2^{2m-2}$$

Proof. By Lemma 5.2, taking into account $m = \frac{Mn}{2}$ and definition of T, for any vector field $v \in \mathrm{Cone}\,(r_1,\varepsilon_1,\Omega)$ we may write

$$(W_{\frac{\delta}{2m}} T_*)(W_{\frac{\delta}{m}} T_*)^{2n-1} W_{\frac{\delta}{2m}} v = \int_{\mathbb{R}^{2(2n-1)}} w_{\frac{\delta}{m}}(t_1)\dots w_{\frac{\delta}{m}}(t_{n-1})(W_{\frac{\delta}{2m}} P_{\hat{t}*}^2 W_{\frac{\delta}{2m}} v)\mathrm{d}\lambda_{4n-2}(t) =$$
$$= \Big(\int_{\mathbb{R}^{4n-2)}\setminus[-\delta,\delta]^{4n-2}} + \int_{[-\delta,\delta]^{4n-2)}}\Big) \prod_{j=1}^{n-1} w_{\frac{\delta}{m}}(t_j)(W_{\frac{\delta}{2m}} P_{\hat{t}*}^2 W_{\frac{\delta}{2m}} v)\mathrm{d}\lambda_{4n-2}(t). \quad (31)$$

By Lemma 5.3 we know that

$$\int_{[-\delta,\delta]^{4n-2}} w_{\frac{\delta}{m}}(t_1)\dots w_{\frac{\delta}{m}}(t_{2n-1})(W_{\frac{\delta}{2m}} P_{\hat{t}*}^2 W_{\frac{\delta}{2m}} v)\mathrm{d}\lambda_{4n-2}(t) \in \widehat{\mathrm{Cone}}\,(r_2',\varepsilon_2',\Omega)\,.$$

We estimate the first term

$$\Big\|\int_{\mathbb{R}^{4n-2}\setminus[-\delta,\delta]^{4n-2}} \prod_{j=1}^{n-1} w_{\frac{\delta}{m}}(t_j)(W_{\frac{\delta}{2m}} P_{\hat{t}*}^2 W_{\frac{\delta}{2m}} v)\mathrm{d}\lambda_{4n-2}(t)\Big\|_{\Omega} \le$$
$$\le \sum_{ij} \frac{2^{-m}}{|\pi_x(\Omega_{ij})|} \int_{\mathbb{R}^{4n-2}\setminus[-\delta,\delta]^{4n-2}} \prod_{j=1}^{n-1} w_{\frac{\delta}{m}}(t_j) \int_{\Omega_{ij}} \Big|W_{\frac{\delta}{2m}} P_{\hat{t}*}^2 W_{\frac{\delta}{2m}} v(z)\Big|\mathrm{d}z\mathrm{d}\lambda_{4n-2}(t) \le$$
$$\le \sup_t \|W_{\frac{\delta}{2m}} P_{\hat{t}*}^2 W_{\frac{\delta}{2m}} v\|_{\Omega} \int_{\mathbb{R}^{4n-2)}\setminus[-\delta,\delta]^{4n-2}} \prod_{j=1}^{n-1} w_{\frac{\delta}{m}}(t_j)\mathrm{d}\lambda_{4n-2}(t).$$

We have an upper bound for the integral:

$$\int_{\mathbb{R}^{4n-2}\setminus[-\delta,\delta]^{4n-2}} \prod_{j=1}^{n-1} w_{\frac{\delta}{m}}(t_j) \mathrm{d}\lambda_{4n-2}(t) \leq (2n-1)e^{-m^2}.$$

We substitute this bound to the estimate for the norm above and use Lemma 7.3:

$$\begin{aligned}
&\Big\| \int_{\mathbb{R}^{4n-2}\setminus[-\delta,\delta]^{4n-2}} \prod_{j=1}^{n-1} w_{\frac{\delta}{m}}(t_j) (W_{\frac{\delta}{2m}} P^2_{\hat{t}*} W_{\frac{\delta}{2m}} v) \mathrm{d}\lambda_{4n-2}(t) \Big\|_\Omega \leq \\
&\leq (2n-1)e^{-m^2} \sup_t \|W_{\frac{\delta}{2m}} P^2_{\hat{t}*} W_{\frac{\delta}{2m}} v\|_\Omega \leq (2n-1)e^{-m^2} \Big(m^2 2^{-2m} \frac{N_\delta}{\delta^2} \Big)^2 \|P^2_{\hat{t}*} v\|_\Omega \leq \\
&\leq 4(2n-1)m^4 e^{-m^2} \frac{2^{-m}}{\inf |\pi_x(\Omega_{ij})|} \|P^2_{\hat{t}*} v\|_{\mathcal{L}_1} \leq 4(2n-1)m^4 e^{-m^2} \frac{2^{-m}}{\inf |\pi_x(\Omega_{ij})|} 2^{2m} \|v\|_{\mathcal{L}_1} \leq \\
&\leq 4(2n-1)m^4 e^{-m^2} \frac{\sup |\pi_x(\Omega_{ij})|}{\inf |\pi_x(\Omega_{ij})|} 2^{2m} \|v\|_\Omega \leq 4(2n-1)m^5 2^{2m} e^{-m^2} \ll \varepsilon_2(m) \|v\|_\Omega.
\end{aligned} \tag{32}$$

■

The following fast dynamo theorem is the main result of the present work.

Theorem 2. *There exists a vector field v on $\mathbb{R}^2$ such that*

$$\lim_{\delta\to 0} \lim_{n\to\infty} \frac{1}{n} \log \big\| (W_\delta T_*)^n v \big\| > 0$$

Proof. One can check by straightforward calculation that for any number ε and any vector field v we have $W_\varepsilon W_\varepsilon v = 4W_{\sqrt{2}\varepsilon} v$. Therefore we may choose $v = W_{\frac{\delta}{2m}} \chi_\square$ and Theorem follows from Lemma 5.4. ■

6. Matrix, approximating the operator $P^2_{\eta*}$.

In this Section we assume that a sequence of vectors $\eta \in \ell_\infty(\mathbb{R}^2)$ is fixed and we study the associated operator $P^2_{\eta*}$ on vector fields on $\mathbb{R}^2$, defined by (18), where the map P_η is given by (17). Our goal is to show that for any sequence η there exist a pair of subspaces $\mathfrak{X}_{\Omega^1}, \mathfrak{X}_{\Omega^2} \subset \mathfrak{X}$ and a linear operator $\mathcal{A}(\eta) \colon \mathfrak{X}_{\Omega^1} \to \mathfrak{X}_{\Omega_2}$ with a simple matrix, approximating $P^2_{\eta*}|_{\mathfrak{X}_{\Omega^1}}$ well enough. Given the operator $\mathcal{A}(\eta)$, we construct a pair of cones $C_1 \subset \mathfrak{X}_{\Omega^1}$ and $C_2 \subset \mathfrak{X}_{\Omega^2}$ such that $\mathcal{A}(\overline{C_1}) \subset C_2$; $C_2 \ll C_1$ and $\|\mathcal{A}\| \,|_{C_1} \geq 2^{m-1}$. We begin with the choice of the operator $\mathcal{A}$.

Let Υ be a chain of partitions associated to the sequence $\eta \in \Sigma_\delta = \{\xi \in \ell_\infty(\mathbb{R}^2) \mid \|\xi\|_\infty \leq \delta\}$. Let $\Omega^1 = \Upsilon^k$ and $\Omega^2 = \Upsilon^{k+1}$ be two consecutive partitions from the chain Υ. We introduce two subspaces $\mathfrak{X}_{\Omega^1}$ and $\mathfrak{X}_{\Omega^2}$ of piecewise-constant vector fields in $\mathfrak{X}$, associated to the partitions Ω^1 and Ω^2, respectively. The subspace $\mathfrak{X}_{\Omega^1}$ has the (canonical) basis

$$\chi^s_{\Omega^1_{ij}} \stackrel{\text{def}}{=} \frac{1}{|\pi_x(\Omega^1_{ij})|}\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\chi_{\Omega^1_{ij}}, \qquad \chi^u_{\Omega^1_{ij}} \stackrel{\text{def}}{=} \frac{1}{|\pi_x(\Omega^1_{ij})|}\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\chi_{\Omega^1_{ij}};$$

and the (canonical) basis of the subspace $\mathfrak{X}_{\Omega^2}$ is

$$\chi^s_{\Omega^2_{ij}} \stackrel{\text{def}}{=} \frac{1}{|\pi_x(\Omega^2_{ij})|}\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\chi_{\Omega^2_{ij}}, \qquad \chi^u_{\Omega^2_{ij}} \stackrel{\text{def}}{=} \frac{1}{|\pi_x(\Omega^2_{ij})|}\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\chi_{\Omega^2_{ij}};$$

both bases have $\mathbb{Z}^2$ elements.

Let $\xi \stackrel{\text{def}}{=} \sigma^{2m(k-1)}\eta$ (see definition of the chain Υ in Subsection 4.3, p. 8). We would like to approximate the operator $P^2_{\xi *}\colon \mathfrak{X} \to \mathfrak{X}$ by a linear operator $\mathcal{A}\colon \mathfrak{X}_{\Omega^1} \to \mathfrak{X}_{\Omega^2}$ chosen so that the averages along the elements of partition Ω^2 are equal for any field $\nu \in \mathfrak{X}_{\Omega^1}$:

$$\int_{\Omega^2_{kl}} A\nu = \int_{\Omega^2_{kl}} P^2_{\xi *}\nu. \tag{33}$$

We write down the action of the operator $\mathcal{A}$ on $\mathfrak{X}_{\Omega^1}$ in matrix form

$$\mathcal{A}\big(\nu^{ij}_s \chi^s_{\Omega^1_{ij}} + \nu^{ij}_u \chi^u_{\Omega^1_{ij}}\big) = \sum_{kl} \nu^{ij}_s \big(SS^{kl}_{ij}\chi^s_{\Omega^2_{kl}} + SU^{kl}_{ij}\chi^u_{\Omega^2_{kl}}\big) + \nu^{ij}_u\big(US^{kl}_{ij}\chi^s_{\Omega^2_{kl}} + UU^{kl}_{ij}\chi^u_{\Omega^2_{kl}}\big), \tag{34}$$

where the four matrices SS, SU, US, and UU are specified as follows, so that (33) holds true (see Lemma 6.14 on p. 35 for details).

$$SS^{kl}_{ij} \stackrel{\text{def}}{=} \frac{1}{|\pi_x(\Omega^1_{ij})|\cdot|\pi_y(\Omega^2_{kl})|}\int_{P_\xi^{-2}(\Omega^2_{kl})\cap\Omega^1_{ij}} \partial_x (P^2_\xi)_x(z)\mathrm{d}z; \tag{35}$$

$$SU^{kl}_{ij} \stackrel{\text{def}}{=} \frac{1}{|\pi_x(\Omega^1_{ij})|\cdot|\pi_y(\Omega^2_{kl})|}\int_{P_\xi^{-2}(\Omega^2_{kl})\cap\Omega^1_{ij}} \partial_x (P^2_\xi)_y(z)\mathrm{d}z; \tag{36}$$

$$US^{kl}_{ij} \stackrel{\text{def}}{=} \frac{1}{|\pi_x(\Omega^1_{ij})|\cdot|\pi_y(\Omega^2_{kl})|}\int_{P_\xi^{-2}(\Omega^2_{kl})\cap\Omega^1_{ij}} \partial_y (P^2_\xi)_x(z)\mathrm{d}z; \tag{37}$$

$$UU^{kl}_{ij} \stackrel{\text{def}}{=} \frac{1}{|\pi_x(\Omega^1_{ij})|\cdot|\pi_y(\Omega^2_{kl})|}\int_{P_\xi^{-2}(\Omega^2_{kl})\cap\Omega^1_{ij}} \partial_y (P^2_\xi)_y(z)\mathrm{d}z. \tag{38}$$

We observe that

$$SS\colon \langle\chi^s_{\Omega^1_{ij}}\rangle \to \langle\chi^s_{\Omega^2_{ij}}\rangle;\ SU\colon \langle\chi^s_{\Omega^1_{ij}}\rangle \to \langle\chi^u_{\Omega^2_{ij}}\rangle;\ US\colon \langle\chi^u_{\Omega^1_{ij}}\rangle \to \langle\chi^s_{\Omega^2_{ij}}\rangle;\ UU\colon \langle\chi^u_{\Omega^1_{ij}}\rangle \to \langle\chi^u_{\Omega^2_{ij}}\rangle.$$

The matrix UU is the most important as it is responsible for the largest eigenvalue of the operator $\mathcal{A}$. We will study it in a great detail in the next Subsection.

Lemma 6.1. *The map $P^2_{\underline{0}}$, corresponding to the zero sequence $\xi \equiv 0$, gives the following matrix elements for any quartet $(i,j,k,l) \in \square \times \square$: $UU^{kl}_{ij} \equiv 1$; $SS^{kl}_{ij} \equiv 2^{-4m}$; $SU^{kl}_{ij} \equiv 0$; $US^{kl}_{ij} \equiv 0$.*

Proof. Each partition of the chain, associated to the zero sequence, is a partition of the unit square $\square$ into 2^{2m+2} equal squares with side length 2^{-m}. Therefore we have that $\Omega^1_{ij} = \left[\frac{i}{2^m}, \frac{i+1}{2^m}\right] \times \left[\frac{j}{2^m}, \frac{j+1}{2^m}\right]$ and $\Omega^2_{kl} = \left[\frac{k}{2^m}, \frac{k+1}{2^m}\right] \times \left[\frac{l}{2^m}, \frac{l+1}{2^m}\right]$.

The preimage of an element $\Omega^2_{kl} \subset \square$ of the partition Ω^2 under P_0^{-2} is equal to 2^m disjoint rectangles $Rec(2, 2^{-3m})$ in $\square$. Thus $|P_{\underline{0}}^{-2}(\Omega^2_{kl}) \cap \Omega^1_{ij}| = 2^{-4m}$. The derivative of $P^2_{\underline{0}}$ on $\square$ is given by the matrix

$$\mathrm{d}P^2_{\underline{0}}(z) \equiv \begin{pmatrix} 2^{-2m} & 0 \\ 0 & 2^{2m} \end{pmatrix} \qquad \text{for all } z \in \square.$$

■

Definition 7. *The matrices*, corresponding to the map $P^2_{\underline{0}}$, we denote by $\overset{\circ}{SS}$, $\overset{\circ}{SU}$, $\overset{\circ}{US}$, and $\overset{\circ}{UU}$, respectively.

Remark 3. Immediately by definition we see that for any quartet (i,j,k,l) such that $(i,j) \in \square$ and $(k,l) \in \mathbb{R}^2 \setminus \square_{1+m\delta}$ or $(i,j) \in \mathbb{R}^2 \setminus \square_{1+m\delta}$ and $(k,l) \in \square$ we have

$$\overset{\circ}{UU}{}^{kl}_{ij} = \overset{\circ}{SU}{}^{kl}_{ij} = \overset{\circ}{US}{}^{kl}_{ij} = \overset{\circ}{SS}{}^{kl}_{ij} = 0 \tag{39}$$

In addition, given $\|\mathrm{d}F_k\| \le \mu_1$, from definition of F_k p. 6, we have

$$\max\big(\|\overset{\circ}{UU}\|_\infty, \|\overset{\circ}{SU}\|_\infty, \|\overset{\circ}{US}\|_\infty, \|\overset{\circ}{SS}\|_\infty\big) \le (1+\mu_1)^{2m}. \tag{40}$$

Remark 4. The condition on the Euclidean norm $\|\mathrm{d}F_k\| \le \mu_1$ implies that there exists a constant M_1 such that for any two partitions Ω^1 and Ω^2 of the class $\mathcal{G}(m,\delta)$,

$$\sup_{(i,j)} \#\big\{(k,l) \in \mathbb{R}^2 \setminus \square_{1+m\delta} \mid P_\xi^{-2}(\Omega^2_{kl}) \cap \Omega^1_{ij} \ne \varnothing\big\} < M_1 \cdot (\mu_1+1)^{2m}. \tag{41}$$

Therefore for any pair $(k,l) \subset \mathbb{R}^2 \setminus \square_{1+m\delta}$ there exist not more than $M_1 \cdot (1+\mu_1)^{2m}$ pairs $(i,j) \subset \mathbb{R}^2 \setminus \square_{1+m\delta}$ such that

$$SS^{kl}_{ij} \cdot SU^{kl}_{ij} \cdot US^{kl}_{ij} \cdot UU^{kl}_{ij} \ne 0.$$

Remark 5. Recall the notations introduced in the beginning of Section 6. There exists a constant M_2, independent of m, such that for $R := M_2 m\delta(1+\mu_1)^{2m} + 1$ and for any quartet (i,j,k,l) where $(i,j) \in \square$ and $(k,l) \in \mathbb{R}^2 \setminus \square_R$ or $(i,j) \in \mathbb{R}^2 \setminus \square_R$ and $(k,l) \in \square$

$$SS^{kl}_{ij} \equiv 0, \quad SU^{kl}_{ij} \equiv 0, \quad US^{kl}_{ij} \equiv 0, \quad UU^{kl}_{ij} \equiv 0.$$

Definition 8. The domains of continuity of the map P^2_ξ we call (P,ξ)-*domains.*

We split (P,ξ)-domains in $P_\xi^{-2}(\Omega^2_{kl})\cap\Omega^1_{ij}\subset\square$ in "good" and "bad" parts:

$$(\Delta^G)^{kl}_{ij} \overset{\text{def}}{=} \big\{\Delta\subset P_\xi^{-2}(\Omega^2_{kl})\cap\Omega^1_{ij} \mid \Delta \text{ is a } (P,\xi)\text{-domain, and } \forall n\le 2m\colon F_\xi^n(\Delta)\subset\square\big\}; \tag{42}$$

$$(\Delta^B)^{kl}_{ij} \overset{\text{def}}{=} \big\{\Delta\subset P_\xi^{-2}(\Omega^2_{kl})\cap\Omega^1_{ij} \mid \Delta \text{ is a } (P,\xi)\text{-domain, and } \exists n\le 2m\colon F_\xi^n(\Delta)\not\subset\square\big\}. \tag{43}$$

Then we may write for $(i,j,k,l)\in\square\times\square$,

$$UU^{kl}_{ij} = (UU^G)^{kl}_{ij} + (UU^B)^{kl}_{ij}, \tag{44}$$

where $UU^G, UU^B\in\mathrm{Mat}(2^m,2^m)$ are given by

$$(UU^G)^{kl}_{ij} \overset{\text{def}}{=} \frac{1}{|\pi_x(\Omega^1_{ij})|}\cdot\frac{1}{|\pi_y(\Omega^2_{kl})|}\cdot\sum_{\Delta\in\Delta^G}\int_\Delta \partial_y(P_\xi^2)_y(z)\,\mathrm{d}z; \tag{45}$$

$$(UU^B)^{kl}_{ij} \overset{\text{def}}{=} \frac{1}{|\pi_x(\Omega^1_{ij})|}\cdot\frac{1}{|\pi_y(\Omega^2_{kl})|}\cdot\sum_{\Delta\in\Delta^B}\int_\Delta \partial_y(P_\xi^2)_y(z)\,\mathrm{d}z. \tag{46}$$

We define three more pairs of matrices $SU^B+SU^G=SU$, $US^B+US^G=US$, $SS^B+SS^G=SS$ in a similar way.

6.1. **Properties of the matrix UU.** The submatrix $UU\colon\langle\chi^u_{\Omega^1_{ij}}\rangle\to\langle\chi^u_{\Omega^2_{ij}}\rangle$ corresponds to a mapping between two subspaces of vector fields parallel to the expanding direction of the Baker's map and associated to two different partitions. It is also responsible for the norm of the operator $\mathcal{A}$. Our goal is to establish the following two facts about the matrix UU.

Proposition 6.1. *The following inequalities hold true for the elements of the matrix UU^G in the canonical bases.*

(1) $\|UU^G\|_\infty=\sup|UU^{kl}_{ij}|\le 4$;
(2) $\#\{(UU^G)^{kl}_{ij}\ne 1\}\le 2^{4\frac12 m}\delta$.

Proposition 6.2. *There exist a constant $\gamma_1<0.01$ such that for M and m sufficiently large and for μ_1 sufficiently small*

$$\max(\|SS\|_\infty,\|US\|_\infty,\|SU\|_\infty,\|UU\|_\infty)\le 2^{\gamma_1 m}.$$

(Recall Condition 3 on F_k: $\|\mathrm{d}F_k\|\le 1+\mu_1$ in the Euclidean operator norm).

By definition, the matrix UU^G is related to subsets of the survivor set $\square\setminus E_{2m}$. To study the set $\square\setminus E_{2m}$, we introduce a simplified system, since the map outside of the unit square is of no importance.

Consider a circle S^1 and a cylinder $\mathcal{C} \stackrel{\text{def}}{=} \mathbb{R} \times S^1 \stackrel{\text{def}}{=} \{(x, y),\ x \in \mathbb{R},\ y \in [-1; 1)\}$. Define a map $h\colon \mathcal{C} \to \mathcal{C}$ by

$$h(z) \stackrel{\text{def}}{=} \begin{cases} \big(\frac{1}{2}(z_x - 1), 2z_y + 1\big), & \text{if } -1 \le z_x \le 0,\ -1 \le z_y \le 1; \\ \big(\frac{1}{2}(z_x + 1), 2z_y - 1\big), & \text{if } 0 \le z_x \le 1,\ -1 \le z_y \le 1; \\ z, & \text{if } |z_x| > 1. \end{cases} \tag{47}$$

Let $\hat{h}\colon \mathcal{C} \times \mathbb{R}^2 \to \mathcal{C}$ be an extension given by

$$\hat{h}(z, w) \stackrel{\text{def}}{=} \begin{cases} \big(\frac{1}{2}(z_x - 1) + w_x, (2z_y + w_y) \bmod 2 - 1\big), & \text{if } -1 \le z_x \le 0,\ -1 \le z_y \le 1; \\ \big(\frac{1}{2}(z_x + 1) + w_x, (2z_y + w_y) \bmod 2 - 1\big), & \text{if } 0 \le z_x \le 1,\ -1 \le z_y \le 1; \\ (z_x + w_x, (z_y + w_y) \bmod 2 - 1), & \text{if } |z_x| > 1. \end{cases} \tag{48}$$

Using the extension $\hat{h}$, we define a small perturbation h_ξ, as described in Subsection 2.

We denote the central part of the cylinder by $\odot \stackrel{\text{def}}{=} \{z \in \mathcal{C}\colon |z_x| \le 1\}$. By rectangle in $\odot$ we understand a subset $Rec(l_x, l_y) = I_x \times I_y$, where $I_x \subset [-1; 1)$ and $I_y \subset S^1 \setminus \{1\}$ are two intervals with $|I_x| = l_x$ and $|I_y| = l_y$.

Lemma 6.2. *Given a sequence $\xi \in \Sigma_\delta$, with $\delta = 2^{-m\alpha}$, for any $1 \le k \le m\alpha - 3$ there exist k rectangles $r_1^{k,\xi}, \ldots, r_k^{k,\xi} \subset \odot$ such that*

$$\big\{z \in \odot \mid \exists 1 \le j \le k\colon h_\xi^j(z) \notin \odot\big\} \subset \bigcup_{j=1}^{k} r_j^{k,\xi}.$$

Moreover, $r_j^{k,\xi} \subset Rec(2^j\delta, 2^{1-j})$ for all $1 \le j \le k$ and for any $a \in \mathbb{R}^2$ with $|a| \le \delta$ the map h_a^{-1} is continuous on the union of the rectangles $\bigcup_{j=1}^{k} r_j^{k,\xi}$.

Proof. By induction in k. Indeed, the conditions $z \in \odot$ and $h_\xi(z) \notin \odot$ are equivalent to $|\pi_x(h_\xi(z))| > 1$ and $z \in \square$. The latter means

$$z \in r_1^\xi \stackrel{\text{def}}{=} \begin{cases} (-1; -1 + 2\xi_x^1) \times (-1; 0) \subset (-1; -1 + 2\delta) \times (-1; 0), & \text{if } \xi_x < 0, \\ (1 - 2\xi_x^1; 1) \times (0; 1) \subset (1 - 2\delta; 1) \times (0; 1), & \text{if } \xi_x > 0. \end{cases} \tag{49}$$

Thus the statement holds true for $k = 1$. Let us add to the induction assumption the following inclusion which is trivial for $k = 1$:

$$\bigcup_{j=1}^{k} r_j^{k,\xi} \subset (1 - 2^k\delta; 1) \times (-1; 1) \cup (-1; -1 + 2^k\delta) \times (-1; 1). \tag{50}$$

We may write

$$\left\{z \in \odot \mid \exists j \le k+1\colon h_\xi^j(z) \notin \odot\right\} \subset$$
$$\subset \left\{z \in \odot \mid h_\xi(z) \notin \odot\right\} \cup \left\{z \in \odot \mid \exists 1 < j \le k+1\colon h_\xi^j(z) \notin \odot\right\} \subset$$
$$\subset r_1^\xi \cup \left\{w = h_{\xi^1}(z) \subset \odot \mid \exists 1 \le j \le k\colon h_{\sigma(\xi)}^j(w) \notin \odot\right\} \subset r_1^\xi \cup h_{\xi^1}^{-1}\Big(\bigcup_{j=1}^k r_j^{k,\sigma(\xi)}\Big).$$

Therefore we may set $r_1^{k+1,\xi} \stackrel{\text{def}}{=} r_1^\xi$ and $r_{j+1}^{k+1,\xi} \stackrel{\text{def}}{=} h_{\xi^1}^{-1}(r_j^{k,\sigma(\xi)}) \cap \Box$ for $j = 1, \dots, k$. Since h_0^{-1} is continuous on every $(r_j^{k,\sigma(\xi)} - \xi^1) \cap \Box$, the sets $r_{j+1}^{k+1,\xi}$ are rectangles. Using supposition (50) we conclude

$$h_{\xi^1}^{-1}\Big(\bigcup_{j=1}^k r_j^{k,\xi}\Big) \subset h_{\xi^1}^{-1}\Big(\big((1-2^k\delta,1) \cup (-1,-1+2^k\delta)\big) \times (-1,1)\Big) \subset$$
$$\subset \big((-1,-1+2^{k+1}\delta) \cup (1-2^k\delta,1)\big) \times (-1,1), \quad (51)$$

and therefore h_a^{-1} is continuous on $\bigcup_{j=1}^{k+1} r_j^{k+1,\xi} - a$ for any $|a| \le \delta$.

Finally, one can check by straightforward calculation that for all $1 \le j \le k$ we have

$$h_{\xi^1}^{-1}(r_j^{k,\sigma(\xi)}) \subset Rec(2^{j+1}\delta, 2^{1-j}).$$

■

Corollary 1. *For any sequence $\xi \in \Sigma_\delta$ there are $\frac{3m}{4}$ rectangles $r_1^\xi, r_2^\xi, \dots, r_{3m/4}^\xi \subset \odot$ such that*

$$\left\{z \in \odot \mid \exists 1 \le j \le \frac{3m}{4}\colon h_\xi^j(z) \notin \odot\right\} \subset \bigcup_{j=1}^{3m/4} r_j^\xi;$$

and the union $\bigcup_{j=1}^{3m/4} r_j^\xi \subset \Box$ may be covered by at most $m^3 2^{2m}\delta$ rectangles $Rec(2^{-5m/4}, 2^{-3m/4})$.

Proof. By Lemma 6.2, there exists $\frac{3m}{4}$ rectangles $r_1^\xi, \dots, r_{3m/4}^\xi \subset \odot$ such that

$$\left\{z \in \odot \mid \exists 1 \le j \le k\colon h_\xi^j(z) \notin \odot\right\} \subset \bigcup_{j=1}^{3m/4} r_j^\xi;$$

moreover, $r_j^\xi \subseteq Rec(2^j\delta, 2^{1-j})$. Therefore, each r_j^ξ may be covered by at most

$$m^2\big((2^{5m/4} \cdot 2^j\delta) \cdot (2^{3m/4} \cdot 2^{1-j}) + 2^{m/4} \cdot 2^{2-j} + 2^{3m/4+j+1}\delta\big) \le 2^{2m} m^3 \delta$$

rectangles $Rec(\frac{2^{-5m/4}}{m}, \frac{2^{-3m/4}}{m})$. Since there are $\frac{3m}{4}$ rectangles $r_1^\xi, \dots, r_{3m/4}^\xi$ their union may be covered by not more than $2^{2m} m^4 \delta$ rectangles $Rec(2^{-5m/4}, 2^{-3m/4})$. ■

We may identify a rectangle on the cylinder $I_x \times I_y \subset \odot$ with a rectangle on the plane $I_x \times I_y \subset \square \subset \mathbb{R}^2$, since we agreed that $I_y \subset S^1 \setminus \{1\}$.

Lemma 6.3. *Under the hypothesis and in the notations of Lemma 6.2 the set* $\bigcup\limits_{j=1}^{m/4} r_j^{m/4,\xi}$ *may be covered by at most* $2^{2m}m^3\delta$ *elements of a partition of the class* $\mathcal{G}(m,\delta)$.

Proof. By definition, all elements of a partition of the class $\mathcal{G}(m,\delta)$ are rectangles. By the second part of Lemma 4.2, $Rec(\frac{2^{-m}}{m}, \frac{2^{-m}}{m}) \subseteq \Omega_{ij} \subset Rec(2^{1-m}, 2^{1-m})$. Therefore any rectangle $Rec(2^k\delta, 2^{1-k})$ may be covered by at most

$$m^2(2^{k+m}\delta \cdot 2^{1+m-k} + 2^{m+2} \cdot 2^{-k} + 2^{m+k+1}\delta) \le m^2 2^{2m+2}\delta$$

elements of the partition. Then all $\frac{m}{4}$ rectangles may be covered by at most $2^{2m}m^3\delta$ elements. ■

We lift the map $h\colon \mathcal{C} \to \mathcal{C}$ to the plane $\mathbb{R}^2$ and obtain

$$H(z) \overset{\text{def}}{=} \begin{cases} \big(\frac{1}{2}(z_x - 1), 2z_y + 1\big), & \text{if } z \in \square,\ -1 \le z_y \le 0; \\ \big(\frac{1}{2}(z_x + 1), 2z_y - 1\big), & \text{if } z \in \square,\ 0 \le z_y \le 1; \\ z, & \text{if } z \notin \square. \end{cases} \tag{52}$$

Let $\widehat{H}\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ be an extension given by

$$\widehat{H}(z,w) \overset{\text{def}}{=} \begin{cases} \big(\frac{1}{2}(z_x - 1), 2z_y + 1\big) + w, & \text{if } z \in \square \text{ and } -1 \le z_y \le 0; \\ \big(\frac{1}{2}(z_x + 1), 2z_y - 1\big) + w, & \text{if } z \in \square \text{ and } 0 \le z_y \le 1; \\ z + w, & \text{if } z \notin \square. \end{cases} \tag{53}$$

Given a sequence $\xi \subset \Sigma_\delta \subset \ell_\infty(\mathbb{R}^2)$ and extension $\widehat{H}$, we define a small perturbation H_ξ, as described in Subsection 2.

Remark 6. Observe that $z \in E_k$ if and only if $\prod\limits_{j=1}^{k} \chi_\square(H_\xi^j(z)) = 0$; where E_k is the k'th escaping set defined by (23), p. 8.

Remark 7. Let p be the doubling map defined by (8) with $s_1 = s_2 = 2$. Let ξ and ς be two sequences defined as in Lemma 4.1. Then for any $k > 0$ the following two diagrams are commutative.

$$\begin{array}{ccc} \square \setminus E_k & \xrightarrow{H_\xi^k} & \mathbb{R}^2 \\ {\scriptstyle \pi_y}\downarrow & & \downarrow{\scriptstyle \pi_y} \\ \mathbb{R} & \xrightarrow{p_{\xi_y}} & \mathbb{R} \end{array} \qquad\qquad \begin{array}{ccc} \square \setminus E_{-k} & \xrightarrow{H_\xi^{-k}} & \mathbb{R}^2 \\ {\scriptstyle \pi_x}\downarrow & & \downarrow{\scriptstyle \pi_x} \\ \mathbb{R} & \xrightarrow{p_{\varsigma_x}} & \mathbb{R} \end{array}$$

Recall the settings, introduced in the beginning of the Section 6, p. 14. Let Υ be a chain of partitions associated to the sequence $\eta \in \Sigma_\delta$. Let $\Omega^1 = \Upsilon^k$ and $\Omega^2 = \Upsilon^{k+1}$ be two consecutive partitions from the chain Υ, and let $\xi = \sigma^{2mk}\eta$ be a shifted sequence.

Lemma 6.4. *The number of elements of the partition Ω^1 inside the square $\square$ possibly escaping in the first $\frac{m}{4}$ iterations is bounded by $2^{\frac{9m}{4}+1}\delta$:*

$$\#\Big\{\Omega^1_{ij} \subset \square \mid \exists\, 1 \le k \le \frac{m}{4} \colon H^k_\xi(\Omega^1_{ij}) \not\subset \square\Big\} \le 2^{\frac{9m}{4}+1}\delta.$$

Proof. By Lemma 6.3

$$\#\Big\{\Omega^1_{ij} \subset \odot \mid \exists\, 1 \le k \le \frac{m}{4} \colon h^k_\xi(\Omega^1_{ij}) \not\subset \odot\Big\} \le 2^{2m} \cdot m^3\delta,$$

which is equivalent to

$$\#\Big\{\Omega^1_{ij} \subset \square \mid \exists\, 1 \le k \le \frac{m}{4} \colon \pi_x(H^k_\xi(\Omega^1_{ij})) \not\subset [-1;1]\Big\} \le 2^{2m} \cdot m^3\delta.$$

Recall the doublin map p defined by (8) with $s_1 = s_2 = 2$. Let $p^k_{\xi_y}$ be a small perturbation as in Lemma 2.1. Then the map p_{ξ_y} $p^k_{\xi_y}$ has exactly 2^k long branches for all $k \le m\alpha$. Therefore we get an upper bound

$$\begin{aligned}\#\Big\{\Omega^1_{ij} \subset \square \mid \forall\, 1 \le k \le \frac{m}{4} \colon \pi_x(H^k_\xi(\Omega^1_{ij})) \subset [-1;1] \text{ and} \\ \exists\, 1 \le k \le \frac{m}{4} \colon \pi_y(H^k_\xi(\Omega^1_{ij})) \not\subset [-1;1]\Big\} \le \\ \le 2^m \cdot \#\Big\{\Omega^1_j \subset [-1;1] \mid \exists\, 1 \le k \le \frac{m}{4} \colon p^k_{\xi_y}(\Omega^1_j) \not\subset [-1;1]\Big\} \le 2^{5m/4+1},\end{aligned}$$

By supposition on α, we know that $2^{2m}m^3\delta \ll 2^{5m/4}$. (In other words, assume that for some $\Omega^1_j \subset [-1;1]$ we have $p^k_{\xi_y}(\Omega^1_j) \subset [-1;1]$ for all $k < k_0$ and $p^{k_0}_{\xi_y}(\Omega^1_j) \not\subset [-1;1]$. Then Ω^1_j is a subset of the domain of a long branch of $p^k_{\xi_y}$ for all $k < k_0$; and the subset of the domain of a main branch that may escape at the iteration k is an interval, i.e. a connected set, of the measure at most $2^{-k}\delta$, which contains at most $2^{m-k}\delta$ intervals of the canonical partition of the perturbation of the doubling map $p^m_{\xi_y}$.) ∎

Remark 8. In Lemma 6.4 above, an alternative upper bound would be $2^{\frac{5m}{4}} \cdot C_\delta$, where C_δ is the maximum number of intervals of the canonical partition for the doubling map in the interval of the length δ. In our case all intervals have the length $|\pi_y(\Omega_{ij})| \le 2^{1-m}$, therefore $2^m\delta > C_\delta > 2^{m-1}\delta$.

Lemma 6.5. *There exists at least $2^{2m} - 2^{9m/4+2}\delta$ elements $\Omega^1_{ij} \subset \square$ of the partition Ω^1 such that $H^k_\xi(\Omega^1_{ij}) \subset \square$ for all $1 \le k \le m$ and*

$$Rec\Big(2^{-m/4}|\pi_x(\Omega^1_{ij})|, 2^{m/4}|\pi_y(\Omega^1_{ij})|\Big) = H^{m/4}_\xi(\Omega^1_{ij}) \subset \square.$$

Proof. By Lemma 6.4 we know that there are at most $2^{9m/4+2}\delta$ elements of the partition Ω^1 such that $H_\xi^{m/4}(\Omega_{ij}^1) \not\subset \Box$. We shall show now that there are at most $2^{5m/4}$ elements of Ω^1 such that $H_\xi^k(\Omega_{ij}^1) \subset \Box$ for all $1 \le k \le \frac{m}{4}$, and yet

$$H_\xi^{m/4}(\Omega_{ij}^1) \not\supseteq Rec\Big(2^{-m/4}|\pi_x(\Omega_{ij}^1)|, 2^{m/4}|\pi_y(\Omega_{ij}^1)|\Big).$$

If $H_\xi^{m/4}(\Omega_{ij}^1)$ is connected, then $H_\xi^{m/4}(\Omega_{ij}^1) = Rec\Big(2^{-m/4}|\pi_x(\Omega_{ij}^1)|, 2^{m/4}|\pi_y(\Omega_{ij}^1)|\Big)$. Thus without loss of generality we may assume that $H_\xi^{m/4}(\Omega_{ij}^1)$ is not a connected set. The latter implies $H_\xi^k(\Omega_{ij}^1) \cap \{z_y = 0\} \neq \varnothing$ for some $1 \le k \le m/4$. Recall the doublin map p defined by (8) with $s_1 = s_2 = 2$. Let $p_{\xi_y}^k$ be a small perturbation as in Lemma 2.1. Since by supposition $H_\xi^k(\Omega_{ij}^1) \subset \Box$ for all $1 \le k \le \frac{m}{4}$, we conclude that $\Omega_j^1 := \pi_y(\Omega_{ij}^1)$ belongs to a main branch of the map $p_{\xi_y}^{m/4}$. We know that the map $p_{\xi_y}^k$ has at most 2^k main branches, and if $\{0\} \in p_{\xi_y}^{k_1}(\Omega_j^1)$, then $\{0\} \notin p_{\xi_y}^{k_2}(\Omega_j^1)$ for all $k_1 < k_2 \le \frac{m}{4}$. So there are at most $2^{m/4+1}$ elements Ω_j^1 such that $\{0\} \in p_{\xi_y}^k(\Omega_j^1)$ for some $1 \le k \le \frac{m}{4}$. Thus there are at most $2^{5m/4}$ elements Ω_{ij}^1 such that $H_\xi^k(\Omega_{ij}^1) \cap \{y = 0\} \neq \varnothing$ for some $1 \le k \le \frac{m}{4}$ and $H_\xi^k(\Omega_{ij}^1) \subset \Box$ for all $1 \le k \le \frac{m}{4}$. ■

Corollary 1. *There exists at least $2^{2m} - 2^{9m/4}\delta$ elements $\Omega_{ij}^1 \subset \Box$ of the partition Ω^1 such that $F_\xi^k(\Omega_{ij})) \subset \Box$ for all $1 \le k \le \frac{m}{4}$ and*

$$Rec\Big(2^{-m/4}|\pi_x(\Omega_{ij}^1)|, 2^{m/4}|\pi_y(\Omega_{ij}^1)|\Big) = F_\xi^{m/4}(\Omega_{ij}^1) \subset \Box.$$

We need the following fact about small perturbations of the doubling map p.

Lemma 6.6. *For any $\frac{m}{2} \le k \le m\alpha - 2$ the perturbation of the doubling map p_ξ^m with $\|\xi\|_\infty$ has at most 2^{k+2} main branches such that their domains $\mathfrak{a}_j^{(m)}$ satisfy $|p_{\xi_y}^m(\mathfrak{a}_j^{(m)})| < 2 - 2^{m-k}\delta$.*

Proof. Let $\mathfrak{a}_j^{(m)} = (a_j^{(m)}; a_{j+1}^{(m)})$ be the domain of a main branch of the map p_ξ^m such that $|p_\xi^m(\mathfrak{a}_j^{(m)})| < 2 - 2^{m-k}\delta$.

We shall show that the interval $\overline{\mathfrak{a}_j^{(m)}}$ is not contained in a domain of a main branch of the map p_ξ^{k+2}.

Assume for a contradiction that for some $\frac{m}{2} \le k \le m\alpha - 2$ there exists a main branch $\mathfrak{a}_i^{(k+2)} \supset \overline{\mathfrak{a}_j^{(m)}}$ of the map p_ξ^{k+2}. By assumption, $a_j^{(m)}$ and $a_{j+1}^{(m)}$ are points of discontinuity of the map p_ξ^m. Since p_ξ^{k+2} is continuous on $\mathfrak{a}_j^{(k+2)}$, we deduce that there exist $k_1, k_2 \ge k+2$ such that $p_\xi^{k_1}(a_j^{(m)}) = 0$ and $p_\xi^{k_2}(a_{j+1}^{(m)}) = 0$. Since $p_\xi^m(\mathfrak{a}_j^{(m)})$ is an interval, we see that either $|p_\xi^m(a_j^{(m)}) + 1| > 2^{m-k-1}\delta$ or $|p_\xi^m(a_{j+1}^{(m)}) - 1| > 2^{m-k-1}\delta$. Without loss of generality, assume the first. Then

$$p_\xi^m(a_j^{(m)}) = p_\xi^{m-k_1}(0) = p_\xi^{m-k_1-1}(-1 + \xi(k_1+1)),$$

and, therefore, $|p_\xi^m(a_{j+1}^{(m)}) + 1| \le 2^{m-k_1+1}\delta$. Thus $k_1 < k+2$. We deduce that the map p_ξ^{k+2} is not continuous on $\overline{\mathfrak{a}_j^{(m)}}$. We know by Lemma 2.1, that for any $1 \le k \le m\alpha$ the map p_ξ^k has exactly 2^k main branches and the Lemma follows.

■

Lemma 6.7. *There exist at least $2^{2m} - 2^{\frac{3}{2}m}$ elements of the partition Ω^1 in the unit square $\square$ such that for some $\breve{\Omega}^1_{ij} \subset \Omega^1_{ij}$ we have $H_\xi^n(\breve{\Omega}^1_{ij}) \subset \square$ for all $1 \le n \le m$ and*

$$H_\xi^m(\breve{\Omega}^1_{ij}) = Rec\big(2^{-m}|\pi_x(\Omega^1_{ij})|, 2 - 2^{\frac{m}{2}}\delta\big).$$

Proof. Let $\eta = \sigma^{m/4}(\xi)$ and let r_j^η, $1 \le j \le \frac{3m}{4}$ be rectangles covering the escaping set $E_{\frac{3m}{4}}$ of the map $F_\eta^{3m/4}$, defined according to Corollary 1 of Lemma 6.2. According to Lemma 6.5 there exist at least $2^{2m} - 2^{\frac{9}{4}m+2}\delta$ elements of the partition Ω^1 such that $H_\xi^k(\Omega^1_{ij}) \subset \square$ for all $1 \le k \le \frac{m}{4}$, and there exists a rectangle $Rec(2^{-m/4}|\pi_x(\Omega^1_{ij})|, 2^{m/4}|\pi_y(\Omega^1_{ij})|) = H_\xi^{m/4}(\Omega^1_{ij}) \subset \square$. It follows from Corollary 1 of Lemma 6.2, that among these elements of the partition Ω^1 one can find at least $2^{2m} - 2^{\frac{9}{4}m+2}\delta - 2^{2m}m^4\delta$ elements that satisfy $H_\xi^{m/4}(\Omega^1_{ij}) \cap \big(\bigcup_{j=1}^{3m/4} r_j^\eta\big) = \varnothing$.

The condition $H_\xi^{m/4}(\Omega^1_{ij}) \cap \big(\bigcup_{j=1}^{3m/4} r_j^\eta\big) = \varnothing$ implies $\pi_x(H_\xi^k(\Omega^1_{ij})) \subset [-1;1]$ for all $1 \le k \le m$, and it follows that $|\pi_x(H_\xi^m(\Omega^1_{ij}))| = 2^{-m}|\pi_x(\Omega^1_{ij})|$. Therefore, $H_\xi^k(\Omega^1_{ij}) \not\subset \square$ for some $\frac{m}{4} < k \le m$ if and only if $\pi_y(H_\xi^k(\Omega^1_{ij})) \not\subset [-1;1]$. By construction, $\Omega^1_j = \pi_y(\Omega^1_{ij})$ is an element of the canonical partition of the map $p_{\xi_y}^m$. By Lemma 6.6 with $k = \frac{m}{2}$, there map $p_{\xi_y}^m$ has at most $2^{\frac{m}{2}+2}$ main branches such that $|p_{\xi_y}^m(\mathfrak{a}_j^{(m)})| \le 2 - 2^{\frac{m}{2}}\delta$. For every Ω^1_{ij}, such that $\pi_x(H_\xi^k(\Omega^1_{ij})) \subset [-1;1]$ and $\pi_y(\Omega^1_{ij})$ contains the domain $\mathfrak{a}_j^{(m)}$ of a main branch of the map $p_{\xi_y}^m$ with $|p_{\xi_y}^m(\mathfrak{a}_j^{(m)})| \ge 2 - 2^{\frac{m}{2}}\delta$, there exists a rectangle $\breve{\Omega}^1_{ij} \stackrel{\text{def}}{=} \pi_x(\Omega^1_{ij}) \times \mathfrak{a}_j^{(m)} \subset \Omega^1_{ij}$ with the property $H_\xi^k(\breve{\Omega}^1_{ij}) \subset \square$ for all $1 \le k \le m$, and, moreover

$$H_\xi^m(\Omega^1_{ij}) \supset H_\xi^m(\breve{\Omega}^1_{ij}) \supset Rec\Big(2^{-m}|\pi_x(\Omega^1_{ij})|, 2 - 2^{m/2}\delta\Big).$$

Therefore, there are at least $2^{2m} - 2^{\frac{9}{4}m+2}\delta - 2^{2m}m^4\delta - 2^{\frac{3}{2}m+2} \ge 2^{2m} - 2^{\frac{3}{2}m+3}$ elements of the partition Ω^1 such that for some $\breve{\Omega}^1_{ij} \subset \Omega^1_{ij}$ which satisfies $H_\xi^k(\breve{\Omega}^1_{ij}) \subset \square$ for all $1 \le k \le m$ we have

$$H_\xi^m(\breve{\Omega}^1_{ij}) = Rec\big(2^{-m}|\pi_x(\Omega^1_{ij})|, 2 - 2^{\frac{m}{2}}\delta\big).$$

In other words, the map H_ξ^m has at least $2^{2m} - 2^{\frac{3}{2}m+3}$ main branches. ■

Corollary 1. *There exist at least $2^{2m} - 2^{\frac{3}{2}m+3}$ elements of the partition Ω^2 such that for some $\breve{\Omega}^2_{ij} \subset \Omega^2_{ij}$ we have $H^{-k}_{\sigma^m\xi}(\breve{\Omega}^2_{ij}) \subset \square$ for all $1 \leq k \leq m$ and*

$$H_\xi^{-m}(\breve{\Omega}^2_{ij}) = Rec(2 - 2^{\frac{m}{2}}\delta, 2^{-m}|\pi_y(\Omega^2_{ij})|).$$

Definition 9. The rectangles $\breve{\Omega}^1_{ij}$ and $\breve{\Omega}^2_{ij}$, constructed in Lemma 6.7 and Corollary 1 of Lemma 6.7 we call *domains of the long branches* of the maps P_ξ and $P^{-1}_{\sigma^m\xi}$, respectively. Their images we call *long branches.*

Lemma 6.8. *For any element Ω^1_{ij} of the partition Ω^1, the set $\Omega^1_{ij} \setminus E_m$ is a union of (disjoint) rectangles. The number of rectangles is equal to the number of main branches of the perturbation $p^m_{\xi_y}$ of the doubling map p.*

Proof. We split the argument into several steps.

Claim 1. The projection $\pi_y(\Omega^1_{ij} \setminus E_m)$ is a union of domains of main branches of the small perturbation $p^m_{\xi_y}$ of the doubling map. First we shall show that the image of the projection $p^n_{\xi_y}(\pi_y(\Omega^1_{ij} \setminus E_m)) \subset [-1;1]$ for all $1 \leq n \leq m$. Indeed, assume for a contradiction that for some $1 < n < m$ we have $p^n_{\xi y}(\pi_y(\Omega^1_{ij} \setminus E_m)) \notin [-1;1]$, and n is the smallest number with this property. Since the horizontal lines $\{y = \text{const}\} \cap \square \setminus E_{m-1}$ are invariant under H^n_ξ, we may conclude that $H^n_\xi(\Omega^1_{ij} \setminus E_m) \not\subset \square$, which is a contradiction. Therefore $\pi_y(\Omega^1_{ij} \setminus E_m)$ is a subset of the domain of a main branch. Let an interval $(a,b) \supset \pi_y(\Omega^1_{ij} \setminus E_m)$ be the domain of the main branch. We shall show that $\Omega^1_i \times (a,b) \subset \Omega^1_{ij} \setminus E_m$. Assume that there exists $z \in \Omega^1_i \times (a,b)$ such that $H^n_\xi(z) \notin \square$ for some $1 \leq n \leq m$. Since $\pi_y(H^n_\xi(z)) = p^n_{\xi_y}(z_y) \in [-1;1]$, we conclude $\pi_x(H^n_\xi(z)) \notin (-1;1)$. Observe that, the lines $\{x = \text{const}\} \cap \square \setminus E_{m-1}$ are invariant with respect to H^n_ξ, we get $H^n_\xi(z_x, \pi_y(\Omega^1_{ij} \setminus E_m)) \notin \square$, which is a contradiction. Therefore $(a,b) \subset \pi_y(\Omega^1_{ij} \setminus E_m)$ and hence $\pi_y(\Omega^1_{ij} \setminus E_m)$ is a union of domains of main main branches.

Claim 2. The set $\{y = \text{const}\} \cap (\Omega^1_{ij} \setminus E_m)$ is connected. Indeed, assume that there are three points $z, u, w \in \{y = \text{const}\} \cap (\Omega^1_{ij} \setminus E_m)$ such that $z_x < u_x < w_x$, with $z, w \in \Omega^1_{ij} \setminus E_m$ and $u \notin \Omega^1_{ij} \setminus E_m$. Then there exists $1 \leq n \leq m$ such that $H^n_\xi(u) \notin \square$, and we may assume that n is the smallest number with such property. Then by invariance of $\{y = \text{const}\} \cap \Omega^1_{ij} \setminus E_{n-1}$ with respect to H^n_ξ, we conclude that either $H^n_\xi(z) \notin \square$ or $H^n_\xi(w) \notin \square$, which is a contradiction.

Claim 3. For any two points $z, w \in \Omega^1_{ij} \setminus E_m$ such that z_y and w_y belong to the same domain of a main branch of $p^m_{\xi_y}$, we have $(z_x, w_y), (w_x, z_y) \in \Omega^1_{ij} \setminus E_m$. Indeed, assume for a contradiction that $(z_x, w_y) \notin \Omega^1_{ij} \setminus E_m$. Then choose the smallest n such that $H^n_\xi(z_x, w_y) \notin \square$. It follows that either $\pi_y(H^n_\xi(z_x, w_y)) \notin [-1; 1]$ or $\pi_x(H^n_\xi(z_x, w_y)) \notin [-1; 1]$, or both. Without loss of generality suppose that projection of the image $\pi_y(H^n_\xi(z_x, w_y) \notin [-1; 1]$. Then due to invariance of $\{x = \text{const}\} \cap \square \setminus E_{m-1}$ we have $\pi_x(H^n_\xi(z)) \notin [-1; 1]$, which is a contradiction.

Summing up, we conclude that the set $\Omega^1_{ij} \setminus E_m$ is a union of rectangles and the number of rectangles is equal to the number of main branches of the map $p^m_{\xi_y}$ in Ω^1_j. ∎

Corollary 1. *In the notation of Lemma 4.1, the set $\Omega^2_{ij} \setminus E_{-m}$ is a union of (disjoint) rectangles for any element Ω^2_{ij} of the partition Ω^2. The number of rectangles is equal to the number of main branches of the perturbation $p^m_{\varsigma_x}$ of the doubling map p.*

Lemma 6.9. *There exist at most $2^{4m}\delta$ quartets (i, j, k, l) such that $H^{-2m}_\xi(\Omega^2_{kl}) \cap \Omega^1_{ij}$ has more than one (P, ξ)-domain Δ that satisfies $H^n_\xi(\Delta) \subset \square$ for all $1 \le n \le 2m$. For any quartet (i, j, k, l) the set $H^{-2m}_\xi(\Omega^2_{kl}) \cap \Omega^1_{ij}$ has at most four (P, ξ)-domains with this property.*

Proof. Let Δ be a (P, ξ)-domain in $H^{-2m}_\xi(\Omega^2_{kl}) \cap \Omega^1_{ij}$ such that $H^n_\xi(\Delta) \subset \square$ for all $1 \le n \le 2m$. Then

$$\begin{aligned}
\#\{\Delta \subset H^{-2m}_\xi(\Omega^2_{kl}) \cap \Omega^1_{ij} \mid H^n_\xi(\Delta) \subset \square \text{ for all } 1 \le n \le 2m\} = \\
= \#\{\Delta \subset H^{-m}_{\sigma^m\xi}(\Omega^2_{kl}) \cap H^m_\xi(\Omega^1_{ij}) \mid H^n_\xi(\Delta) \subset \square \text{ for all } -m \le n \le m\} = \\
= \#\big\{\Delta \subset \big(\Omega^2_{kl} \setminus E_{-m}\big) \cap \big(\Omega^1_{ij} \setminus E_m\big)\big\}.
\end{aligned}$$

By Lemma 6.8 and Corollary 1 of Lemma 6.8, both sets $\Omega^2_{kl} \setminus E_{-m}$ and $\Omega^1_{ij} \setminus E_m$ are unions of rectangles, and the number of rectangles equal to the number of main branches of the corresponding doubling maps on the associated intervals. By Theorem 1 there are at most $2^m\delta$ intervals Ω_i or Ω_l that contain two main branches. Thus there are at most $2^{4m}\delta$ quartets (i, j, k, l) such that Ω_i or Ω_l or both contain two main branches of the maps $p^m_{\varsigma_x}$ and $p^m_{\xi_y}$, respectively; and the Lemma follows. ∎

Using Lemmas 6.7 and 6.9 and Corollary 1 of Lemma 6.7, we get

Corollary 1. *Let Υ be a chain of partitions associated to the sequence $\eta \in \Sigma_\delta$. Let $\Omega^1 = \Upsilon^k$ and $\Omega^2 = \Upsilon^{k+1}$ be two consecutive partitions from the chain Υ, and let $\xi = \sigma^{2m(k-1)}\eta$ be a shifted sequence. Then*

(1) *There exist at least* $2^{2m} - 2^{\frac{3}{2}m+3}$ *elements* Ω^1_{ij} *such that for some* $\breve{\Omega}^1_{ij} \subset \Omega^1_{ij}$ *we have*

$$P_\xi(\breve{\Omega}_{ij}) = Rec\Big(2^{-m}|\pi_x(\Omega_{ij})|, 2 - 2^{\frac{m}{2}}\delta\Big) \qquad \textit{and} \qquad \mathrm{d}P_\xi\big|_{\breve{\Omega}_{ij}} = \begin{pmatrix} 2^{-m} & 0 \\ 0 & 2^m \end{pmatrix}.$$

(2) *There exist at least* $2^{2m} - 2^{\frac{3}{2}m+3}$ *elements* Ω^2_{ij} *such that for some* $\breve{\Omega}^2_{kl} \subset \Omega^2_{kl}$ *we have*

$$P^{-1}_\xi(\breve{\Omega}_{ij}) = Rec\Big(2 - 2^{\frac{m}{2}}\delta, 2^{-m}|\pi_y(\Omega_{ij})|\Big) \qquad \textit{and} \qquad \mathrm{d}P^{-1}_\xi\big|_{\breve{\Omega}_{ij}} = \begin{pmatrix} 2^{m} & 0 \\ 0 & 2^{-m} \end{pmatrix}.$$

(3) *There exists at most* $2^{4m}\delta$ *quartets* (i,j,k,l) *such that the set* $P^{-2}_\xi(\Omega^2_{kl}) \cap \Omega^1_{ij}$ *contains more than one* (P,ξ)*-domain* Δ *that satisfies* $d_y(P^2_\xi)_y\big|_\Delta = 2^{2m}$.

Proof. Observe that for any $1 \le k \le 2m$ and for any $z \in \Box \backslash E_k$ we have $F^k_\xi(z) = H^k_\xi(z)$. ∎

Lemma 6.10. *The area of a good* (P,ξ)*-domain* Δ *is very small. More precisely, we have an upper bound* $|\Delta| \le 2^{2-4m}$.

Proof. Recall the definition of good connected components (42) and observe

$$\begin{aligned}(\Delta^G)^{kl}_{ij} &= \big\{\Delta \subset P^{-2}_\xi(\Omega^2_{kl}) \cap \Omega^1_{ij} \mid \Delta \text{ is a } (P,\xi)\text{-domain}, \forall 1 \le n \le 2m : F^n_\xi(\Delta) \subset \Box\big\} = \\ &= \big\{\Delta \subset P^{-2}_\xi(\Omega^2_{kl}) \cap \Omega^1_{ij}) \mid \Delta \text{ is a } (P,\xi)\text{-domain}, \forall 1 \le n \le 2m : H^n_\xi(\Delta) \subset \Box\big\}.\end{aligned}$$

We shall show that for any $\Delta \in \Delta^G$ the area $|\Delta| \le 2^{-2m} \cdot |\pi_x(\Omega^1_{ij})| \cdot |\pi_y(\Omega^2_{kl})|$. Indeed, consider the image $\Delta' = P_\xi(\Delta)$. Since P_ξ is area-preserving, $|\Delta'| = |\Delta|$. Since $P_\xi(\Delta') \subset \Omega^2_{kl}$, the length $|\pi_y(\Delta')| \le 2^{-m} \cdot |\pi_y(\Omega^2_{kl})|$; and $P^{-1}_{\sigma^m\xi}(\Delta') \subset \Omega^1_{ij}$ implies $|\pi_x(\Delta')| \le 2^{-m} \cdot |\pi_x(\Omega^1_{ij})|$. Thus

$$|\Delta| = |\Delta'| \le 2^{-2m}|\pi_x(\Omega^1_{ij})| \cdot |\pi_y(\Omega^2_{kl})| \le 2^{2-4m}. \tag{54}$$

∎

Corollary 1. *The matrix* SS^G *is small. More precisely,*

$$\sum_\Box \sum_\Box \big|(SS^G)^{kl}_{ij}\big| \le 2^{4-2m}.$$

Proof. By straightforward calculation, using Lemma 6.10,

$$\sum_{\square}\sum_{\square}|(SS^G)_{ij}^{kl}| = \sum_{\square}\sum_{\square}\frac{1}{|\pi_x(\Omega^1_{ij})|}\cdot\frac{1}{|\pi_y(\Omega^2_{kl})|}\cdot\sum_{\Delta\in\Delta^G}\int_\Delta \partial_x(P^2_\xi)_x(z)\mathrm{d}z \le$$

$$\le \sum_{\square}\sum_{\square}\frac{1}{|\pi_x(\Omega^1_{ij})|}\cdot\frac{1}{|\pi_y(\Omega^2_{kl})|}\cdot\sum_{\Delta\in\Delta^G} 2^{-4m}|\Delta| \le$$

$$\le \sum_{\square}\sum_{\square}\frac{1}{|\pi_x(\Omega^1_{ij})|}\cdot\frac{1}{|\pi_y(\Omega^2_{kl})|}\cdot 4(2^{-2m}|\pi_x(\Omega^1_{ij})|\cdot|\pi_y(\Omega^2_{kl})|)\cdot 2^{-4m} \le 2^{4-2m}.$$

∎

Now we are ready to prove

Proposition 6.1. The matrix UU^G has the following properties

(1) $\|UU^G\|_\infty \le 4$;

(2) $\#\{(UU^G)_{ij}^{kl} \ne 1\} \le 2^{4\frac{1}{2}m}\delta$.

Proof. By Lemma 6.9, for any $(i,j,k,l) \in \square\times\square$ we have $\#(\Delta^G)_{ij}^{kl} \le 4$, and by Lemma 6.10 we know $|\Delta| \le 2^{-2m}\cdot|\pi_x(\Omega^1_{ij})|\cdot|\pi_y(\Omega^2_{kl})|$. We calculate

$$|(UU^G)_{ij}^{kl}| \le \sum_{\Delta\in(\Delta^G)_{ij}^{kl}} |\Delta|\cdot|\partial_y(P^2_\xi)_y|\cdot|\pi_x(\Omega^1_{ij})|^{-1}\cdot|\pi_y(\Omega^2_{kl})|^{-1} \le$$

$$\le 4\cdot(2^{-2m}|\pi_x(\Omega^1_{ij})|\cdot|\pi_y(\Omega^2_{kl})|)\cdot 2^{2m}\cdot|\pi_x(\Omega^1_{ij})|^{-1}\cdot|\pi_y(\Omega^2_{kl})|^{-1} = 4.$$

To prove the second part, we recall that by Lemma 6.7 there are at least $2^{2m}-2^{\frac{3}{2}m+3}$ elements Ω^1_{ij} such that for some $\breve\Omega_{ij}\subset\Omega^1_{ij}$ the image is a rectangle $P_\xi(\breve\Omega_{ij}) = Rec(2^{-m}|\pi_x(\Omega_{ij})|), 2-2^{\frac{m}{2}}\delta)$ and $H^n_\xi(\breve\Omega_{ij}) \subset \square$ for all $1\le n\le m$. Similarly by Corollary 1 of Lemma 6.7, there are at least $2^{2m}-2^{\frac{3}{2}m+3}$ elements Ω^2_{kl} such that for some small rectangle $\breve\Omega_{kl}\subset\Omega^2_{kl}$ the preimage $P_\xi^{-1}(\breve\Omega_{kl}) = Rec(2-2^{\frac{m}{2}}\delta, 2^{-m}|\pi_y(\Omega_{ij})|)$ and $H^{-n}_\xi(\breve\Omega_{kl})\subset\square$ for all $1\le n\le m$. Then there are at least $(2^{2m}-2^{\frac{3}{2}m+3}-2^{\frac{5}{2}m}\delta)^2$ pairs Ω^1_{ij}, Ω^2_{kl} such that $P_\xi(\breve\Omega_{ij})\cap P^{-1}_\xi(\breve\Omega_{kl}) \ne \varnothing$ which correspond to $(UU^G)_{ij}^{kl}\ne 0$. If $(\Delta^G)_{ij}^{kl}$ has only one element, then it is $\Delta = P_\xi(\breve\Omega_{ij})\cap P^{-1}_{\sigma^m\xi}(\breve\Omega_{kl})$ and $|\Delta| = 2^{-2m}\cdot|\pi_x(\Omega^1_{ij})|\cdot|\pi_y(\Omega^2_{kl})|$. Therefore

$$(UU^G)_{ij}^{kl} = \frac{1}{|\pi_x(\Omega^1_{ij})|}\cdot\frac{1}{|\pi_y(\Omega^2_{kl})|}\int_\Delta 2^{2m} = 1.$$

Summing up, there are at least $2^{4m}-2^{\frac{9}{2}m+1}\delta$ elements $(UU^G)_{ij}^{kl} = 1$. By Lemma 6.9 the set $(\Delta^G)_{ij}^{kl}$ has more than one connected component for not more that $2^{4m}\delta$ quartets (i,j,k,l). Therefore at most $2^{4m}\delta$ elements satisfy $1 < (UU^G)_{ij}^{kl} \le 4$. The other elements are zeros. ∎

Now we proceed to the supremum norm of the matrix UU. Our goal is to prove the following

Proposition 6.2. There exist a constant $\gamma_1 < 0.01$ such that for M and m sufficiently large and for μ sufficiently small

$$\max(\|SS\|_\infty, \|SU\|_\infty, \|US\|_\infty, \|UU\|_\infty) \leq 2^{\gamma_1 m}.$$

We define two functions on the unit square

$$t_{\text{in}} \colon \square \to \mathbb{N} \qquad t_{\text{in}}(z) = \sum_{j=0}^{2m} \chi_\square(F_\xi^j(z)); \tag{55}$$

$$t_{\text{ex}} \colon \square \to \mathbb{N} \cap \left[1; \frac{2m}{M}\right] \qquad t_{\text{ex}}(z) = \#\{1 \leq n \leq 2m \colon F_\xi^{n-1}(z) \in \square \text{ and } F_\xi^n(z) \notin \square\}. \tag{56}$$

Given a sequence $\imath \in \{0,1\}^{\mathbb{N}}$ we define a subset of the unit square $\square$

$$\Delta_\imath \stackrel{\text{def}}{=} \left\{ z \in \square \colon \chi_\square(F_\xi^n(z)) = \imath_n \text{ for all } n \in \{0, 1, \ldots, 2m\} \right\}.$$

Note that some of $\Delta_\imath$ may be empty and they are not necessary connected.

Lemma 6.11. *There are at most $\frac{2m}{M} 2e^{\frac{2m+M}{2+M}}$ non-empty disjoint subsets $\Delta_\imath \subset \square$.*

Proof. We know the total number of sequences that correspond to the points with $t_{\text{ex}} \equiv s$:

$$\#\{\imath \in \{0,1\}^{\mathbb{N}} \mid t_{\text{ex}}(\imath) = s\} = \binom{2m - (s-1)M}{s}.$$

Observe that the number of disjoint subsets $\Delta_\imath \subset \Delta$ is equal to the number of different sequences, which we can estimate in the following way. It is well known that $\binom{2n}{n} > \binom{k}{s}$ for all $1 \leq k \leq 2n$ and $1 \leq s \leq k$. The equality $2m - (s-1)M = 2s$ has the solution $s_0 = \frac{2m+M}{2+M}$ so we conclude $\binom{2m-(s-1)M}{s} \leq \binom{2s_0}{s_0}$ for all $s > s_0 = \frac{2m+M}{2+M}$. Using the Stirling formula, we calculate

$$\binom{2s_0}{s_0} \leq \text{const} \cdot \frac{(2s_0)^{2s_0}}{s_0^{2s_0}} = \text{const} \cdot 2^{2s_0} = \text{const} \cdot 2^{\frac{4m+2M}{2+M}}$$

We also may write for all $s < s_0$

$$\binom{2m - (s-1)M}{s} = \frac{(2m - (s-1)M)!}{s!(2m - (s-1)M - s)!} \leq (2m - (s-1)M)^s \left(\frac{e}{s}\right)^s.$$

By straightforward calculation

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big(\frac{(2m-(s-1)M)e}{s}\Big)^s = \\ =\Big(\frac{(2m-(s-1)M)e}{s}\Big)^s \cdot \Big(\ln\frac{2m-(s-1)M}{s} - \frac{s}{2m-(s-1)M}\Big) > 0$$

for all $s \in (1; s_0)$, because

$$\ln\frac{2m-(s-1)M}{s} > \ln\frac{2m-(s_0-1)M}{s_0} = \ln 2 > \\ > \frac{1}{2} = \frac{s_0}{2m-(s_0-1)M} > \frac{s}{2m-(s-1)M}.$$

We conclude that for $s < s_0$

$$\binom{2m-(s-1)M}{s} \le (2s_0)^{s_0}\frac{e^{s_0}}{s_0^{s_0}} = (2e)^{\frac{2m+M}{2+M}}.$$

Summing up,

$$\sum_{j=1}^{\frac{m}{M}}\binom{2m-(s-1)M}{s} \le \frac{m}{M}(2e)^{\frac{2m+M}{2+M}}.$$

■

Given a sequence $\jmath \in \{-1,0,1\}^{\mathbb{N}}$ we define a subset of the unit square

$$\Delta_\jmath \overset{\mathrm{def}}{=} \big\{ z \in \square \colon \chi_\square(F_\xi^n(z)) \cdot \operatorname{sgn} \pi_y(F_\xi^n(z)) = \jmath_n \text{ for all } n \in \{0,1,\dots,2m\} \big\}.$$

Note that some of $\Delta_\jmath$ may be empty, and they are not necessary connected.

Definition 10. We introduce to projections of the tower to the zero floor:

$$\pi_x \colon X \to X \qquad \pi_x(z,n) = ((z_x,0),0);$$
$$\pi_y \colon X \to X \qquad \pi_y(z,n) = ((0,z_y),0).$$

Lemma 6.12. *Given a quartet (i,j,k,l) and a subset $B_\imath \overset{\mathrm{def}}{=} \Delta_\imath \cap \Omega^1_{ij} \cap P_\xi^{-2}(\Omega^2_{kl})$, there are at most $6^{\frac{2m}{M}}$ disjoint subsets $\Delta_\jmath$ such that $\Delta_\jmath \cap B_\imath \ne \varnothing$.*

Proof. Consider a first half of the sequence $\imath$ of the length m, the subsequence $\imath_1, \imath_2, \dots, \imath_m$. It may contain not more than $\frac{m}{M}$ "blocks" of 1's. We shall show by induction in number of blocks that

(1) There are not more than $6^{\frac{m}{M}}$ different sequences $\jmath_1,\dots,\jmath_m$ such that $\Delta_\jmath \cap B_\imath \ne \varnothing$.
(2) The projection of the image $\pi_y(P_\xi(B_\imath))$ may be covered by not more than $6^{\frac{m}{M}}$ intervals of the total length not more than 2.

In order to use induction, we need to study the original map $F\colon X \to X$ of the tower X defined on p. 6; we also recall that by definition $P_\xi = F_\xi^m\colon \mathbb{R}^2 \to \mathbb{R}^2$.

Given a sequence $\imath$, there are two possibilities.

Case 1. All blocks of 1's in $\imath$ are not longer than $m\alpha - 1$.

Case 2. There are blocks of 1's in $\imath$ of the length $m\alpha$ or longer.

Case 1. Assume that all blocks of 1's in the sequence $\imath$ are not longer than $m\alpha - 1$.

The base of induction. Assume that there is only one block of 1's. Then there exist two numbers $1 \le t_1 \le s_1 \le m$, $s_1 - t_1 \le m\alpha$:

$$\imath_k = \begin{cases} 1, & \text{if } t_1 \le k \le s_1; \\ 0, & \text{otherwise.} \end{cases}$$

We deduce that $\pi_y(P_\xi^{t_1-1}(\Delta_\imath))$ belongs to a union of domains of main branches of the perturbation $p^{s_1-t_1}_{\sigma^{t_1-1}\xi_y}$ of the doubling map p. We know by Lemma 2.1 that the map $p^{s_1-t_1}_{\sigma^{t_1-1}\xi_y}$ has exactly $2^{s_1-t_1}$ main branches, all of them are long and their domains have the length at least $2^{t_1-s_1} > 2^{-m\alpha}$. In addition, since

$$\operatorname{diam}(B_\imath) = \operatorname{diam}(\Delta_\imath \cap \Omega^1_{ij} \cap P_\xi^{-2}(\Omega^2_{kl})) \le \operatorname{diam}(\Omega^1_{ij}) < 2^{2-m}$$

we conclude that there exists an interval $I \subset [-1;1]$ such that $\pi_y(F_\xi^{t_1}(B_\imath)) \subset I$ and[1] the length $|I| < 2^{2-m} \cdot (1+\mu_1)^{t_1-1} < 2^{-m\alpha} < 2^{t_1-s_1}$. Thus the interval I may intersect not more than 2 domains of main branches of the map $p^{s_1-t_1}_{\sigma^{t_1-1}\xi_y}$ and therefore there are not more than 4 sequences $\jmath_k$, $1 \le k \le m$ corresponding to the sequence $\imath_k$, $1 \le k \le m$. In addition, we observe that the image $\pi_y(F_\xi^{s_1}(B_\imath))$ may be covered by 4 intervals of the total length not more than $2^{-m} \cdot 2^{s_1-t_1} \cdot (1+\mu_1)^m$.

Now assume that there are n blocks of 1's. Namely, there exist

$$1 \le t_1 \le s_1 < t_2 \le s_2 < \ldots < t_n \le s_n \le m \tag{57}$$

such that $t_{i+1} - s_i \ge M$ and $s_i - t_i \le m\alpha - 1$, where

$$\imath_k = \begin{cases} 1, & \text{if } t_i \le k \le s_i, \text{ for } i = 1, \ldots, n; \\ 0, & \text{otherwise .} \end{cases} \tag{58}$$

Since $s_n - t_n < m\alpha$, by Lemma 2.1 the doubling map $p^{s_n-t_n}_{\sigma^{t_n-1}\xi}$ has exactly $2^{s_n-t_n}$ main branches, all of which are long, and their domains have length at least $2^{t_n-s_n}$. By

[1]We may safely assume that $2^\alpha > 1 + \mu_1$.

induction assumption, the set $\pi_y(F_\xi^{t_n-1}(B_\imath))$ may be covered by 4^{n-1} intervals of the total length

$$2^{-m}\cdot\prod_{k=1}^{n-1}2^{s_k-t_k}\cdot(1+\mu_1)^m\le 2^{-m}\cdot 2^{m-(s_n-t_n)-M(n-1)}=2^{t_n-s_n}\cdot 2^{-M(n-1)}.$$

Therefore it may intersect not more than $\min(2\cdot 4^{n-1},2^{s_n-t_n})$ domains of the main branches of the map $p^{s_n-t_n}_{\sigma^{t_n-1}\xi}$. Consequently, there are at most 4^n different sequences $\jmath$ of the length m and the projection of the image $\pi_y(F_\xi^{s_n}(B_\imath))$ may be covered by 4^n intervals of the total length $2^{-m}\cdot\prod\limits_{k=1}^{n}2^{s_k-t_k}\cdot(1+\mu_1)^m\le 2^{-M(n-1)}(1+\mu_1)^m$.
Case 2. There exists a subsequence of 1's of the length $m\alpha$ or longer. Then there is only one subsequence with this property (since $\alpha>\frac{15}{16}$). There are two possibilities.

(2A) In the notations introduced in (57) and (58) above, $s_1-t_1>m\alpha$.

(2B) In the notations introduced in (57) and (58) above, $s_n-t_n>m\alpha$ for some $n>1$.

In the case 2A, the map $p^{s_1-t_1}_{\sigma^{t_1-1}\xi}$ has at least $2^{s_1-t_1-2}$ long branches, and their domains have length at least $2^{t_1-s_1}$. At the same time the projection of the image $\pi_y(F_\xi^{t_1-1}(B_\imath))$ is contained in an interval I of the length $|I|<2^{-m}\cdot(1+\mu_1)^{t_1}<2^{t_1-s_1}$. By Lemma 2.1, the distance between any two domains of the main branches of the map $p^{s_1-t_1}_{\sigma^{s_1-1}\xi}$ which are not long, is at least $2^{m(\alpha-1)}>2^{t_1-s_1}$. Therefore the interval I may intersect not more than three domains of main branches (two long and one more) of the map $p^{s_1-t_1}_{\sigma^{t_1-1}\xi}$. Thus we conclude that there are not more than 6 different sequences $\jmath_{t_1},\dots\jmath_{s_1}$, corresponding to the sequence $\imath_{t_1},\dots,\imath_{s_1}$. The induction step then follows as above, giving $6^{\frac{m}{M}}$ sequences.

In the case 2B, the map $p^{s_n-t_n}_{\sigma^{t_n-1}\xi}$ has at least $2^{s_n-t_n-2}$ long branches, and their domains have length at least $2^{t_n-s_n}$. Then by induction from the Case 1, we know that there are $4^{n-1}<4^{m(1-\alpha)-M}$ sequences corresponding to the sequence $\imath_1,\dots,\imath_{t_n-1}$ and the image of the set $\pi_y(P_\xi^{t_n-1}(B_\imath))$ may be covered by 4^{n-1} intervals of the total length not more than $2^{t_n-s_n-M}$. We see that the total number of long branches of the map $p^{s_n-t_n}_{\sigma^{t_n-1}\xi}$ is greater than the number of intervals covering the image

$$2^{m\alpha-2}>4^{m(1-\alpha)-M},$$

and the total length of intervals is shorter than a domain of any long branch. Therefore, each of the intervals may intersect not more than three domains of main branches, and we get at most $6\cdot 4^{k-1}$ different sequences. In addition, we notice that the image $\pi_y(F_\xi^{s_n}(B_\imath))$ may be covered by $6\cdot 4^{k-1}$ intervals.

To complete the proof of the Lemma, we need to calculate number of different sequences $\jmath_{m+1}, \dots, \jmath_{2m}$ such that $\Delta_\jmath \cap B_\imath \neq \varnothing$. We would like to apply the argument above to the inverse map $F^{-m}_{\sigma^m\xi} = P^{-1}_{\sigma^m\xi}$. Let us consider the image $P_\xi(B_\jmath) \subset \Omega^2_{kl}$. Define a sequence $\jmath'$, associated to the iterations of the inverse map $P_{\sigma^m}\xi^{-1}$.

$$\jmath' \colon z \to \{-1,0,1\}^{\mathbb{N}} \qquad \jmath'_k(z) = \begin{cases} 1, & \text{if } F^{-k+1}_{\sigma^{2m-k}\xi}(z) \in \square + \xi_x^{2m+1-k},\ z_x > \xi_x^{2m-k}; \\ -1, & \text{if } F^{-k+1}_{\sigma^{2m-k}\xi}(z) \in \square + \xi_x^{2m+1-k},\ z_x < \xi_x^{2m-k}; \\ 0, & \text{if } F^{-k+1}_{\sigma^{2m-k}\xi}(z) \notin \square + \xi_x^{2m+1-k}. \end{cases} \tag{59}$$

We see that

$$\jmath'_k(P^2_\xi z) = \jmath_{2m-k+1}(z) \text{ for all } 0 \leq k \leq m.$$

We may associate the sequence $\jmath'$ to main branches of the doubling map p_{ς_x}, defined as in Lemma 4.1 p. 9, in the following way.

$$\left\{\jmath'_k \equiv 1,\ \text{for } 0 \leq t_1 \leq k \leq t_2 \leq m,\ t_1 < t_2\right\} \Longleftrightarrow$$
$$\left\{\pi_x(F^{-t_1}_{\sigma^{2m-t_1}}(z)) \text{ in a domain of a main branch of } p^{t_2-t_1-1}_{\sigma^{t_1}\varsigma_x}\right\}.$$

Indeed, if, say, $\jmath'_{t_1} = 1$, then by definition, $F^{-t_1+1}_{\sigma^{2m-t_1}\xi}(z) \in \square + \xi_x^{2m+1-t_1}$ and $z_x > \xi_x^{2m-t_1}$. Consequently, $F^{-l}_{\sigma^{2m-l-1}\xi}(z) \in \square$ for all $t_1 \leq l \leq t_2$, and therefore $\pi_x(P^{-t_1}_{\sigma^{2m-t_1-1}\xi}(z))$ is in a domain of a long branch of $p^{t_2-t_1}_{\sigma^{2m-l-1}\varsigma_x}$.

In the case $t_1 = t_2 = 1$, i.e. a block of the length 1, we get two sequences corresponding to a given $\jmath_{t_1} = 1$ and $\jmath = -1$, similarly to the previous case.

It follows that to any sequence $\imath$ of the length $2m$ correspond $6^{2m/M}$ sequences $\jmath$.

■

Corollary 1. *Among all sequences $\jmath$, there are at most $\frac{2m}{M} \cdot 6^{\frac{2m}{M}}(2e)^{\frac{2m+M}{2+M}}$ pairwise disjoint segments $\Delta_\jmath$ such that $P^{-2}_\xi(\Omega^2_{kl}) \cap \Omega^1_{ij} \cap \Delta_\jmath \neq \varnothing$.*

Now we are ready to prove

Proposition 6.2. There exist a constant $\gamma_1 < 0.01$ such that for M and m sufficiently large

$$\max(\|UU\|_\infty, \|SU\|_\infty, \|US\|_\infty, \|SS\|_\infty) \leq 2^{\gamma_1 m}.$$

Proof. Recall the definition of the matrices, for instance

$$UU^{kl}_{ij} = \frac{1}{|\pi_x(\Omega^1_{ij})|} \cdot \frac{1}{|\pi_y(\Omega^2_{kl})|} \int_{P^{-2}_\xi(\Omega^2_{kl}) \cap \Omega^1_{ij}} \partial_y(P^2_\xi)_y(z)\mathrm{d}z$$

and the other three are defined using another three partial derivatives, according to (35)–(37). Consider a vertical line segment $\Delta_c = \{z_x = c\} \cap P_\xi^{-2}(\Omega^2_{kl}) \cap \Omega^1_{ij}$. Recall that according to condition 5 the composition of maps outside of the unit square $F_{i_1} \circ \ldots \circ F_{i_M}$, where $i_1, \ldots, i_M \in \{1, \ldots, M\}$ is a polynomial of degree at most d. Since P_ξ^2 is smooth on each $\Delta_\jmath \cap \Delta_c$ and $P_\xi^2(\Delta_\jmath \cap \Delta_c) \subset \Omega^2_{kl}$. We can estimate the length of the image using condition 5, p. 6:

$$|P_\xi^2(\Delta_\jmath \cap \Delta_c)| \le \operatorname{diam}(\Omega^2_{kl}) \cdot d^{\frac{2m}{M}} \le 2^{\frac{m}{500}}; \tag{60}$$

since the preimage with respect to any of the orthogonal projections π_x and π_y has at most $d^{\frac{2m}{M}}$ connected components.

$$\max\Big(\int_{\Delta_\jmath} |\partial_y (P_\xi^2)_y(z)| \mathrm{d}z, \int_{\Delta_\jmath} |\partial_x (P_\xi^2)_y(z)| \mathrm{d}z, \int_{\Delta_\jmath} |\partial_y (P_\xi^2)_x(z)| \mathrm{d}z, \int_{\Delta_\jmath} |\partial_x (P_\xi^2)_x(z)| \mathrm{d}z\Big) \le$$
$$\le \big|P_\xi^2(\Delta_\jmath \cap \Delta_c)\big| \le d^{\frac{2m}{M}} \operatorname{diam}(\Omega^2_{kl}).$$

Therefore

$$\int_{P_\xi^{-2}(\Omega^2_{kl}) \cap \Omega^1_{ij}} \max\big(|\partial_y (P_\xi^2)_y(z)|, |\partial_x (P_\xi^2)_y(z)|, |\partial_y (P_\xi^2)_x(z)|, |\partial_x (P_\xi^2)_x(z)|\big) \mathrm{d}z =$$
$$= \int_{\pi_y(\Omega^1_{ij})} \int_{\Delta_c} \max\big(|\partial_y (P_\xi^2)_y(z)|, |\partial_x (P_\xi^2)_y(z)|, |\partial_y (P_\xi^2)_x(z)|, |\partial_x (P_\xi^2)_x(z)|\big) \mathrm{d}z \mathrm{d}c =$$
$$= \int_{\pi_y(\Omega^1_{ij})} \sum_{\Delta_\jmath \subset \Delta_c} \int_{\Delta_\jmath} \max\big(|\partial_y (P_\xi^2)_y(z)|, |\partial_x (P_\xi^2)_y(z)|, |\partial_y (P_\xi^2)_x(z)|, |\partial_x (P_\xi^2)_x(z)|\big) \mathrm{d}z \mathrm{d}c \le$$
$$\le \frac{2m}{M} (2e)^{\frac{2m+M}{2+M}} \cdot 6^{\frac{2m}{M}} \cdot \operatorname{diam}(\Omega^2_{kl}) \cdot d^{\frac{2m}{M}} \cdot |\pi_y(\Omega^1_{ij})|.$$

Finally,

$$\int_{P_\xi^{-2}(\Omega^1_{ij}) \cap \Omega^2_{kl}} \max\big(|\partial_y (P_\xi^2)_y(z)|, |\partial_x (P_\xi^2)_y(z)|, |\partial_y (P_\xi^2)_x(z)|, |\partial_x (P_\xi^2)_x(z)|\big) \mathrm{d}z \le$$
$$\le |\pi_x(\Omega^1_{ij})| \cdot |(\pi_y(\Omega^2_{kl})| \cdot \frac{2m}{M} (2e)^{\frac{2m+M}{2+M}} \cdot 6^{\frac{2m}{M}} \cdot d^{\frac{2m}{M}}.$$

We can choose μ_1 and μ_2 sufficiently small so that for m and M large enough and for some $\gamma_1 \le 0.01$

$$\frac{2m}{M} (2e)^{\frac{2m+M}{2+M}} \cdot 6^{\frac{2m}{M}} \cdot d^{\frac{2m}{M}} \le 2^{\gamma_1 m}.$$

■

Lemma 6.13. *The sum of elements of the matrix $|(UU^B)^{kl}_{ij}|$ with $(i,j,k,l) \in \square \times \square$ is at most $2^{2m} \cdot 8m\delta$.*

Proof. Indeed, recall that for any $\Delta \subset \Delta_\xi^B$ there exists $1 \le n \le 2m$ such that $F_\xi^n(\Delta) \not\subset \square$ and thus

$$\bigcup_{ij}\bigcup_{kl}\bigcup_{(\Delta^B)_{ij}^{kl}} \Delta = \bigcup_{ij}\bigcup_{kl}\Big\{\Delta \text{ is a } (P,\xi)\text{-domain}\Big|\, F_\xi^n(\Delta) \not\subset \square \text{ for some } 1 \le n \le 2m\Big\} =$$

$$= \{z \in \square \mid \exists 1 \le n \le 2m\colon F_\xi^n(z) \notin \square\} =: B.$$

We get $|B| \le 8m\delta$ by induction in number of iterations and conclude

$$\sum_{ij}\sum_{kl} |(UU^B)_{ij}^{kl}| \le \int_B |\partial_y (P_\xi^2)_y(z)| \mathrm{d}z \le 2^{2m} \cdot 8m\delta.$$

■

Remark 9. It follows from the condition 3 on the map F (see p. 6) that partial derivatives are essentially bounded $\|\partial_y (P_\xi^2)_x\|_\infty \le (1+\mu)^{2m}$, $\|\partial_x (P_\xi^2)_y\|_\infty \le (1+\mu)^{2m}$, $\|\partial_x (P_\xi^2)_x\|_\infty \le (1+\mu)^{2m}$. Thus by the same argument as in Lemma 6.13 we get

$$\sum_{\square}\sum_{\square} |(US^B)_{ij}^{kl}| \le (1+\mu)^{2m} m\delta; \tag{61}$$

$$\sum_{\square}\sum_{\square} |(SU^B)_{ij}^{kl}| \le (1+\mu)^{2m} m\delta; \tag{62}$$

$$\sum_{\square}\sum_{\square} |(SS^B)_{ij}^{kl}| \le (1+\mu)^{2m} m\delta. \tag{63}$$

6.2. **The operators $W_\delta \mathcal{A}$ and $W_\delta P_{\xi*}^2$ are close on $\mathfrak{X}$.** We keep the notation introduced in the first paragraph of this Section.

Let Υ be a chain of partitions associated to the sequence $\eta \in \Sigma_\delta$. Let $\Omega^1 = \Upsilon^k$ and $\Omega^2 = \Upsilon^{k+1}$ be two consecutive partitions from the chain Υ. Let $\xi \stackrel{\text{def}}{=} \sigma^{2m(k-1)}\eta$ (cf. Definition of the chain Υ in subsection 4.3, p. 8). Let $\mathcal{A}\colon \mathfrak{X}_{\Omega_1} \to \mathfrak{X}_{\Omega_2}$ be a linear operator, approximating the operator $P_{\xi*}^2$, defined according to (34).

In this section we establish the following

Proposition 6.3. *The operators $W_\delta \mathcal{A}$ and $W_\delta P_{\xi*}^2$ are close. Namely, for any $\nu \in \mathfrak{X}_{\Omega^1}$,*

$$\|W_\delta (P_{\xi*}^2 - \mathcal{A})\nu\|_{\Omega^2, \mathcal{L}_1} \le \frac{4 \sup \operatorname{diam}(\Omega_{kl}^2)}{\delta} \cdot 2^{2m}, \tag{64}$$

$$\|W_\delta (P_{\xi*}^2 - \mathcal{A})\nu\|_\infty \le \frac{4 \sup \operatorname{diam}(\Omega_{kl}^2)}{\delta} \cdot 2^{(2+\gamma_1)m}; \tag{65}$$

where γ_1 is defined by Proposition 6.2.

We start with

Lemma 6.14. *For any element Ω^2_{kl} of the partition Ω^2, and for any $\nu \in \mathfrak{X}_{\Omega^1}$,*

$$\int_{\Omega^2_{kl}} P^2_{\xi*}\nu = \int_{\Omega^2_{kl}} \mathcal{A}\nu.$$

Proof. Let $\nu = \sum_{ij} \nu^{ij}_s \chi^s_{\Omega^1_{ij}} + \sum_{ij} \nu^{ij}_u \chi^u_{\Omega^1_{ij}}$. Then

$$\begin{aligned}
P^2_{\xi*}\nu(z) &= \mathrm{d}P^2_\xi(P^{-2}_\xi z)\cdot \nu(P^{-2}_\xi z) = \\
&= \sum_{ij} \nu^{ij}_s \mathrm{d}P^2_\xi(P^{-2}_\xi z)\chi^s_{\Omega^1_{ij}}(P^{-2}_\xi z) + \sum_{ij} \nu^{ij}_u \mathrm{d}P^2_\xi(P^{-2}_\xi z)\chi^u_{\Omega^1_{ij}}(P^{-2}_\xi z) = \\
&= \sum_{ij} \nu^{ij}_s \big(\partial_x(P^2_\xi)_x(P^{-2}_\xi z) + \partial_x(P^2_\xi)_y(P^{-2}_\xi z)\big)\cdot \frac{1}{|\pi_x(\Omega^1_{ij})|}\chi_{\Omega^1_{ij}}(P^{-2}_\xi z) + \\
&\quad + \sum_{ij} \nu^{ij}_u \big(\partial_y(P^2_\xi)_x(P^{-2}_\xi z) + \partial_y(P^2_\xi)_y(P^{-2}_\xi z)\big)\cdot \frac{1}{|\pi_x(\Omega^1_{ij})|}\chi_{\Omega^1_{ij}}(P^{-2}_\xi z).
\end{aligned}$$

We may integrate

$$\begin{aligned}
&\frac{1}{|\Omega^2_{kl}|}\int_{\Omega^2_{kl}} \partial_x(P^2_\xi)_x(P^{-2}_\xi z)\cdot \frac{1}{|\pi_x(\Omega^1_{ij})|}\cdot \chi_{\Omega^1_{ij}}(P^{-2}_\xi z)\mathrm{d}z = \\
&\quad = \frac{1}{|\pi_x(\Omega^2_{kl})|}\cdot\frac{1}{|\pi_y(\Omega^2_{kl})|}\cdot\frac{1}{|\pi_x(\Omega^1_{ij})|}\int_{P^{-2}_\xi(\Omega^2_{kl})\cap\Omega^1_{ij}} \partial_x(P^2_\xi)_x(z)\mathrm{d}z = \frac{1}{|\pi_x(\Omega^2_{kl})|}SS^{kl}_{ij}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{1}{|\pi_x(\Omega^2_{kl})|}US^{kl}_{ij} &= \frac{1}{|\Omega^2_{kl}|}\int_{\Omega^2_{kl}} \partial_y(P^2_\xi)_x(P^{-2}_\xi z)\frac{1}{|\pi_x(\Omega^1_{ij})|}\chi_{\Omega^1_{ij}}(P^{-2}_\xi z)\mathrm{d}z;\\
\frac{1}{|\pi_x(\Omega^2_{kl})|}SU^{kl}_{ij} &= \frac{1}{|\Omega^2_{kl}|}\int_{\Omega^2_{kl}} \partial_x(P^2_\xi)_y(P^{-2}_\xi z)\frac{1}{|\pi_x(\Omega^1_{ij})|}\chi_{\Omega^1_{ij}}(P^{-2}_\xi z)\mathrm{d}z;\\
\frac{1}{|\pi_x(\Omega^2_{kl})|}UU^{kl}_{ij} &= \frac{1}{|\Omega^2_{kl}|}\int_{\Omega^2_{kl}} \partial_y(P^2_\xi)_y(P^{-2}_\xi z)\frac{1}{|\pi_x(\Omega^1_{ij})|}\chi_{\Omega^1_{ij}}(P^{-2}_\xi z)\mathrm{d}z.
\end{aligned}$$

So we may write

$$\frac{1}{|\Omega^2_{kl}|}\int_{\Omega^2_{kl}} P^2_{\xi*}\nu(z)\mathrm{d}z = \sum_{ij}\nu^{ij}_s\frac{1}{|\pi_x(\Omega^2_{kl})|}(SS^{kl}_{ij}+US^{kl}_{ij}) + \sum_{ij}\nu^{ij}_u\frac{1}{|\pi_x(\Omega^2_{kl})|}(SU^{kl}_{ij}+UU^{kl}_{ij}).$$

Observe that for any Ω^2_{kl}, by definition of the operator $\mathcal{A}$ (34) on p. 15,

$$\frac{1}{|\Omega^2_{kl}|}\int_{\Omega^2_{kl}} \mathcal{A}\nu = \frac{1}{|\pi_x(\Omega^2_{kl})|}\Big(\sum_{ij}\nu^{ij}_s(SS^{kl}_{ij}+US^{kl}_{ij}) + \sum_{ij}\nu^{ij}_u(SU^{kl}_{ij}+UU^{kl}_{ij})\Big).$$

∎

Lemma 6.15. *For any partition Ω of the plane $\mathbb{R}^2$ into rectangles we have*

$$\int_{\mathbb{R}^2}\Big|\max_{t\in\Omega_{ij}} w_\delta(z-t) - \min_{t\in\Omega_{ij}} w_\delta(z-t)\Big|\mathrm{d}z \le \frac{4\sup\mathrm{diam}(\Omega_{ij})}{\pi\delta}.$$

Proof. Given a compact convex subset $A \subset \mathbb{R}^2$, let $\gamma(A)$ be the longest line segment connecting the points where the function $w_\delta(t)$ achieves its maximum and minimum in A. By straightforward calculation

$$\max_{t\in\Omega_{ij}} w_\delta(z-t) - \min_{t\in\Omega_{ij}} w_\delta(z-t) = \max_{t\in\Omega_{ij}-z} w_\delta(t) - \min_{t\in\Omega_{ij}-z} w_\delta(t) \le \int_{\gamma(\Omega_{ij}-z)} |\nabla w_\delta(t)| \mathrm{d}t.$$

Thus

$$\begin{aligned}
\int_{\mathbb{R}^2} \big|\max_{\Omega_{ij}} w_\delta(z-t) - \min_{t\in\Omega_{ij}} w_\delta(z-t)\big| \mathrm{d}z &\le \int_{\mathbb{R}^2}\int_{\gamma(\Omega_{ij}-z)} \big|\nabla w_\delta(t)\big| \mathrm{d}t\mathrm{d}z = \\
&= \int_{\mathbb{R}^2}\int_{\gamma(\Omega_{ij})} \big|\nabla w_\delta(t-z)\big|\mathrm{d}t\mathrm{d}z = \int_{\gamma(\Omega_{ij})}\int_{\mathbb{R}^2} \big|\nabla w_\delta(t-z)\big|\mathrm{d}z\mathrm{d}t = \\
&= \int_{\gamma(\Omega_{ij})}\int_{\mathbb{R}^2} \big|\nabla w_\delta(z)\big|\mathrm{d}z\mathrm{d}t \le \mathrm{diam}(\Omega_{ij}) \int_{\mathbb{R}^2} \big|\nabla w_\delta(z)\big|\mathrm{d}z = \\
&= \mathrm{diam}(\Omega_{ij}) \int_{\mathbb{R}^2} \frac{1}{\pi^2\delta^4}\sqrt{z_x^2+z_y^2}\cdot e^{\frac{-z_x^2-z_y^2}{2\delta^2}} \mathrm{d}z \le \\
&\le \mathrm{diam}(\Omega_{ij}) \int_{\mathbb{R}^2} \frac{1}{\pi^2\delta^4}(|z_x|+|z_y|)\cdot e^{\frac{-z_x^2-z_y^2}{2\delta^2}} \mathrm{d}z \le \\
&\le \mathrm{diam}(\Omega_{ij})\Big(\int_{\mathbb{R}} \frac{|z_x|}{\pi^2\delta^3} e^{-\frac{z_x^2}{2\delta^2}}\mathrm{d}z_x + \int_{\mathbb{R}} \frac{|z_y|}{\pi^2\delta^3} e^{-\frac{z_y^2}{2\delta^2}}\mathrm{d}z_y\Big) \le \\
&\le \frac{4\mathrm{diam}(\Omega_{ij})}{\pi\delta}.
\end{aligned}$$

■

Lemma 6.16. *Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a bounded integrable function. Assume that for any element Ω^1_{ij} of a partition Ω^1 of the class (m,δ) we have $\int_{\Omega_{ij}} f \equiv 0$. Then for any partition Ω^2 of the class (m,δ)*

$$\|W_\delta f\|_{\Omega^2,\mathcal{L}_1} \le 8\frac{\sup \mathrm{diam}(\Omega^1_{ij})}{\delta}\|f\|_{\Omega^1,\mathcal{L}_1}; \qquad (6.16.1)$$

$$\|W_\delta f\|_\infty \le 8\frac{\sup \mathrm{diam}(\Omega^1_{ij})}{\delta}\|f\|_\infty. \qquad (6.16.2)$$

Proof. By straightforward calculation

$$\|W_\delta f\|_{\mathcal{L}_1} = \int_{\mathbb{R}^2}\Big|\int_{\mathbb{R}^2} w_\delta(z-t)f(t)\mathrm{d}t\Big|\mathrm{d}z = \int_{\mathbb{R}^2}\Big|\sum_{ij}\int_{z-\Omega_{ij}} w_\delta(t)f(z-t)\mathrm{d}t\Big|\mathrm{d}z.$$

We recall $\int_{z-\Omega_{ij}} f(z-t)\mathrm{d}t = \int_{\Omega_{ij}} f(t)\mathrm{d}t = 0$ and so $\int_{z-\Omega_{ij}} f(z-t)\int_{z-\Omega_{ij}} w_\delta(s)\mathrm{d}s\mathrm{d}t = 0$. Hence we conclude

$$\begin{aligned}
\|W_\delta f\|_{\Omega^2,\mathcal{L}_1} \le \frac{2^{-m}}{\inf|\pi_y(\Omega^2_{kl})|}\int_{\mathbb{R}^2}\Big|\sum_{ij}\int_{z-\Omega^1_{ij}}\Big(w_\delta(t)-\frac{1}{|\Omega^1_{ij}|}\int_{z-\Omega^1_{ij}} w_\delta(s)\mathrm{d}s\Big)f(z-t)\mathrm{d}t\Big|\mathrm{d}z \le \\
\le \frac{2^{-m}}{\inf|\pi_y(\Omega^2_{kl})|}\int_{\mathbb{R}^2}\sum_{ij}\int_{z-\Omega^1_{ij}}\Big|w_\delta(t)-\frac{1}{|\Omega^1_{ij}|}\int_{z-\Omega^1_{ij}} w_\delta(s)\mathrm{d}s\Big|\cdot\big|f(z-t)\big|\mathrm{d}t\mathrm{d}z \le \\
\le \frac{2^{-m}}{\inf|\pi_y(\Omega^2_{kl})|}\int_{\mathbb{R}^2}\sum_{ij}\int_{z-\Omega^1_{ij}}\big|\max_{s\in z-\Omega^1_{ij}} w_\delta(s)-\min_{s\in z-\Omega^1_{ij}} w_\delta(s)\big|\cdot|f(z-t)|\mathrm{d}t\mathrm{d}z \le \\
\le \frac{2^{-m}}{\inf|\pi_y(\Omega^2_{kl})|}\int_{\mathbb{R}^2}\sum_{ij}\big|\max_{s\in z-\Omega^1_{ij}} w_\delta(s)-\min_{s\in z-\Omega^1_{ij}} w_\delta(s)\big|\cdot\int_{z-\Omega^1_{ij}}|f(z-t)|\mathrm{d}t\mathrm{d}z = \\
= \frac{2^{-m}}{\inf|\pi_y(\Omega^2_{kl})|}\int_{\mathbb{R}^2}\sum_{ij}\big|\max_{s\in z-\Omega^1_{ij}} w_\delta(s)-\min_{s\in z-\Omega^1_{ij}} w_\delta(s)\big|\cdot\int_{\Omega^1_{ij}}|f(t)|\mathrm{d}t\mathrm{d}z = \\
\le \frac{2^{-m}}{\inf|\pi_y(\Omega^2_{kl})|}\sum_{ij}\int_{\mathbb{R}^2}\big|\max_{s\in z-\Omega^1_{ij}} w_\delta(s)-\min_{s\in z-\Omega^1_{ij}} w_\delta(s)\big|\cdot\int_{\Omega^1_{ij}}|f(t)|\mathrm{d}t\mathrm{d}z \le \\
\le \frac{2^{-m}}{\inf|\pi_y(\Omega^2_{kl})|}\sum_{ij}\frac{4\mathrm{diam}(\Omega^1_{ij})}{\pi\delta}\int_{\Omega^1_{ij}}|f(t)|\mathrm{d}t \le \\
\le \frac{\sup|\pi_y(\Omega^1_{ij})|}{\inf|\pi_y(\Omega^2_{kl})|}\cdot\frac{4\sup\mathrm{diam}(\Omega^1_{ij})}{\pi\delta}\|f\|_{\Omega^1,\mathcal{L}_1},
\end{aligned}$$

by Lemma 6.15.

Similarly for the supremum norm

$$\begin{aligned}
\sup|W_\delta f| = \sup_z\Big|\int_{\mathbb{R}^2} w_\delta(z-t)f(t)\mathrm{d}t\Big| \le \sup_z\Big|\sum_{ij}\int_{\Omega^1_{ij}} w_\delta(z-t)f(t)\mathrm{d}t\Big| = \\
= \sup_z\Big|\sum_{ij}\Big(\int_{\Omega^1_{ij}} w_\delta(z-t)-\frac{1}{|\Omega^1_{ij}|}\int_{\Omega^1_{ij}} w_\delta(z-s)\mathrm{d}s\Big)f(t)\mathrm{d}t\Big| \le \\
\le \sup_z\sum_{ij}\int_{\Omega^1_{ij}}\big|\max_{t\in\Omega^1_{ij}} w_\delta(z-t)-\min_{t\in\Omega^1_{ij}} w_\delta(z-t)\big|\cdot|f(t)|\mathrm{d}t \le \\
\le \sup|f|\sup_z\sum_{ij}|\Omega^1_{ij}|\cdot\big|\max_{t\in\Omega^1_{ij}} w_\delta(z-t)-\min_{t\in\Omega^1_{ij}} w_\delta(z-t)\big| \le \\
\le \sup|f|\sup_z\sum_{ij}|\Omega^1_{ij}|\cdot\sup_{t\in\Omega^1_{ij}}|\nabla_z w_\delta(z-t)|\cdot\mathrm{diam}(\Omega^1_{ij}) \le \\
\le \sup|f|\sup\mathrm{diam}(\Omega^1_{ij})\cdot\int_{\mathbb{R}^2}|\nabla_z w_\delta(z)|\mathrm{d}z \le \frac{\sup\mathrm{diam}(\Omega^1_{ij})}{\delta}\cdot\sup|f|.
\end{aligned}$$

■

Lemma 6.17. *Let Ω^1 and Ω^2 be two partitions of the class $\mathcal{G}(m,\delta)$. Then for any $\xi \in \ell_\infty(\mathbb{R}^2)$ we have*

$$\|(P^2_{\xi *}\nu)\|_2 \le 2^{2m+2}\|\nu\|_1,$$

Proof. Upper bound for the supremum norm is obvious. Indeed, we have for the first coordinate

$$\begin{aligned}\|(P^2_{\xi *}\nu)_y\|_{\mathcal{L}_1} &= \int_{\mathbb{R}^2}\Big|(P^2_{\xi *}\nu)_y(z)\Big|\mathrm{d}z = \\ &= \int_{\mathbb{R}^2}\Big|\partial_x(P^2_\xi)_y(P^{-2}_\xi z)\nu_s(P^{-2}_\xi z) + \partial_y(P^2_\xi)_y(P^{-2}_\xi z)\nu_u(P^{-2}_\xi z)\Big|\mathrm{d}z = \\ = \int_{\mathbb{R}^2}&\Big|\partial_x(P^2_\xi)_y(z)\nu_s(z) + \partial_y(P^2_\xi)_y(z)\nu_u(z)\Big|\mathrm{d}z \le 2^{2m+1}\int_{\mathbb{R}^2}|\nu_s(z)| + |\nu_u(z)|\mathrm{d}z. \quad (66)\end{aligned}$$

For the second coordinate we have got

$$\begin{aligned}\|(P^2_{\xi *}\nu)_x\|_{\mathcal{L}_1} &= \int_{\mathbb{R}^2}\Big|(P^2_{\xi *}\nu)_x(z)\Big|\mathrm{d}z = \\ &= \int_{\mathbb{R}^2}\Big|\partial_x(P^2_\xi)_x(P^{-2}_\xi z)\nu_s(P^{-2}_\xi z) + \partial_y(P^2_\xi)_x(P^{-2}_\xi z)\nu_u(P^{-2}_\xi z)\Big|\mathrm{d}z = \\ = \int_{\mathbb{R}^2}&\Big|\partial_x(P^2_\xi)_x(z)\nu_s(z) + \mathrm{d}_y(P^2_\xi)_x(z)\nu_u(z)\Big|\mathrm{d}z \le 2^{2m+1}\int_{\mathbb{R}^2}|\nu_s(z)| + |\nu_u(z)|\mathrm{d}z. \quad (67)\end{aligned}$$

Therefore

$$\|P^2_{\xi *}\nu\|_2 = \sum_{ij}\frac{2^{-m}}{|\pi_y(\Omega^2_{ij})|}\int_{\Omega_{ij}}\Big|P^2_{\xi *}\nu(z)\Big|\mathrm{d}z \le \frac{2^{-m}}{\inf|\pi_y(\Omega^2_{ij})|}\|P^2_{\xi *}\nu\|_{\mathcal{L}_1} \le m2^{2m+1}\|\nu\|_1.$$

■

Lemma 6.18. *In the notations introduced in the beginning of this subsection 6.2, p. 34, the following inequalities on the norm of operators hold true for M and m large enough.*

$$\|UU\nu_u\|_{\Omega^2,\mathcal{L}_1} \le 4\cdot 2^{2m}\|\nu\|_1, \quad (6.18.1)$$

$$\max(\|SU\nu_u\|_{\Omega^2,\mathcal{L}_1}, \|US\nu_s\|_{\Omega^2,\mathcal{L}_1}, \|SS\nu_s\|_{\Omega^2,\mathcal{L}_1}) \le 2^{\gamma_2 m}\|\nu\|_1; \quad (6.18.2)$$

where the constant γ_2 satisfies

$$1 < \gamma_2 = \frac{9}{4} + \gamma_1 + 2\log_2(1+\mu_1) - \alpha < \frac{3}{2}. \quad (68)$$

Proof. Let $\nu_u = \sum_{ij}\nu_u^{ij}\chi^u_{\Omega^1_{ij}} \in \Phi_{\Omega^1}$ be the y-component of a field with the unit norm

$$\|\nu_u\| = \max\Big(\sum_{ij}|\nu_u^{ij}|\cdot|\pi_y(\Omega_{ij})|, 2^{\frac{3}{4}m}\sup|\nu_u^{ij}|\Big) = 1,$$

therefore we will be assuming that $\sum_{ij} |\nu_u^{ij}| \le 2^{m-1}$ and $\sup |\nu_u^{ij}| \le 2^{-\frac{3}{4}m}$. We write down the formal action of the operator UU on ν_u

$$UU\nu_u = \sum_{kl}\sum_{ij} UU_{ij}^{kl}\nu_u^{ij}\chi^u_{\Omega^2_{kl}} = \sum_{\square}\sum_{\square}(UU_{ij}^{kl}-1)\nu_u^{ij}\chi^u_{\Omega^2_{kl}} + \sum_{\square}\sum_{\square}\nu_u^{ij}\chi^u_{\Omega^2_{kl}} + \\ + \Big(\sum_{\mathbb{R}^2\setminus\square}\sum_{\square} + \sum_{\square}\sum_{\mathbb{R}^2\setminus\square}\Big) UU_{ij}^{kl}\nu_u^{ij}\chi^u_{\Omega^2_{kl}} + \sum_{\mathbb{R}^2\setminus\square}\sum_{\mathbb{R}^2\setminus\square} UU_{ij}^{kl}\nu_u^{ij}\chi^u_{\Omega^2_{kl}}. \quad (69)$$

We estimate the norm of each of the four terms separately. Recall that by the choice of the basis $\chi^u_{\Omega^2_{kl}} = \frac{1}{|\pi_x(\Omega^2_{kl})|}\chi_{\Omega^2_{kl}}\binom{0}{1}$ and therefore

$$\|\chi^u_{\Omega^2_{kl}}\|_{\Omega^2,\mathcal{L}_1} = \frac{2^{-m}}{|\pi_y(\Omega^2_{kl})|}\int_{\Omega^2_{kl}} \chi^u_{\Omega^2_{kl}} = 2^{-m}.$$

$$\Big\|\sum_{\square}\sum_{\square}(UU_{ij}^{kl}-1)\nu_u^{ij}\chi^u_{\Omega^2_{kl}}\Big\|_{\Omega^2,\mathcal{L}_1} = \sum_{\square}\Big|\sum_{\square}(UU_{ij}^{kl}-1)\nu_u^{ij}\Big|\cdot 2^{-m} \le \\ \le 2^{-m}\sum_{\square}\sum_{\square}|UU_{ij}^{kl}-1|\cdot|\nu_u^{ij}| \le 2^{-m}\sup|\nu_u^{ij}|\sum_{\square}\sum_{\square}|UU_{ij}^{kl}-1| \le \\ \le 2^{1-\frac{7}{4}m}\cdot\sum_{\square}\sum_{\square}|(UU^G)_{ij}^{kl} + (UU^B)_{ij}^{kl} - 1| \le \\ \le 2^{1-\frac{7}{4}m}\cdot\sum_{\square}\sum_{\square}|(UU^G)_{ij}^{kl} - 1| + |(UU^B)_{ij}^{kl}| \le \\ \le 2^{-\frac{7}{4}m}(2^{2m}\delta + 2^{\frac{9}{2}m}\delta) \le 2^{2\frac{3}{4}m}\delta, \quad (70)$$

using Lemma 6.13 and the second part of Proposition 6.1.

The second part of (69) has the following upper bound, since $\sum_{ij}|\nu_u^{ij}| \le 2^m$,

$$\Big\|\sum_{\square}\sum_{\square}\nu_u^{ij}\chi^u_{\Omega^2_{kl}}\Big\|_{\Omega^2,\mathcal{L}_1} = 2^{-m}\sum_{\square}\Big|\sum_{\square}\nu_u^{ij}\Big| \le 2^{2m}\cdot 2^m\cdot 2^{1-m} \le 2^{2m+1}.$$

The last sum has only finite number of non-zero terms and can be estimated via the supremum norm. Recall Remark 5: for $R = M_2(1+\mu_1)^{2m}\cdot m\delta+1$, any quartet (i,j,k,l) such that $(i,j)\in\square$ and $(k,l)\in\mathbb{R}^2\setminus\square_R$ or $(i,j)\in\mathbb{R}^2\setminus\square_R$ and $(k,l)\in\square$

$$SS_{ij}^{kl}\equiv 0, \quad SU_{ij}^{kl}\equiv 0, \quad US_{ij}^{kl}\equiv 0, \quad UU_{ij}^{kl}\equiv 0.$$

Therefore

$$\Big\|\Big(\sum_{\mathbb{R}^2\setminus\square}\sum_{\square}+\sum_{\square}\sum_{\mathbb{R}^2\setminus\square}\Big)UU_{ij}^{kl}\nu_u^{ij}\chi^u_{\Omega^2_{kl}}\Big\|_{\Omega^2,\mathcal{L}_1}=$$

$$=\Big\|\Big(\sum_{\square_R\setminus\square}\sum_{\square}+\sum_{\square}\sum_{\square_R\setminus\square}\Big)UU_{ij}^{kl}\nu_u^{ij}\chi^u_{\Omega^2_{kl}}\Big\|_{\Omega^2,\mathcal{L}_1}\leq$$

$$\leq\Big(\sum_{\square_R\setminus\square}\sum_{\square}+\sum_{\square}\sum_{\square_R\setminus\square}\Big)\sup|UU_{ij}^{kl}|\cdot\sup|\nu_u^{ij}|2^{-m}\leq$$

$$\leq 4(R^2-1)m^4 2^{4m}\cdot 2^{\gamma_1 m}\cdot 2^{-\frac{3}{4}m}\cdot 2^{1-m}\leq$$

$$\leq M_2 m^5\delta 2^{(\gamma_1+\frac{9}{4})m}(1+\mu_1)^{2m}. \quad (71)$$

We have for the last term, using the bound (41) (p. 16)

$$\Big\|\sum_{\mathbb{R}^2\setminus\square}\sum_{\mathbb{R}^2\setminus\square}UU_{ij}^{kl}\nu_u^{ij}\chi^u_{\Omega^2_{kl}}\Big\|_{\Omega^2,\mathcal{L}_1}\leq\sum_{\mathbb{R}^2\setminus\square}|\nu_u^{ij}|\cdot 2^{-m}\cdot\sup|UU_{ij}^{kl}|\cdot M_1(1+\mu_1)^{2m}\leq$$

$$\leq 2^{m-1}\cdot 2^{-m}\cdot 2^{\gamma_1 m}\cdot M_1(1+\mu_1)^{2m}=M_1\cdot 2^{\gamma_1 m}\cdot(1+\mu_1)^{2m}. \quad (72)$$

Summing up the last four together, we get an upper bound $\|UU\nu_u\|_{\Omega^2,\mathcal{L}_1}\leq 2^{2+2m}$.

Now we proceed to the last inequality (6.18.2). We would like to show that there exists a constant γ_2 satisfying (68) such that for M and m large enough:

$$\max\big(\|SU\nu_u\|_{\Omega^2,\mathcal{L}_1},\|US\nu_s\|_{\Omega^2,\mathcal{L}_1},\|SS\nu_s\|_{\Omega^2,\mathcal{L}_1}\big)\leq 2^{\gamma_2 m}\|\nu\|_{\Omega^1,\mathcal{L}_1}.$$

We shall show that it holds true for the matrix SU, the argument for the matrix US is similar.

As before, we may assume for the first component of the vector field $\nu_s\in\Phi_{\Omega^1}$ that[1]

$$\max\Big(\sum_{ij}\nu_s^{ij}\cdot|\pi_y(\Omega_{ij})|, 2^{\frac{3}{4}m}\sup|\nu_s^{ij}|\Big)=1,$$

and, consequently, $\sum_{ij}|\nu_s^{ij}|\leq 2^{m-1}$ and $\sup|\nu_s^{ij}|\leq 2^{-\frac{3}{4}m}$. We recall the definition of "good" and "bad" connected components (42) and (43) :

$$(\Delta^G)_{ij}^{kl}\overset{\text{def}}{=}\{\Delta\subset P_\xi^{-2}(\Omega^2_{kl})\cap\Omega^1_{ij}\mid\Delta\text{ is a }(P,\xi)\text{-domain},\forall 1\leq n\leq 2m: F_\xi^n(\Delta)\subset\square\};$$

$$(\Delta^B)_{ij}^{kl}\overset{\text{def}}{=}\{\Delta\subset P_\xi^{-2}(\Omega^2_{kl})\cap\Omega^1_{ij}\mid\Delta\text{ is a }(P,\xi)\text{-domain},\exists 1\leq n\leq 2m: F_\xi^n(\Delta)\not\subset\square\}.$$

We may write, similarly to (44)

$$(SU)_{ij}^{kl}=(SU^G)_{ij}^{kl}+(SU^B)_{ij}^{kl},$$

[1] We denote the space of essentially bounded, absolutely integrable, piece-wise constant functions, associated to the partition Ω^1 of $\mathbb{R}$ by Φ_{Ω^1}.

where

$$(SU^G)^{kl}_{ij}: \ = \frac{1}{|\pi_x(\Omega^1_{ij})|}\cdot\frac{1}{|\pi_y(\Omega^2_{kl})|}\cdot\sum_{\Delta\in\Delta^G}\int_\Delta \partial_y(P^2_\xi)_y(z)\mathrm{d}z;$$
$$(SU^B)^{kl}_{ij}: \ = \frac{1}{|\pi_x(\Omega^1_{ij})|}\cdot\frac{1}{|\pi_y(\Omega^2_{kl})|}\cdot\sum_{\Delta\in\Delta^B}\int_\Delta \partial_y(P^2_\xi)_y(z)\mathrm{d}z.$$

Obviously, $(SU^G)^{kl}_{ij}\equiv 0$. We also recall $B=\{z\in\square \mid \exists 1\le n\le 2m\colon F^n_\xi(z)\notin\square\}$ and observe that

$$\sum_\square\sum_\square |(SU^B)^{kl}_{ij}| \le \int_B |\partial_x(P^2_\xi)_y(z)|\mathrm{d}z = 2^{2m}\cdot 8m\delta.$$

We may write the action of SU on ν_s

$$SU\nu_s = \sum_{kl}\sum_{ij} SU^{kl}_{ij}\nu^{ij}_s\chi^u_{\Omega^2_{kl}} = \sum_\square\sum_\square (SU^B)^{kl}_{ij}\nu^{ij}_s\chi^u_{\Omega^2_{kl}} +$$
$$+\sum_{\mathbb{R}^2\setminus\square}\sum_\square SU^{kl}_{ij}\nu^{ij}_s\chi^u_{\Omega^2_{kl}} + \sum_\square\sum_{\mathbb{R}^2\setminus\square} SU^{kl}_{ij}\nu^{ij}_s\chi^u_{\Omega^2_{kl}} + \sum_{\mathbb{R}^2\setminus\square}\sum_{\mathbb{R}^2\setminus\square} SU^{kl}_{ij}\nu^{ij}_s\chi^u_{\Omega^2_{kl}}.$$

We have the following upper bound for the first term, corresponding to the central part of the matrix

$$\Big\|\sum_\square\sum_\square (SU^B)^{kl}_{ij}\nu^{ij}_s\chi^u_{\Omega^2_{kl}}\Big\|_{\Omega^2,\mathcal{L}_1} = \sum_\square\sum_\square |(SU^B)^{kl}_{ij}|\cdot|\nu^{ij}_s|\cdot 2^{-m}\le$$
$$\le \sup|\nu^{ij}_s|\cdot 2^{-m}\cdot\sum_\square\sum_\square|(SU^B)^{kl}_{ij}| \le 2^{2m}m\delta\cdot 2^{-\frac{3}{4}m}\cdot 2^{-m}\le 2^{\frac{m}{4}}m\delta.$$

Repeating the estimates (71) and (72) above, since $\|SU\|_\infty\le\|UU\|_\infty\le 2^{\gamma_1 m}$ and using the upper bounds $\|\nu_s\|_\infty\le 2^{-\frac{3}{4}m}$ we obtain

$$\Big\|\Big(\sum_{\mathbb{R}^2\setminus\square}\sum_\square + \sum_\square\sum_{\mathbb{R}^2\setminus\square} + \sum_{\mathbb{R}^2\setminus\square}\sum_{\mathbb{R}^2\setminus\square}\Big) SU^{kl}_{ij}\nu^{ij}_s\chi^u_{\Omega^2_{kl}}\Big\|_{\Omega^2,\mathcal{L}_1}\le$$
$$\le \sup\big|SU^{kl}_{ij}\big|\cdot\sup|\nu^{ij}_s|\cdot(1+\mu_1)^{2m}(M_1+M_2m^5\delta\cdot 2^{\frac{5}{2}m})\le$$
$$\le 2^{\gamma_1 m}\cdot 2^{\frac{5}{2}m-\frac{3}{4}m}(1+\mu_1)^{2m}\cdot M_2m^5\delta \le 2^{\gamma_1 m}\cdot 2^{\frac{7}{4}m}(1+\mu_1)^{2m}\cdot M_2m^5\delta.$$

Summing up altogether, we get

$$\|SU\nu_s\|_{\Omega^2,\mathcal{L}_1}\le 2^{\gamma_1 m}\cdot 2^{\frac{9}{4}m}(1+\mu_1)^{2m}\cdot M_2m^5\delta + 2^{\frac{m}{4}}m\delta \le 2^{\gamma_2 m}.$$

Similarly, $\|US\nu_y\|\le 2^{\gamma_2 m}$. It only remains to show that for $\gamma_2=\gamma_1+\frac{9}{4}+2\log_2(1+\mu_1)-\alpha$ and for M and m sufficiently large

$$\|SS\nu_s\|_{\Omega^2,\mathcal{L}_1}\le 2^{\gamma_2 m}. \tag{73}$$

Recall Corollary 1 of Lemma 6.10:

$$\sum_{\square}\sum_{\square}\left|(SS^G)_{ij}^{kl}\right| \le 2^{4-2m}.$$

We can get an upper bound for the central part

$$\Big\|\sum_{\square}\sum_{\square}(SS^G)_{ij}^{kl}\nu_s^{ij}\chi^u_{\Omega^2_{kl}}\Big\|_{\Omega^2,\mathcal{L}_1} = \sum_{\square}\sum_{\square}\left|(SS^G)_{ij}^{kl}\right|\cdot|\nu_s^{ij}|\cdot 2^{-m} \le$$
$$\le \sup|\nu_s^{ij}|\cdot 2^{-m}\cdot\sum_{\square}\sum_{\square}\left|(SS^G)_{ij}^{kl}\right| \le 2^{4-2m}\cdot 2^{-\frac{3}{4}m}\cdot 2^{-m} < 4\cdot 2^{-3m/2}.$$

Repeating the estimates (71) and (72) for the matrix SS and taking into account an upper bound $\|SS\|_\infty \le 2^{\gamma_1 m}$ from Proposition 6.2, we get

$$\Big\|\Big(\sum_{\mathbb{R}^2\setminus\square}\sum_{\square}+\sum_{\square}\sum_{\mathbb{R}^2\setminus\square}+\sum_{\mathbb{R}^2\setminus\square}\sum_{\mathbb{R}^2\setminus\square}\Big)SS_{ij}^{kl}\nu_s^{ij}\chi^u_{\Omega^2_{kl}}\Big\|_{\Omega^2,\mathcal{L}_1} \le$$
$$\le \sup\left|SU_{ij}^{kl}\right|\cdot(1+\mu_1)^{2m}(M_1+M_2m^5 2^{\frac{5}{2}m}\delta) \le 2^{(\gamma_1+\frac{5}{2})m}\cdot(1+\mu_1)^{2m}\cdot M_2m^5\delta.$$

Thus

$$\|SS\nu_s\|_{\Omega^2,\mathcal{L}_1} = \Big\|\sum_{kl}\sum_{ij}SS_{ij}^{kl}\nu_s^{ij}\chi^s_{\Omega^2_{kl}}\Big\|_{\Omega^2,\mathcal{L}_1} \le$$
$$\le 2^{(\gamma_1+\frac{5}{2})m}\cdot(1+\mu_1)^{2m}\cdot M_2m^5\delta + 2^{3-3m/2} \le 2^{\gamma_2 m}.$$

■

Corollary 1. *Under the hypothesis and in the notations of Lemma 6.18, the norm of the operator* $\|\mathcal{A}\|_{\Omega^2} \le 2^{2m+2}$. *Namely,* $\|\mathcal{A}\nu\|_2 \le 2^{2m+2}\|\nu\|_1$.

Proof. Recall the definition (34) of the operator $\mathcal{A}\colon \mathfrak{X}_{\Omega^1} \to \mathfrak{X}_{\Omega^2}$

$$\mathcal{A}\nu = \sum_{ij}\mathcal{A}\big(\nu_s^{ij}\chi^s_{\Omega^1_{ij}} + \nu_u^{ij}\chi^u_{\Omega^1_{ij}}\big) =$$
$$= \sum_{ij}\sum_{kl}\Big(\nu_s^{ij}\big(SS_{ij}^{kl}\chi^s_{\Omega^2_{kl}} + SU_{ij}^{kl}\chi^u_{\Omega^2_{kl}}\big) + \nu_u^{ij}\big(US_{ij}^{kl}\chi^s_{\Omega^2_{kl}} + UU_{ij}^{kl}\chi^u_{\Omega^2_{kl}}\big)\Big),$$

The upper bound for $\mathcal{L}_1$-norm follows from the parts 6.18.1 and 6.18.2 of Lemma 6.18.

Now we proceed to the supremum norm.

$$\sup_z |\mathcal{A}\nu(z)| =$$
$$= \sup_z \Big| \sum_{ij} \sum_{kl} \Big(\nu_s^{ij} \big(SS_{ij}^{kl} \chi^s_{\Omega^2_{kl}}(z) + SU_{ij}^{kl} \chi^u_{\Omega^2_{kl}}(z) \big) + \nu_u^{ij} \big(US_{ij}^{kl} \chi^s_{\Omega^2_{kl}}(z) + UU_{ij}^{kl} \chi^u_{\Omega^2_{kl}}(z) \big) \Big) \Big| \le$$
$$\le \frac{1}{\inf |\pi_x(\Omega^2_{kl})|} \sup_{kl} \Big| \sum_{ij} \Big(\nu_s^{ij} \big(SS_{ij}^{kl} + SU_{ij}^{kl} \big) + \nu_u^{ij} \big(US_{ij}^{kl} + UU_{ij}^{kl} \big) \Big) \Big| \le$$
$$\le \frac{1}{\inf |\pi_x(\Omega^2_{kl})|} \cdot (\|SS\|_\infty + \|SU\|_\infty + \|US\|_\infty + \|UU\|_\infty) \cdot \Big(\sum_{ij} (|\nu_s^{ij}| + |\nu_u^{ij}|) \Big) \le$$
$$\le 2^m \cdot 4 \cdot 2^{\gamma_1 m} \cdot 2^m \le 2^{2+(2+\gamma_1)m}.$$

The Corollary follows from the definition of the norm on p. 8. ■

The result we were seeking follows immediately

Proposition 6.3. The operators $W_\delta \mathcal{A}$ and $W_\delta P^2_{\xi*}$ are close. Namely,

$$\|W_\delta(P^2_{\xi*} - \mathcal{A})\nu\|_{\Omega^2, \mathcal{L}_1} \le \frac{4 \sup \operatorname{diam}(\Omega^2_{kl})}{\delta} \cdot \frac{\sup |\pi_y(\Omega^1_{ij})|}{\inf |\pi_y(\Omega^2_{kl})|} \cdot 2^{2m} \|\nu\|_1; \tag{74}$$
$$\|W_\delta(P^2_{\xi*} - \mathcal{A})\nu\|_\infty \le \frac{4 \sup \operatorname{diam}(\Omega^2_{kl})}{\delta} \cdot 2^{(2+\gamma_1)m} \|\nu\|_1. \tag{75}$$

Proof. Follows from Lemma 6.14, Lemma 6.16, the first and second parts of Lemma 6.18, and Corollary 1 of Lemma 6.18. ■

Corollary 2.

$$\|W_\delta(P^2_{\xi*} - \mathcal{A})\nu\|_2 \le \frac{8 \sup \operatorname{diam}(\Omega^2_{kl})}{\delta} \cdot 2^{2m} \|\nu\|_1$$

6.3. **A pair of cones for the operator $\mathcal{A}$.** In this Subsection we construct two cones $C_1 \subset \mathfrak{X}_{\Omega^1}$ and $C_2 \subset \mathfrak{X}_{\Omega^2}$ such that $\mathcal{A}(\overline{C_1}) \subset C_2$, $C_2 \ll C_1$, and $\|\mathcal{A}|_{C_1}\| \ge 2^{m-1}$. This is the main result of Section 6, which is presented in Preliminary Dynamo Theorem 3 below.

Lemma 6.19. *The operator UU is a small perturbation of the operator $\mathring{UU}$. Namely*

$$\|(UU - \mathring{UU})\nu\|_2 \le 2^{(\gamma_1 + 2\frac{3}{4})m} \delta \|\nu\|_1.$$

Proof. We begin with $(\Omega^2, \mathcal{L}_1)$-norm. Consider a vector field $\nu \in \mathfrak{X}_{\Omega_1}$ with $\|\nu\|_1 = 1$. We may assume that $\sum\limits_{ij} |\nu_u^{ij}| \leq 2^m$ and $\sup |\nu_u^{ij}| \leq 2^{-\frac{3}{4}m}$. Then

$$
\begin{aligned}
\|(UU - \mathring{UU})\nu\|_{\Omega^2,\mathcal{L}_1} &= \Big\| \sum_{kl} \sum_{ij} (UU_{ij}^{kl} - \mathring{UU}_{ij}^{kl}) \nu_u^{ij} \chi^u_{\Omega^2_{kl}} \Big\|_{\Omega^2,\mathcal{L}_1} = \\
&= \sum_{kl} \Big| \sum_{ij} (UU_{ij}^{kl} - \mathring{UU}_{ij}^{kl}) \nu_u^{ij} \Big| \cdot 2^{-m} \leq \sum_{kl} \sum_{ij} \big|UU_{ij}^{kl} - \mathring{UU}_{ij}^{kl}\big| \cdot \big|\nu_u^{ij}\big| \cdot 2^{-m} \leq \\
&\leq \sum_{\square} \sum_{\square} \big|UU_{ij}^{kl} - \mathring{UU}_{ij}^{kl}\big| \cdot \big|\nu_u^{ij}\big| \cdot 2^{-m} + \\
&+ \Big(\sum_{\mathbb{R}^2\setminus\square} \sum_{\square} + \sum_{\square} \sum_{\mathbb{R}^2\setminus\square} + \sum_{\mathbb{R}^2\setminus\square} \sum_{\mathbb{R}^2\setminus\square} \Big) \big|UU_{ij}^{kl} - \mathring{UU}_{ij}^{kl}\big| \cdot \big|\nu_u^{ij}\big| \cdot 2^{-m}
\end{aligned}
$$

We have for the first term

$$
\begin{aligned}
\sum_{\square} \sum_{\square} \big|UU_{ij}^{kl} - \mathring{UU}_{ij}^{kl}\big| \cdot \big|\nu_u^{ij}\big| \cdot 2^{-m} &= \sum_{\square} \sum_{\square} \big|UU_{ij}^{kl} - 1\big| \cdot \big|\nu_u^{ij}\big| \cdot 2^{-m} \leq \\
&\leq \|UU\|_\infty \cdot \#\{(i,j,k,l) \in \square \times \square \mid UU_{ij}^{kl} \neq 1\} \cdot \sup |\nu_u^{ij}| \cdot 2^{-m} \leq \\
&\leq 2^{\gamma_1 m} \cdot 2^{4\frac{1}{2}m} \delta \cdot 2^{-\frac{3}{4}m} \cdot 2^{-m} \leq 2^{(2\frac{3}{4} + \gamma_1 - \alpha)m}.
\end{aligned}
$$

Recall Remark 5: for $R = M_2(1+\mu_1)^{2m} \cdot m\delta + 1$, any quartet (i,j,k,l) such that $(i,j) \in \square$ and $(k,l) \in \mathbb{R}^2 \setminus \square_R$ or $(i,j) \in \mathbb{R}^2 \setminus \square_R$ and $(k,l) \in \square$

$$
SS_{ij}^{kl} \equiv 0, \quad SU_{ij}^{kl} \equiv 0, \quad US_{ij}^{kl} \equiv 0, \quad UU_{ij}^{kl} \equiv 0.
$$

Since $\mathring{UU}_{ij}^{kl} \equiv 0$ for all $(i,j,k,l) \in \square \times (\mathbb{R}^2 \setminus \square) \cup (\mathbb{R}^2 \setminus \square) \times \square$ we may write for the second term

$$
\begin{aligned}
\Big(\sum_{\mathbb{R}^2\setminus\square} \sum_{\square} + \sum_{\square} \sum_{\mathbb{R}^2\setminus\square} \Big) &\big|UU_{ij}^{kl} - \mathring{UU}_{ij}^{kl}\big| \cdot \big|\nu_u^{ij}\big| \cdot |\pi_y(\Omega^2_{kl})| = \\
&= \Big(\sum_{\mathbb{R}^2\setminus\square} \sum_{\square} + \sum_{\square} \sum_{\mathbb{R}^2\setminus\square} \Big) \big|UU_{ij}^{kl}\big| \cdot \big|\nu_u^{ij}\big| \cdot |\pi_y(\Omega^2_{kl})| = \\
&= \Big(\sum_{\square_R\setminus\square} \sum_{\square} + \sum_{\square} \sum_{\square_R\setminus\square} \Big) \big|UU_{ij}^{kl}\big| \cdot \big|\nu_u^{ij}\big| \cdot |\pi_y(\Omega^2_{kl})| \leq \\
&\leq 2^{1-m} \# \big\{ (i,j,k,l) \in \square \times (\square_R \setminus \square) \cup (\square_R \setminus \square) \times \square \big\} \cdot \|UU\|_\infty \cdot \sup |\nu_u^{ij}| \leq \\
&\leq 2^{1-m} \cdot 2^{4m} (1+\mu_1)^{2m} \cdot M_2 m^5 \delta \cdot 2^{\gamma_1 m} \cdot 2^{-\frac{3}{4}m} \leq m^2 2^{(\frac{9}{4} + \gamma_1 - \alpha)m},
\end{aligned}
$$

where $\gamma_2 = \frac{5}{2} + \gamma_1 - \alpha + 2\log(1+\mu_1)$. Finally, for the last term we calculate

$$\sum_{\mathbb{R}^2\setminus\square}\sum_{\mathbb{R}^2\setminus\square}\left|UU_{ij}^{kl} - \overset{\circ}{U}U_{ij}^{kl}\right| \cdot \left|\nu_u^{ij}\right| \cdot 2^{-m} \le 2^{-m} \cdot \sum_{\mathbb{R}^2\setminus\square}\sum_{\mathbb{R}^2\setminus\square}\left(\left|UU_{ij}^{kl}\right| + \left|\overset{\circ}{U}U_{ij}^{kl}\right|\right) \cdot \left|\nu_u^{ij}\right| \le$$
$$\le 2^{-m} \cdot 2M_1(1+\mu_1)^{2m}\|UU\|_\infty \sum_{\mathbb{R}^2\setminus\square}|\nu_u^{ij}| \le (1+\mu_1)^{2m} \cdot 2^{\gamma_1 m}.$$

Summing up,

$$\|(UU - \overset{\circ}{U}U)\nu\|_{\Omega^2,\mathcal{L}_1} \le m^2 2^{(2\frac{3}{4}+\gamma_1)}\delta\|\nu\|_1.$$

The upper bound for the supremum norm is easy:

$$\|(UU - \overset{\circ}{U}U)\nu\|_\infty = \sup_z\left\|\sum_{kl}\sum_{ij}(UU_{ij}^{kl} - \overset{\circ}{U}U_{ij}^{kl})\nu_u^{ij}\chi^u_{\Omega^2_{kl}}(z)\right\| \le$$
$$\le \frac{1}{\inf|\pi_x(\Omega^2_{kl})|} \cdot \sup_{kl}\sum_{ij}\left|UU_{ij}^{kl} - \overset{\circ}{U}U_{ij}^{kl}\right| \cdot \left|\nu_u^{ij}\right| \le$$
$$\le \frac{2\|UU\|_\infty}{\inf|\pi_x(\Omega^2_{kl})|} \cdot \sum_{ij}|\nu_u^{ij}| \le 2^{(\gamma_1+2)m+1}.$$

Then

$$\max(\|(UU - \overset{\circ}{U}U)\nu\|_{\Omega^2,\mathcal{L}_1}, 2^{-\frac{3}{4}m}\|(UU - \overset{\circ}{U}U)\nu\|_\infty) \le 2^{(2\frac{3}{4}+\gamma_1)}\delta\|\nu\|_1.$$

■

Let Υ be a chain of partitions associated to the sequence $\eta \in \Sigma_\delta$. Let $\Omega^1 = \Upsilon^k$, $\Omega^2 = \Upsilon^{k+1}$, and $\Omega^3 = \Upsilon^{k+2}$ be three consecutive partitions from the chain Υ. Consider the sequence $\xi \overset{\text{def}}{=} \sigma^{2m(k-1)}\eta$ (See definition of the chain Υ in subsection 4.3, p. 8). Let $\mathcal{A}\colon \mathfrak{X}_{\Omega_1} \to \mathfrak{X}_{\Omega_2}$ be a linear operator, approximating the operator $P^2_{\xi*}$, defined according to (34). Consider $\text{Cone}\,(1,\Omega^1) \subset \mathfrak{X}_{\Omega^1}$ and $\text{Cone}\left(2^{(\frac{3}{4}+\gamma_1-\alpha)m},\Omega^2\right) \subset \mathfrak{X}_{\Omega^2}$; defined according to the general definition from p. 8.

$$\text{Cone}\,(1,\Omega^1) \overset{\text{def}}{=} \left\{\nu = \left(\begin{smallmatrix}0\\d\end{smallmatrix}\right)\chi_\square + \psi,\ \psi\in\Omega^1,\ \|\psi\|_1 \le d,\ \sum_\square \psi_u^{ij} = 0\right\}; \tag{76}$$

$$\text{Cone}\left(2^{(\frac{3}{4}+\gamma_1-\alpha)m},\Omega^2\right) \overset{\text{def}}{=} \left\{\nu = \left(\begin{smallmatrix}0\\d\end{smallmatrix}\right)\chi_\square + \psi,\ \psi\in\Omega^2,\ \|\psi\|_2 \le d2^{(\gamma_1+\frac{3}{4}-\alpha)m},\ \sum_\square \psi_u^{ij} = 0\right\}. \tag{77}$$

Theorem 3 (Preliminary Dynamo Theorem)**.** *In the notations introduced above for arbitrary partition Ω^3 of the class $\mathcal{G}(m,\delta)$,*

$$\mathcal{A}\colon \text{Cone}\,(1,\Omega^1) \to \text{Cone}\left(2^{(\frac{3}{4}+\gamma_1-\alpha)m},\Omega^2\right)$$

Proof. Consider a piecewise constant vector field $\nu \in \mathrm{Cone}\,(1, \Omega^1)$. By definition of the Cone $(1, \Omega^1)$, we may write $\nu = \binom{0}{d}\chi_\square + \psi$, where $\|\psi\| \leq d$ and $\sum_\square \psi_u^{ij} = 0$. We deduce $\|\nu_s\| = \|\psi_s\| \leq d$ and $\|\psi_u\| \leq d$. Moreover, since

$$\int_\square \mathring{U}U\psi_u = \int_\square \sum_\square \sum_{ij} \mathring{U}U_{ij}^{kl}\psi_u^{ij}\chi_{\Omega_{kl}^2}^u = \sum_\square \int_\square \sum_{kl} \psi_u^{ij}\chi_{\Omega_{kl}^2}^u =$$
$$= \sum_\square \psi_u^{ij} \int_\square \sum_\square \frac{1}{|\pi_x(\Omega_{kl})|}\chi_{\Omega_{kl}^2} = \sum_\square \psi_u^{ij} \sum_\square |\pi_y(\Omega_{kl}^2)| = 2^{m+1}\sum_\square \psi_u^{ij}. \quad (78)$$

We conclude that the condition $\int_\square \mathring{U}U\psi_u = 0$ is equivalent to

$$\sum_\square \psi_u^{ij} = 0 \quad (79)$$

By definition of $\mathcal{A}$ we write

$$\mathcal{A}\nu = \sum_{ij} \mathcal{A}\big(\nu_s^{ij}\chi_{\Omega_{ij}^1}^s + \nu_u^{ij}\chi_{\Omega_{ij}^1}^u\big) =$$
$$= \sum_{ij}\sum_{kl}\Big(\nu_s^{ij}\big(SS_{ij}^{kl}\chi_{\Omega_{kl}^2}^s + SU_{ij}^{kl}\chi_{\Omega_{kl}^2}^u\big) + \nu_u^{ij}\big(US_{ij}^{kl}\chi_{\Omega_{kl}^2}^s + UU_{ij}^{kl}\chi_{\Omega_{kl}^2}^u\big)\Big) =$$
$$= \sum_{ij}\sum_{kl}\Big(\nu_s^{ij}\big(SS_{ij}^{kl}\chi_{\Omega_{kl}^2}^s + SU_{ij}^{kl}\chi_{\Omega_{kl}^2}^u\big) + \nu_u^{ij}US_{ij}^{kl}\chi_{\Omega_{kl}^2}^s\Big) +$$
$$+ \sum_{kl}\sum_{ij}(UU_{ij}^{kl} - \mathring{U}U_{ij}^{kl})\nu_u^{ij}\chi_{\Omega_{kl}^2}^u + \sum_{kl}\sum_{ij}\mathring{U}U_{ij}^{kl}\nu_u^{ij}\chi_{\Omega_{kl}^2}^u. \quad (80)$$

By Lemma 6.19 we know

$$\Big\|\sum_{kl}\sum_{ij}(UU_{ij}^{kl} - \mathring{U}U_{ij}^{kl})\nu_u^{ij}\chi_{\Omega_{kl}^2}^u\Big\|_2 \leq 2^{2\frac{3}{4}+\gamma_1-\alpha}d. \quad (81)$$

Using the third equality of Lemma 6.18, we get (recall $\gamma_2 = \gamma_1 + 2\frac{1}{4} + 2\log_2(1+\mu_1) - \alpha$)

$$\Big\|\sum_{ij}\sum_{kl}\Big(\nu_s^{ij}\big(SS_{ij}^{kl}\chi_{\Omega_{kl}^2}^s + SU_{ij}^{kl}\chi_{\Omega_{kl}^2}^u\big) + \nu_u^{ij}US_{ij}^{kl}\chi_{\Omega_{kl}^2}^s\Big)\Big\|_{\Omega^2,\mathcal{L}_1} \leq 3\cdot 2^{\gamma_2 m}d.$$

The supremum norm estimate is similar to the supremum norm of $\mathcal{A}$

$$\sup_z \Big| \sum_{ij} \sum_{kl} \Big(\nu_s^{ij} \big(SS_{ij}^{kl} \chi^s_{\Omega^2_{kl}}(z) + SU_{ij}^{kl} \chi^u_{\Omega^2_{kl}}(z) \big) + \nu_u^{ij} US_{ij}^{kl} \chi^s_{\Omega^2_{kl}}(z) \Big) \Big| \leq$$
$$\leq \frac{1}{\inf |\pi_x(\Omega^2_{kl})|} \sup_{kl} \Big| \sum_{ij} \Big(\nu_s^{ij} \big(SS_{ij}^{kl} + SU_{ij}^{kl} \big) + \nu_u^{ij} US_{ij}^{kl} \Big) \Big| \leq$$
$$\leq \frac{1}{\inf |\pi_x(\Omega^2_{kl})|} \cdot \left(\|SS\|_\infty + \|SU\|_\infty + \|US\|_\infty \right) \cdot \Big(\sum_{ij} (|\nu_s^{ij}| + |\nu_u^{ij}|) \Big) \leq$$
$$\leq 2^m \cdot 4 \cdot 2^{\gamma_1 m} \cdot 2^m d \leq 2^{2+(2+\gamma_1)m} d.$$

Thus

$$\Big\| \sum_{ij} \sum_{kl} \Big(\nu_s^{ij} \big(SS_{ij}^{kl} \chi^s_{\Omega^2_{kl}} + SU_{ij}^{kl} \chi^u_{\Omega^2_{kl}} \big) + \nu_u^{ij} US_{ij}^{kl} \chi^s_{\Omega^2_{kl}} \Big) \Big\|_2 \leq$$
$$\leq \max(2^{2+(\frac{3}{2}+\gamma_1)m}, 3 \cdot 2^{\gamma_2 m}) d = 3 \cdot 2^{\gamma_2 m} d. \quad (82)$$

We expand $\nu_u = d\chi_\square + \psi_u$ and observe, using Lemma 6.1 and equality (39)

$$\mathring{U}U \chi_\square = \sum_\square \sum_\square \mathring{U}U_{ij}^{kl} \chi_{\Omega^2_{kl}} = 2^{2m} \chi_\square. \quad (83)$$

By definition of the $(\Omega^2, \mathcal{L}_1)$-norm,

$$\left\| \chi^u_{\Omega^2_{kl}} \right\|_{\Omega^2, \mathcal{L}_1} = \frac{2^{-m}}{|\pi_y(\Omega_{kl})|} \int_{\Omega^2_{kl}} \frac{\chi_{\Omega^2_{kl}}(z)}{|\pi_x(\Omega^2_{kl})|} \mathrm{d}z = 2^{-m}.$$

Using (79), we calculate the norm

$$\Big\| \sum_{kl} \sum_{ij} \mathring{U}U_{ij}^{kl} \psi_u^{ij} \chi^u_{\Omega^2_{kl}} \Big\|_{\Omega^2, \mathcal{L}_1} \leq \Big\| \sum_\square \sum_\square \mathring{U}U_{ij}^{kl} \psi_u^{ij} \chi^u_{\Omega^2_{kl}} \Big\|_{\Omega^2, \mathcal{L}_1} + \quad (84)$$
$$+ \Big\| \Big(\sum_{\mathbb{R}^2 \setminus \square} \sum_\square + \sum_\square \sum_{\mathbb{R}^2 \setminus \square} + \sum_{\mathbb{R}^2 \setminus \square} \sum_{\mathbb{R}^2 \setminus \square} \Big) \mathring{U}U_{ij}^{kl} \psi_u^{ij} \chi^u_{\Omega^2_{kl}} \Big\|_{\Omega^2, \mathcal{L}_1} \leq$$
$$\leq 2^{-m} \sum_\square \Big| \sum_\square \psi_u^{ij} \Big| + 2^{-m} \sum_{\mathbb{R}^2 \setminus \square} \sum_{\mathbb{R}^2 \setminus \square} |\mathring{U}U_{ij}^{kl}| \cdot |\psi_u^{ij}| \cdot |\pi_y(\Omega^2_{kl})| \leq$$
$$\leq 2^{-m} (1+\mu_1)^{2m} \sup_{\mathbb{R}^4 \setminus \square \times \square} |\mathring{U}U_{ij}^{kl}| \cdot \sup |\psi_u^{ij}| \leq 2^{-m} (1+\mu_1)^{2m} \cdot 2^{\gamma_1 m} \cdot d 2^{-m/2} \leq$$
$$\leq d 2^{-3m/2} (1+\mu_1)^{2m} \cdot 2^{\gamma_1 m}. \quad (85)$$

We shall estimate the supremum norm as well

$$\sup_z\Big|\sum_{kl}\sup_{ij}\mathring{U}U^{kl}_{ij}\psi^{ij}_u\chi^u_{\Omega^2_{kl}}(z)\Big|\le\frac{1}{|\pi_x(\Omega^2_{kl})|}\sup_{kl}\Big|\sum_{ij}\mathring{U}U^{kl}_{ij}\psi^{ij}_u\Big|\le$$
$$\le\frac{1}{|\pi_x(\Omega^2_{kl})|}\cdot\sup|\mathring{U}U^{kl}_{ij}|\cdot\sum_{ij}|\psi^{ij}_u|\le d(1+\mu)^{2m}\cdot 2^{2m}.$$

Then

$$\Big\|\sum_{kl}\sum_{ij}\mathring{U}U^{kl}_{ij}\psi^{ij}_u\chi^u_{\Omega^2_{kl}}\Big\|_{\Omega^2}\le$$
$$\le d\cdot\max\big(2^{-3m/2}(1+\mu_1)^{2m}\cdot 2^{\gamma_1 m},(1+\mu)^{2m}\cdot 2^{3m/2}\big)=d(1+\mu)^{2m}\cdot 2^{3m/2}.\quad(86)$$

Now we substitute (81), (82), and (83) to (80) and obtain $\mathcal{A}\nu=d2^{2m}\chi_\square+\psi^1$, where

$$\psi^1=\sum_{ij}\sum_{kl}\Big(\nu^{ij}_s\big(SS^{kl}_{ij}\chi^s_{\Omega^2_{kl}}+SU^{kl}_{ij}\chi^u_{\Omega^2_{kl}}\big)+\nu^{ij}_uUS^{kl}_{ij}\chi^s_{\Omega^2_{kl}}\Big)+$$
$$+\sum_{kl}\sum_{ij}(UU^{kl}_{ij}-\mathring{U}U^{kl}_{ij})\nu^{ij}_u\chi^u_{\Omega^2_{kl}}+\sum_{kl}\sum_{ij}\mathring{U}U^{kl}_{ij}\psi^{ij}_u\chi^u_{\Omega^2_{kl}}.\quad(87)$$

with the norm (recall $\gamma_2=\gamma_1+\frac{9}{4}+2\log_2(1+\mu_1)-\alpha$).

$$\|\psi^1\|_{\Omega^2}\le d2^{\gamma_2 m}+d2^{(2\frac{3}{4}+\gamma_1-\alpha)m}+d(1+\mu)^{2m}2^{\frac{3}{2}m}\le d2^{1+(2\frac{3}{4}+\gamma_1-\alpha)m}\le$$
$$\le d\|\mathring{U}U\chi_\square\|_{\Omega^2}\cdot 2^{(\frac{3}{4}+\gamma_1-\alpha)m}.$$

We would like to write ψ^1_y as a sum $\psi^1_y=b\chi_\square+\phi$ with $\int_\square\mathring{U}U\phi=0$. We may choose

$$b=\frac{\int_\square\mathring{U}U\psi^1_y}{\int_\square\mathring{U}U\chi_\square}.\quad(88)$$

Using (83) we get $\int_\square\mathring{U}U\chi_\square=2^{2m+2}$. Using (87) we get

$$\psi^1_y=\sum_{kl}\sum_{ij}\nu^{ij}_sSU^{kl}_{ij}\chi^u_{\Omega^2_{kl}}+\sum_{kl}\sum_{ij}(UU^{kl}_{ij}-\mathring{U}U^{kl}_{ij})\nu^{ij}_u\chi^u_{\Omega^2_{kl}}+\sum_{kl}\sum_{ij}\mathring{U}U^{kl}_{ij}\psi^{ij}_u\chi^u_{\Omega^2_{kl}}.\quad(89)$$

Apply (78) to ψ^1_y

$$\int_\square\psi^1_y=2^{m+1}\sum_{kl}\sum_{ij}\big(SU^{kl}_{ij}\nu^{ij}_u+(UU^{kl}_{ij}-\mathring{U}U^{kl}_{ij})\nu^{ij}_u+\mathring{U}U^{kl}_{ij}\psi^{ij}_u\big).$$

We may obtain an upper bound

$$\Big|\int_\square\psi^1_y\Big|\le 2^{m+1}\Big(\Big|\sum_{kl}\sum_{ij}SU^{kl}_{ij}\nu^{ij}_s\Big|+\Big|\sum_{kl}\sum_{ij}(UU^{kl}_{ij}-\mathring{U}U^{kl}_{ij})\nu^{ij}_u\Big|+\Big|\sum_{kl}\sum_{ij}\mathring{U}U^{kl}_{ij}\psi^{ij}_u\Big|\Big).$$

From Lemma 6.19 it follows that

$$\Big|\sum_{kl}\sum_{ij}(UU_{ij}^{kl}-\mathring{U}U_{ij}^{kl})\nu_u^{ij}\Big| \le 2^m \cdot 2^{(2\frac{3}{4}+\gamma_1-\alpha)m}d.$$

Using (85) we deduce

$$\Big|\sum_{kl}\sum_{ij}\mathring{U}U_{ij}^{kl}\psi_u^{ij}\Big| \le d(1+\mu)^{2m}\cdot 2^{-\frac{m}{2}}\cdot 2^{\gamma_1 m}(1+\mu)^{2m}.$$

From the third part (6.18.2) of Lemma 6.18 we get

$$\Big|\sum_{ij}\sum_{kl}SU_{ij}^{kl}\nu_s^{ij}\Big| \le 2^{(\gamma_2+1)m}d.$$

Summing up the last three together, we get

$$\Big|\int_{\square}\psi_y^1\Big| \le 3d\cdot 2^{(3\frac{3}{4}+\gamma_1-\alpha)m}. \tag{90}$$

We conclude that the ratio (88) is bounded by $b \le 2^{(1\frac{3}{4}+\gamma_1-\alpha)m} \ll 2^{2m}$.

Therefore $\mathcal{A}\nu = d(2^{2m}+b)\binom{0}{1}\chi_\square+\binom{\psi_x^1}{\phi} \in \mathrm{Cone}\left(2^{(\frac{3}{4}+\gamma_1-\alpha)m},\Omega^2\right)$ and $\|\mathcal{A}\nu\| \ge d2^{2m-1}$. ■

7. An invariant cone for the operator $W_{\frac{\delta}{2m}}P_{t*}^2W_{\frac{\delta}{2m}}$

The main goal of this Section is to get rid of the dependence of the sequence in the Preliminary Dynamo Theorem. We exploit properties of the Weierstrass transform, and construct an invariant cone for the operator $W_{\frac{\delta}{2m}}P_{t*}^2W_{\frac{\delta}{2m}}$, which is independent of the choice of $\|t\| \le \delta = 2^{-m\alpha}$.

7.1. Discretization and the Weierstrass transform toolbox. In this Subsection we establish the fact that the image of the Weierstrass transform may be very well approximated by piecewise-constant vector fields associated to some canonical partition.

Two-dimensional discretization operator on vector fields on the real plane, associated to a partition Ω, we define by

$$D_\Omega\colon \mathcal{L}(\mathbb{R}^2)\cap\mathcal{L}_\infty(\mathbb{R}^2)\to\mathfrak{X} \qquad\qquad D_\Omega v \stackrel{\text{def}}{=} \sum_{ij}\big(d_s^{ij}\chi_{\Omega_{ij}}^s + d_u^{ij}\chi_{\Omega_{ij}}^u\big), \tag{91}$$

where

$$d_s^{ij} \stackrel{\text{def}}{=} \frac{1}{|\pi_y(\Omega_{ij})|}\int_{\Omega_{ij}} v_s \quad\text{and}\quad d_u^{ij} \stackrel{\text{def}}{=} \frac{1}{|\pi_y(\Omega_{ij})|}\int_{\Omega_{ij}} v_u. \tag{92}$$

In this section we assume that Ω^1, Ω^2, and Ω^3, are three arbitrary partitions of the class $\mathcal{G}(m,\delta)$, defined on p. 9. In particular, all three partitions satisfy Lemma 4.2.

Lemma 7.1. *Let $\nu \in \mathfrak{X}$ be a bounded vector field with absolutely integrable components in $\mathbb{R}^2$. Then there exists a constant $\gamma_3 > 0$, that depends on δ and on the size of partition elements, such that*

$$\|W_{\frac{\delta}{m}}\nu - D_{\Omega^2}W_{\frac{\delta}{m}}\nu\|_2 \leq 2^{-\gamma_3 m}\|\nu\|_1.$$

One may choose $\gamma_3 = 1 - \frac{\log_2 \delta}{m} + \frac{2\log_2 m}{m} = 1 - \alpha + \frac{2\log_2 m}{m} < 1 - \alpha + \gamma_1$.

Proof. We shall show that the inequality holds true for any bounded and integrable function $f\colon \mathbb{R}^2 \to \mathbb{R}$ first. We may write by definition

$$W_\delta f(z) = \int_{\mathbb{R}^2} w_\delta(z-t)f(t)\mathrm{d}t,$$

and for the discretization operator we have that

$$\begin{aligned} D_{\Omega^2}W_\delta f(z) &= \sum_{ij}\frac{1}{|\pi_y(\Omega^2_{ij})|}\int_{\Omega^2_{ij}}\int_{\mathbb{R}^2} w_\delta(s-t)f(t)\mathrm{d}t\mathrm{d}s\cdot\chi^u_{\Omega^2_{ij}}(z) = \\ &= \int_{\mathbb{R}^2} f(t)\sum_{ij}\frac{1}{|\Omega^2_{ij}|}\int_{\Omega^2_{ij}} w_\delta(s-t)\mathrm{d}s\cdot\chi_{\Omega^2_{ij}}(z)\mathrm{d}t. \end{aligned}$$

Therefore, $(\Omega^2, \mathcal{L}_1)$ norm may be bounded as following:

$$\begin{aligned} &\|W_\delta f - D_{\Omega^2}W_\delta f\|_{\Omega^2,\mathcal{L}_1} = \sum_{ij}\frac{2^{-m}}{|\pi_y(\Omega^2_{ij})|}\int_{\Omega^2_{ij}}\big|W_\delta f(z) - W_\delta D_{\Omega^2}f(z)\big|\mathrm{d}z = \\ &= \sum_{ij}\frac{2^{-m}}{|\pi_y(\Omega^2_{ij})|}\int_{\Omega^2_{ij}}\bigg|\int_{\mathbb{R}^2} f(t)\Big(w_\delta(z-t) - \sum_{kl}\frac{1}{|\Omega^2_{kl}|}\int_{\Omega^2_{kl}} w_\delta(s-t)\mathrm{d}s\cdot\chi_{\Omega^2_{kl}}(z)\Big)\mathrm{d}t\bigg|\mathrm{d}z \leq \\ &\leq \int_{\mathbb{R}^2}|f(t)|\sum_{ij}\frac{2^{-m}}{|\pi_y(\Omega^2_{ij})|}\int_{\Omega^2_{ij}}\bigg|w_\delta(z-t) - \sum_{kl}\frac{1}{|\Omega^2_{kl}|}\int_{\Omega^2_{kl}} w_\delta(s-t)\mathrm{d}s\cdot\chi_{\Omega^2_{kl}}(z)\bigg|\mathrm{d}z\mathrm{d}t \leq \\ &\leq \int_{\mathbb{R}^2}|f(t)|\mathrm{d}t\cdot\sup_t\sum_{ij}\frac{2^{-m}}{|\pi_y(\Omega^2_{ij})|}\int_{\Omega^2_{ij}}\bigg|w_\delta(z-t) - \sum_{kl}\frac{1}{|\Omega^2_{kl}|}\int_{\Omega^2_{kl}} w_\delta(s-t)\mathrm{d}s\cdot\chi_{\Omega^2_{kl}}(z)\bigg|\mathrm{d}z \leq \\ &\leq \int_{\mathbb{R}^2}|f(t)|\mathrm{d}t\cdot\sup_t\sum_{ij}\frac{2^{-m}}{|\pi_y(\Omega^2_{ij})|}\int_{\Omega^2_{ij}}\bigg|w_\delta(z-t) - \frac{1}{|\Omega^2_{ij}|}\int_{\Omega^2_{ij}} w_\delta(s-t)\mathrm{d}s\bigg|\mathrm{d}z \leq \\ &\leq \int_{\mathbb{R}^2}|f(t)|\mathrm{d}t\cdot\sup_t\sum_{ij}\frac{2^{-m}}{|\pi_y(\Omega^2_{ij})|}\int_{\Omega^2_{ij}}\Big|\max_z w_\delta(z-t) - \min_z w_\delta(z-t)\Big|\mathrm{d}z. \quad (93) \end{aligned}$$

We have to find an upper bound for the last term:

$$
\begin{aligned}
\sup_t \sum_{ij} \frac{2^{-m}}{|\pi_y(\Omega^2_{ij})|} \int_{\Omega^2_{ij}} \Big|\max_{s\in\Omega^2_{ij}} w_\delta(s-t) - \min_{s\in\Omega^2_{ij}} w_\delta(s-t)\Big| \mathrm{d}z \le \\
\le 2^{-m} \sup_t \sum_{ij} |\pi_x(\Omega^2_{ij})| \cdot \big|\max_{z\in\Omega^2_{ij}} w_\delta(z-t) - \min_{z\in\Omega^2_{ij}} w_\delta(z-t)\big| \le \\
\le 2^{-m} \sup_t \sum_{ij} |\pi_x(\Omega^2_{ij})| \cdot |\mathrm{diam}(\Omega^2_{ij})| \cdot \sup_{z\in\Omega^2_{ij}} |\nabla_z w_\delta(z-t)| = \\
= 2^{-m} \sup_t \sum_{ij} \frac{|\mathrm{diam}(\Omega^2_{ij})|}{|\pi_y(\Omega^2_{ij})|} \cdot \sup_{z\in\Omega^2_{ij}} |\nabla_z w_\delta(z-t)| \cdot |\Omega^2_{ij}| \le \\
\le 2^{-m} \sup_{ij} \frac{|\mathrm{diam}(\Omega^2_{ij})|}{|\pi_y(\Omega^2_{ij})|} \cdot \sup_t \sum_{ij} \sup_{z\in\Omega^2_{ij}} |\nabla_z w_\delta(z-t)| \cdot |\Omega^2_{ij}| \le \\
\le 2^{-m} \sup_{ij} \frac{|\mathrm{diam}(\Omega^2_{ij})|}{|\pi_y(\Omega^2_{ij})|} \cdot \int_{\mathbb{R}^2} |\nabla_z w_\delta(z)| \mathrm{d}z \le \frac{2^{-m}}{\delta} \sup_{ij} \frac{|\mathrm{diam}(\Omega^2_{ij})|}{|\pi_y(\Omega^2_{ij})|}. \qquad (94)
\end{aligned}
$$

Therefore substituting (94) to (93) we conclude

$$
\|W_\delta f - D_{\Omega^2} W_\delta f\|_{\Omega^2,\mathcal{L}_1} \le \frac{\sup |\pi_y(\Omega^1_{ij})|}{\delta} \cdot \sup_{kl} \frac{|\mathrm{diam}(\Omega^2_{kl})|}{|\pi_y(\Omega^2_{kl})|} \|f\|_{\Omega^1,\mathcal{L}_1}. \qquad (95)
$$

Similarly, for the supremum norm

$$
\begin{aligned}
&\|D_{\Omega^2} W_\delta \nu_s - W_\delta \nu_s\|_\infty = \\
&= \sup_s \Big| \int_{\mathbb{R}^2} w_\delta(s-t)\nu_s(t)\mathrm{d}t - \sum_{ij} \frac{1}{|\Omega^2_{ij}|} \int_{\mathbb{R}^2} w_\delta(z-t)\mathrm{d}t\mathrm{d}z \chi_{\Omega^2_{ij}}(s) \Big| = \\
&= \sup_{ij} \sup_{s\in\Omega^2_{ij}} \Big| \int_{\mathbb{R}^2} w_\delta(s-t)\nu_s(t)\mathrm{d}t - \frac{1}{|\Omega^2_{ij}|} \int_{\Omega^2_{ij}} \int_{\mathbb{R}^2} w_\delta(z-t)\nu_s(t)\mathrm{d}t\mathrm{d}z \Big| = \\
&= \sup_{ij} \Big| \max_{s\in\Omega^2_{ij}} \int_{\mathbb{R}^2} w_\delta(s-t)\nu_s(t)\mathrm{d}t - \min_{s\in\Omega^2_{ij}} \int_{\mathbb{R}^2} w_\delta(s-t)\nu_s(t)\mathrm{d}t \Big| \le \\
&\le \sup_{\Omega^2_{ij}} \int_{\gamma(\Omega^2_{ij})} \Big|\nabla \int_{\mathbb{R}^2} w_\delta(s-t)\nu_s(t)\mathrm{d}t\Big| \mathrm{d}s, \qquad (96)
\end{aligned}
$$

where $\gamma(\Omega^2_{ij})$ is a line segment connecting the points of maxima and minima of the integrand in Ω^2_{ij}. We proceed therefore

$$\begin{aligned}
\|D_{\Omega^2}W_\delta\nu_s - W_\delta\nu_s\|_\infty &\leq \sup \operatorname{diam}(\Omega^2_{ij}) \cdot \sup_s \Big|\nabla_s \int_{\mathbb{R}^2} w_\delta(s-t)\nu_s(t)\mathrm{d}t\Big| \leq \\
&\leq \sup \operatorname{diam}(\Omega^2_{ij}) \cdot \sup|\nu| \cdot \sup_s \int_{\mathbb{R}^2} \Big|\nabla_s w_\delta(s-t)\Big|\mathrm{d}t \leq \\
&\leq \sup \operatorname{diam}(\Omega^2_{ij}) \cdot \sup|\nu| \cdot \int_{\mathbb{R}^2} \frac{1}{\pi^2\delta^4}\sqrt{t_x^2+t_y^2}\cdot e^{-\frac{t_x^2-t_y^2}{2\delta^2}}\mathrm{d}t \leq \\
&\leq \frac{\sup \operatorname{diam}(\Omega^2_{ij})}{\pi\delta}\|\nu\|_\infty. \qquad (97)
\end{aligned}$$

We put (95) and (97) together, and conclude that we may find a constant $\gamma_3 > 0$ such that

$$\max\Big(\frac{m\sup|\pi_y(\Omega^1_{ij})|}{\delta}\cdot\sup_{kl}\frac{|\operatorname{diam}(\Omega^2_{kl})|}{|\pi_y(\Omega^2_{kl})|}, \frac{m\sup\operatorname{diam}(\Omega^2_{ij})}{\delta}\Big) = 2^{-\gamma_3 m}.$$

■

Remark 10. It follows from the properties of partitions of the class $\mathcal{G}(m,\delta)$, Lemma 4.2, that $\gamma_3 < 1-\alpha$ and it may be chosen arbitrary close to $1-\alpha$.

Lemma 7.2. *Let Ω be a partition of $\mathbb{R}^2$ the class $\mathcal{G}(m,\delta)$. Then*

$$\|W_\delta\chi_\square - D_\Omega W_\delta\chi_\square\|_\Omega \leq 2^{-m/4}; \qquad (7.2.1)$$

$$\|W_\delta\chi_\square - \chi_\square\|_\Omega \leq 2^{-m/4}. \qquad (7.2.2)$$

Proof. We start with the first inequality. The upper bound for the supremum norm is trivial. Indeed, observe that for any non-negative integrable function f and any element Ω_{ij}

$$\sup_{\Omega_{ij}} f \geq \frac{1}{|\Omega_{ij}|}\int_{\Omega_{ij}} f > 0,$$

and, consequently,

$$\sup_{\Omega_{ij}}\Big|f - \frac{1}{|\Omega_{ij}|}\int_{\Omega_{ij}} f\Big| \leq \sup_{\Omega_{ij}}|f|.$$

Therefore

$$\sup_z\Big|\int_\square w_\delta(z-t)\mathrm{d}t - \sum_{ij}\frac{1}{|\Omega_{ij}|}\int_{\Omega_{ij}}\int_\square w_\delta(z-t)\mathrm{d}t\mathrm{d}s\chi_{\Omega_{ij}}(z)\Big| =$$

$$= \sup_{ij}\sup_{z\in\Omega_{ij}}\Big|\int_\square w_\delta(z-t)\mathrm{d}t - \frac{1}{|\Omega_{ij}|}\int_{\Omega_{ij}}\int_\square w_\delta(s-t)\mathrm{d}t\mathrm{d}s\Big| \leq \sup_{ij}\sup_{z\in\Omega_{ij}}\Big|\int_\square w_\delta(z-t)\mathrm{d}t\Big| \leq 1.$$

Now we consider $(\Omega, \mathcal{L}_1)$-norm. Let k be such that $e^k > 2^m$ and $k < m$. Introduce three sets of indices:

$$\begin{aligned} r_1 \colon &= \{(i,j) \in \mathbb{Z}^2 \mid \Omega_{ij} \subset \square_{1-k\delta}\}; \\ r_2 \colon &= \{(i,j) \in \mathbb{Z}^2 \mid \Omega_{ij} \subset \square_{1+k\delta},\, \Omega_{ij} \not\subset \square_{1-k\delta}\}; \\ r_3 \colon &= \{(i,j) \in \mathbb{Z}^2 \mid \Omega_{ij} \not\subset \square_{1+k\delta}\}. \end{aligned}$$

We split the sum of integrals in three parts:

$$\begin{aligned} &\sum_{kl} \frac{2^{-m}}{|\pi_y(\Omega_{kl}|} \int_{\Omega_{kl}} \Big| \int_{\square} w_\delta(z-t)\mathrm{d}t - \sum_{ij} \frac{1}{|\Omega_{ij}|} \int_{\Omega_{ij}} \int_{\square} w_\delta(s-t)\mathrm{d}t\mathrm{d}s \chi_{\Omega_{ij}}(z) \Big| \mathrm{d}z = \\ &\quad \sum_{ij} \frac{2^{-m}}{|\pi_y(\Omega_{ij}|} \int_{\Omega_{ij}} \int_{\Omega_{ij}} \Big| \int_{\square} w_\delta(z-t)\mathrm{d}t - \frac{1}{|\Omega_{ij}|} \int_{\Omega_{ij}} \int_{\square} w_\delta(s-t)\mathrm{d}t\mathrm{d}s \Big| \mathrm{d}z \\ &= \Big(\sum_{r_1} + \sum_{r_2} + \sum_{r_3} \Big) \frac{2^{-m}}{|\pi_y(\Omega_{ij}|} \int_{\Omega_{ij}} \Big| \int_{\square} w_\delta(z-t)\mathrm{d}t - \frac{1}{|\Omega_{ij}|} \int_{\Omega_{ij}} \int_{\square} w_\delta(s-t)\mathrm{d}t\mathrm{d}s \Big| \mathrm{d}z. \end{aligned} \tag{98}$$

We estimate the three sums separately.

Observe that for any $(i,j) \in r_1$ and any $z \in \Omega_{ij} \subset \square_{1-k\delta}$

$$1 > \int_{\square} w_\delta(z-t)\mathrm{d}t = \int_{\square - z} w_\delta(t)\mathrm{d}t \geq \int_{-k\delta}^{k\delta} \int_{-k\delta}^{k\delta} w_\delta(t)\mathrm{d}t_x \mathrm{d}t_y = \int_{-k}^{k} \int_{-k}^{k} w_1(t)\mathrm{d}t \geq 1 - 4e^{-k}.$$

Therefore

$$\begin{aligned} &\sum_{r_1} \frac{2^{-m}}{|\pi_y(\Omega_{ij})|} \int_{\Omega_{ij}} \Big| \int_{\square} w_\delta(z-t)\mathrm{d}t - \frac{1}{|\Omega_{ij}|} \int_{\Omega_{ij}} \int_{\square} w_\delta(s-t)\mathrm{d}t\mathrm{d}s \Big| \mathrm{d}z \leq \\ &\quad \leq \sum_{r_1} \frac{2^{-m}}{|\pi_y(\Omega_{ij})|} \int_{\Omega_{ij}} \Big| 1 - \frac{1}{|\Omega_{ij}|} \int_{\Omega_{ij}} (1 - 4e^{-k}) \Big| \mathrm{d}z \leq \frac{2^{-m}}{\inf |\pi_y(\Omega_{ij})|} \sum_{r_1} 4e^{-k} |\Omega_{ij}| \leq \\ &\qquad \leq \frac{(1-k\delta)^2}{2^m e^k \inf |\pi_y(\Omega_{ij})|} \leq \sup \operatorname{diam} |\Omega_{ij}|. \end{aligned} \tag{99}$$

Observe that for any $(i,j)\in r_2$ and any $z\in\Omega_{ij}$

$$\sup_{z\in\Omega_{ij}}\left|\int_\square w_\delta(z-t)\mathrm{d}t-\frac{1}{|\Omega_{ij}|}\int_{\Omega_{ij}}\int_\square w_\delta(s-t)\mathrm{d}t\mathrm{d}s\right|\leq$$
$$\leq\sup_{z\in\Omega_{ij}}\left|\nabla_z\int_\square w_\delta(z-t)\mathrm{d}t\right|\cdot\mathrm{diam}(\Omega_{ij})\leq\sup_{z\in\Omega_{ij}}\int_\square\left|\nabla_z w_\delta(z-t)\right|\mathrm{d}t\cdot\mathrm{diam}(\Omega_{ij})=$$
$$=\sup_{z\in\Omega_{ij}}\int_\square\frac{1}{\pi^2\delta^4}\sqrt{(z_x-t_x)^2+(z_y-t_y)^2}\cdot e^{-\frac{(z_x-t_x)^2-(z_y-t_y)^2}{2\delta^2}}\mathrm{d}t\cdot\mathrm{diam}(\Omega_{ij})\leq$$
$$\leq\sup_{z\in\Omega_{ij}}\int_\square\frac{1}{\pi^2\delta^4}\big(|z_x-t_x|+|z_y-t_y|\big)\cdot e^{-\frac{(z_x-t_x)^2-(z_y-t_y)^2}{2\delta^2}}\mathrm{d}t\cdot\mathrm{diam}(\Omega_{ij})\leq\frac{4\mathrm{diam}(\Omega_{ij})}{\pi^2\delta}.$$

Therefore

$$\sum_{r_2}\int_{\Omega_{ij}}\left|\int_\square w_\delta(z-t)\mathrm{d}t-\frac{1}{|\Omega_{ij}|}\int_{\Omega_{ij}}\int_\square w_\delta(s-t)\mathrm{d}t\mathrm{d}s\cdot\chi_{\Omega_{ij}}(z)\right|\mathrm{d}z\leq$$
$$\leq\sum_{r_2}|\Omega_{ij}|\frac{4\mathrm{diam}(\Omega_{ij})}{\delta}\leq\big((1+k\delta)^2-(1-k\delta)^2\big)\frac{4\sup\mathrm{diam}(\Omega_{ij})}{\delta}\leq 16k\sup\mathrm{diam}(\Omega_{ij}).\tag{100}$$

Finally, for the third term we cut r_3 into squared annuli

$$\mathrm{ai}_n:=\{(i,j)\in r_1,|\,\Omega_{ij}\subset\square_{1+(k+n)\delta},\,\Omega_{ij}\not\subset\square_{1+(k+n-1)\delta}\}.$$

Obviously, $\bigcup_{n=0}^{\infty}\mathrm{ai}_n=r_1$, and $\sum_{\mathrm{ai}_n}|\Omega_{ij}|\leq 2\delta+\delta^2(2k+2n-1)$. Therefore,

$$\sum_{r_1}\int_{\Omega_{ij}}\left|\int_\square w_\delta(z-t)\mathrm{d}t-\frac{1}{|\Omega_{ij}|}\int_{\Omega_{ij}}\int_\square w_\delta(s-t)\mathrm{d}t\mathrm{d}s\right|\mathrm{d}z=$$
$$=\sum_{n=0}^{\infty}\sum_{\mathrm{ai}_n}\int_{\Omega_{ij}}\left|\int_\square w_\delta(z-t)\mathrm{d}t-\frac{1}{|\Omega_{ij}|}\int_{\Omega_{ij}}\int_\square w_\delta(s-t)\mathrm{d}t\mathrm{d}s\right|\mathrm{d}z=$$
$$=\sum_{n=0}^{\infty}\sum_{\mathrm{ai}_n}|\Omega_{ij}|\cdot\sup_{z\in\Omega_{ij}}\left|\int_\square w_\delta(z-t)\mathrm{d}t-\frac{1}{|\Omega_{ij}|}\int_{\Omega_{ij}}\int_\square w_\delta(s-t)\mathrm{d}t\mathrm{d}s\right|\leq$$
$$\leq\sum_{n=0}^{\infty}\sum_{\mathrm{ai}_n}|\Omega_{ij}|\cdot\mathrm{diam}(\Omega_{ij})\cdot\sup_{z\in\Omega_{ij}}\int_\square\left|\nabla_z w_\delta(z-t)\right|\mathrm{d}t\leq$$
$$\leq\sum_{n=0}^{\infty}\sum_{\mathrm{ai}_n}|\Omega_{ij}|\cdot\mathrm{diam}(\Omega_{ij})\cdot\frac{4}{\pi^2\delta}\cdot e^{-\frac{(k+n)^2}{2}}\leq 4\sup\mathrm{diam}(\Omega_{ij}).\tag{101}$$

Substituting up (99), (100), and (101) to (98):

$$\|W_{\frac{\delta}{m}}\chi_\square-D_\Omega W_{\frac{\delta}{m}}\chi_\square\|_\Omega<32m\sup\mathrm{diam}(\Omega_{ij}).$$

We conclude, using the second part of Lemma 4.2: $\Omega_{ij} \subset Rec(2^{1-m}, 2^{1-m})$

$$\|W_\delta \chi_\square - D_\Omega W_\delta \chi_\square\|_\Omega \le \max(32m \sup \operatorname{diam}(\Omega_{ij}), 2^{-m/4}) = 2^{-m/4}.$$

Now we consider the second inequality (7.2.2). Obviously, $\|W_\delta \chi_\square - \chi_\square\|_\infty \le 1$. We proceed to the weighted $(\Omega, \mathcal{L}_1)$-norm. We shall show that

$$\|W_\delta \chi_\square - \chi_\square\|_{\Omega,\mathcal{L}_1} = \sum_{ij} \frac{2^{-m}}{|\pi_y(\Omega_{ij})|} \int_{\Omega_{ij}} |W_\delta \chi_\square - \chi_\square| \le \frac{12\delta}{2^m \inf |\pi_y(\Omega_{ij})|}. \tag{102}$$

By straightforward calculation

$$\sum_{ij} \frac{2^{-m}}{|\pi_y(\Omega_{ij})|} \int_{\Omega_{ij}} |W_\delta \chi_\square - \chi_\square| = \sum_{ij} \frac{2^{-m}}{|\pi_y(\Omega_{ij})|} \int_{\Omega_{ij}} \Big| \int_\square w_\delta(z-t) \mathrm{d}t - \chi_\square(z) \Big| \mathrm{d}z \le$$
$$\le \frac{2^{-m}}{\inf |\pi_y(\Omega_{ij})|} \cdot \Big(\int_{\mathbb{R}^2 \setminus \square} \Big| \int_\square w_\delta(z-t) \mathrm{d}t \Big| \mathrm{d}z + \int_\square \Big| \int_\square w_\delta(z-t) \mathrm{d}t - 1 \Big| \mathrm{d}z \Big). \tag{103}$$

Recall the error function

$$\operatorname{erf}(z) \colon = \int_0^z \frac{2}{\sqrt{\pi}} e^{-\frac{x^2}{2}} \mathrm{d}x;$$

and its antiderivative

$$\int \operatorname{erf}(z) \mathrm{d}z = z \operatorname{erf}(z) + \frac{e^{-z^2}}{\sqrt{\pi}}.$$

We estimate each of two terms of (103) separately.

$$\int_\square \frac{1}{\sqrt{2\pi}\delta} e^{-\frac{(x_1-t_1)^2}{2\delta^2}} \mathrm{d}x_1 \mathrm{d}t_1 =$$
$$= \int_{-1}^1 \int_{-1-t_1}^{1-t_1} \frac{1}{\sqrt{2\pi}\delta} e^{-\frac{x_1^2}{2\delta^2}} \mathrm{d}x_1 \mathrm{d}t_1 = \int_{-1}^1 \int_{\frac{-1-t_1}{\sqrt{2}\delta}}^{\frac{1-t_1}{\sqrt{2}\delta}} \frac{1}{\sqrt{\pi}} e^{-x_1^2} \mathrm{d}x_1 \mathrm{d}t_1 =$$
$$= \frac{1}{2} \int_{-1}^1 \Big(\int_0^{\frac{1-t_1}{\sqrt{2}\delta}} \frac{2}{\sqrt{\pi}} e^{-x_1^2} \mathrm{d}x_1 + \int_0^{\frac{1+t_1}{\sqrt{2}\delta}} \frac{2}{\sqrt{\pi}} e^{-x_1^2} \mathrm{d}x_1 \Big) \mathrm{d}t_1 =$$
$$= \frac{1}{2} \int_{-1}^1 \operatorname{erf}\Big(\frac{1-t_1}{\sqrt{2}\delta} \Big) + \operatorname{erf}\Big(\frac{1+t_1}{\sqrt{2}\delta} \Big) \mathrm{d}t_1 =$$
$$= \frac{\delta}{\sqrt{2}} \Big(\int_0^{\sqrt{2}/\delta} \operatorname{erf}(z) \mathrm{d}z - \int_{-\sqrt{2}/\delta}^0 \operatorname{erf}(z) \mathrm{d}z \Big) =$$
$$= \frac{\delta}{\sqrt{2}} \Big(\Big(z \operatorname{erf}(z) + \frac{e^{-z^2}}{\sqrt{\pi}} \Big) \Big|_0^{\sqrt{2}/\delta} - \Big(z \operatorname{erf}(z) + \frac{e^{-z^2}}{\sqrt{\pi}} \Big) \Big|_{-\sqrt{2}/\delta}^0 \Big) =$$
$$= 2 \operatorname{erf}\Big(\frac{\sqrt{2}}{\delta} \Big) + \sqrt{\frac{2}{\pi}} \delta (e^{-2/\delta^2} - 1) \ge (2-\delta)(1 - e^{-2/\delta^2}).$$

Therefore for the first term of (103) we have

$$\int_{\square}\Big|\int_{\square} w_\delta(z-t)\mathrm{d}t - 1\Big|\mathrm{d}z = \int_{\square}\Big(1-\int_{\square} w_\delta(z-t)\mathrm{d}t\Big)\mathrm{d}z =$$
$$= 4-\Big(\int_{\square}\frac{1}{\sqrt{2\pi}\delta}e^{-\frac{(x_1-t_1)^2}{2\delta^2}}\mathrm{d}x_1\mathrm{d}t_1\Big)^2 \le 4-(2-\delta)^2(1-e^{-2/\delta^2})^2 \le 4\delta. \quad (104)$$

We claim

$$\int_{\mathbb{R}^2\setminus\square}\int_{\square} w_\delta(z-t)\mathrm{d}t\mathrm{d}z \le 8\delta. \quad (105)$$

Indeed, using approximation $\mathrm{erf}(x) = 1-\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-x^2}\mathrm{d}x \ge 1-e^{-x}$ for large x,

$$\int_{-1}^{1}\int_{1}^{+\infty}\frac{1}{\sqrt{2\pi}\delta}e^{-\frac{(x_1-t_1)^2}{2\delta^2}}\mathrm{d}x_1\mathrm{d}t_1 = \frac{1}{2}\int_{-1}^{1}\int_{\frac{1-t_1}{\sqrt{2}\delta}}^{+\infty}\sqrt{2}\pi e^{-x_1^2}\mathrm{d}x_1\mathrm{d}t_1 =$$
$$= 1-\frac{1}{2}\int_{-1}^{1}\mathrm{erf}\Big(\frac{1-t_1}{\sqrt{2}\delta}\Big)\mathrm{d}t_1 = 1+\frac{1}{2}\int_{-\sqrt{2}/\delta}^{0}\mathrm{erf}(z)\mathrm{d}z = 1+\frac{1}{2}\Big(z\,\mathrm{erf}(z)+\frac{e^{-z^2}}{\sqrt{\pi}}\Big)\Big|_{-\sqrt{2}/\delta}^{0} =$$
$$= 1+\frac{1}{2}\Big(\frac{1}{\sqrt{\pi}}-\frac{\sqrt{2}}{\delta}\mathrm{erf}\Big(\frac{\sqrt{2}}{\delta}\Big)-\frac{e^{-2/\delta^2}}{\sqrt{\pi}}\Big) \le 1-\big(1-e^{-2/\delta^2}\big)+\frac{\delta}{\sqrt{2\pi}}-\frac{e^{-2/\delta^2}}{\sqrt{2\pi}} \le \delta.$$

Therefore,

$$\int_{1}^{+\infty}\int_{1}^{+\infty}\int_{\square} w_\delta(z-t)\mathrm{d}t\mathrm{d}z \le 4\delta^2;$$

and, similarly,

$$\int_{1}^{+\infty}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} w_\delta(z-t)\mathrm{d}t\mathrm{d}z \le 2\delta.$$

The claim (105) follows and hence the inequality (102). ■

Lemma 7.3. *Let Ω^1 and Ω^2 be two arbitrary partitions of $\mathbb{R}^2$ of the class $\mathcal{G}(m,\delta)$. Then An upper bound for the norm of the Weierstrass transform is given by*

$$\|W_\delta\nu\|_2 \le \sup|\pi_y(\Omega^2_{kl})|\cdot\sup|\pi_x(\Omega^1_{ij})|\cdot m^2\frac{N_\delta}{\delta^2}\cdot\|\nu\|_1.$$

Proof. Consider a function $f\in\mathcal{L}_1(\mathbb{R}^2)\cap\mathcal{L}_\infty(\mathbb{R}^2)$ with $\|f\|_{\Omega^1}=1$. Then

$$\sum_{ij}\frac{2^{-m}}{|\pi_y(\Omega^1_{ij})|}\int_{\Omega^1_{ij}}|f|\le 1; \qquad \sup|f|\le 2^{\frac{m}{4}}.$$

By straightforward calculation

$$\begin{aligned}\|W_\delta f\|_{\Omega^2,\mathcal{L}_1} &= \sum_{ij}\frac{2^{-m}}{|\pi_y(\Omega^2_{ij})|}\int_{\Omega^2_{ij}}\Big|\sum_{kl}\int_{\Omega^1_{kl}} w_\delta(z-t)f(t)\mathrm{d}t\Big|\mathrm{d}z \le \\ &\le 2^{-m}\sum_{kl}\int_{\Omega^1_{kl}}|f(t)|\sum_{ij}\frac{1}{|\pi_y(\Omega^2_{ij})|}\int_{\Omega^2_{ij}} w_\delta(z-t)\mathrm{d}z\mathrm{d}t \le \\ &\le 2^{-m}\sum_{kl}\int_{\Omega^1_{kl}}|f(t)|\Big(\sum_{|\Omega^1_{kl}-\Omega^2_{ij}|>m\delta}+\sum_{|\Omega^1_{kl}-\Omega^2_{ij}|<m\delta}\Big)\frac{1}{|\pi_y(\Omega^2_{ij})|}\int_{\Omega^2_{ij}} w_\delta(z-t)\mathrm{d}z\mathrm{d}t.\end{aligned} \tag{106}$$

We have to estimate two sums separately. We know that $\|w_\delta\|_\infty \le \frac{1}{\delta^2}$; thus

$$\frac{1}{|\pi_y(\Omega^2_{ij})|}\int_{\Omega^2_{ij}} w_\delta(z-t)\mathrm{d}z \le \frac{|\pi_x(\Omega^2_{ij})|}{\delta^2}.$$

Therefore, since for a fixed Ω^1_{kl}, the total number of elements of another partition Ω^2_{ij} satisfying $|\Omega^2_{ij}-\Omega^1_{kl}| < m\delta$ is bounded by $m^2 N_\delta$:

$$\sum_{|\Omega^1_{kl}-\Omega^2_{ij}|<m\delta}\frac{1}{|\pi_y(\Omega^2_{ij})|}\int_{\Omega^2_{ij}} w_\delta(z-t)\mathrm{d}z \le \sup|\pi_x(\Omega^2_{ij})|\cdot m^2\cdot\frac{N_\delta}{\delta^2}. \tag{107}$$

We also observe that for any $t\in\Omega^1_{kl}$

$$\sum_{|\Omega^1_{kl}-\Omega^2_{ij}|>m\delta}\frac{1}{|\pi_y(\Omega^2_{ij})|}\int_{\Omega^2_{ij}} w_\delta(z-t)\mathrm{d}z \le \frac{1}{\inf|\pi_y(\Omega^2_{ij})|}\int_{\mathbb{R}^2\setminus\square_{1+m\delta}} w_\delta(z-t)\mathrm{d}z \le \frac{4e^{-m}}{\inf|\pi_y(\Omega^2_{ij})|}. \tag{108}$$

Substituting (107) and (108) to (106) we get

$$\begin{aligned}\|W_\delta f\|_{\Omega^2,\mathcal{L}_1} &= 2^{-m}\sum_{kl}\int_{\Omega^1_{kl}}|f(t)|\Big(\frac{4e^{-m}}{\inf|\pi_y(\Omega^2_{ij})|}+\sup|\pi_x(\Omega^2_{ij})|\cdot m^2\frac{N_\delta}{\delta^2}\Big)\mathrm{d}t \le \\ &\le \sup|\pi_x(\Omega^2_{ij})|\cdot\sup|\pi_y(\Omega^1_{kl})|\cdot m^2\frac{N_\delta}{\delta^2}\|f\|_{\Omega^1,\mathcal{L}_1}.\end{aligned}$$

The upper bound of the supremum norm is easy

$$\|W_\delta f\|_\infty = \sup_{z\in\mathbb{R}^2}\Big|\int_{\mathbb{R}^2} w_\delta(z-t)f(t)\mathrm{d}t\Big| \le \sup_{z\in\mathbb{R}^2}|f(z)|.$$

The upper bound for the vector fields follows immediately.

■

7.2. **Constructing an invariant cone.** In this Subsection we use approximations we obtained earlier and two cones constructed for the operator $\mathcal{A}$ (Section 6, Theorem 3) to get an invariant cone in the space $\mathfrak{X}$ for the operator $W_{\frac{\delta}{2m}} P^2_{\xi *} W_{\frac{\delta}{2m}}$. The main result is Theorem 4. We shall prove two Lemmas first.

Lemma 7.4. *There exists $\gamma_4 > 0$ such that for any $\nu \in \mathrm{Cone}\left(2^{(\frac{3}{4}+\gamma_1-\alpha)m}, \Omega^1\right)$ and for arbitrary partition Ω^2 of $\mathbb{R}^2$ of the class $\mathcal{G}(m,\delta)$:*

$$\|D_{\Omega^2} W_\delta \nu\|_2 \geq (1 - 2^{-\gamma_4 m})\|\nu\|_1.$$

(See p. 8 for a general definition of a cone in $\mathfrak{X}$.)

Proof. Let $\nu \in \mathfrak{X}_{\Omega^2}$ be a bounded and integrable vector field. Then similarly to one-dimensional case, by Lemma 7.3

$$\|W_{\frac{\delta}{m}}\nu\|_{\Omega^2,\mathcal{L}_1} = \sum_{ij} \frac{2^{-m}}{|\pi_y(\Omega^2_{ij})|} \int_{\Omega^2_{ij}} \left| \int_{\mathbb{R}^2} w_\delta(z-t)\nu(t)\mathrm{d}t \right| \mathrm{d}z \leq$$

$$\leq \sup|\pi_y(\Omega^2_{kl})| \cdot \sup|\pi_x(\Omega^2_{ij})| \cdot m^4 \frac{N_\delta}{\delta^2} \cdot \|\nu\|_{\Omega^1,\mathcal{L}_1}.$$

By Lemma 7.2 we know that

$$\|W_\delta(\begin{smallmatrix}0\\1\end{smallmatrix})\chi_\square - D_{\Omega^2} W_\delta(\begin{smallmatrix}0\\1\end{smallmatrix})\chi_\square\|_2 \leq 2^{-\frac{m}{4}}.$$

Now we find a lower bound for the norm of $\|D_{\Omega^2} W_\delta(\begin{smallmatrix}0\\1\end{smallmatrix})\chi_\square\|_{\Omega^2}$. Observe that the integral over the unit square $\int_\square w_\delta(z)\mathrm{d}z \geq 1 - e^{-1/\delta^2}$.

$$\|D_{\Omega^2} W_\delta(\begin{smallmatrix}0\\1\end{smallmatrix})\chi_\square\|_2 \geq \|W_\delta(\begin{smallmatrix}0\\1\end{smallmatrix})\chi_\square\|_2 - \|W_\delta(\begin{smallmatrix}0\\1\end{smallmatrix})\chi_\square - D_{\Omega^2} W_\delta(\begin{smallmatrix}0\\1\end{smallmatrix})\chi_\square\|_2 \geq 1 - 2^{-\frac{m}{4}} - e^{-1/\delta^2}.$$

Consider $\psi \in \mathfrak{X}_{\Omega^1}$, with $\|\psi\|_1 \leq d 2^{(\frac{3}{4}+\gamma_1-\alpha)m}$, $\int_\square \mathring{U} U \psi_u = 0$. Then by Lemma 7.1

$$\|W_{\frac{\delta}{m}}\psi - D_{\Omega^2} W_{\frac{\delta}{m}}\psi\|_2 \leq d \cdot 2^{(\frac{3}{4}+\gamma_1-\gamma_3-\alpha)m}, \tag{109}$$

where $\gamma_3 = 1 - \alpha + \frac{2\log_2 m}{m}$; and thus by Lemma 7.3

$$\|D_{\Omega^2} W_{\frac{\delta}{m}}\psi\|_2 \leq \|W_{\frac{\delta}{m}}\psi\|_2 + d \cdot 2^{(\frac{3}{4}+\gamma_1-\gamma_3-\alpha)m} \leq$$

$$\leq d \cdot \sup|\pi_y(\Omega^2_{ij})| \cdot \sup|\pi_x(\Omega^1_{ij})| \cdot \frac{m^2 N_\delta}{\delta^2} + d \cdot 2^{(\frac{3}{4}+\gamma_1-\gamma_3-\alpha)m}. \tag{110}$$

We use Lemma 7.2 and (109), to estimate the approximation error for the field $W_\delta \nu$:

$$\|W_{\frac{\delta}{m}}\nu - D_{\Omega^2} W_{\frac{\delta}{m}}\nu\|_2 \leq d\|W_{\frac{\delta}{m}}(\begin{smallmatrix}0\\1\end{smallmatrix})\chi_\square - D_{\Omega^2} W_{\frac{\delta}{m}}(\begin{smallmatrix}0\\1\end{smallmatrix})\chi_\square\|_2 + \|W_{\frac{\delta}{m}}\psi - D_{\Omega^2} W_{\frac{\delta}{m}}\psi\|_2 \leq$$

$$\leq d 2^{-m/4} + d \cdot 2^{(\frac{3}{4}+\gamma_1-\gamma_3-\alpha)m}.$$

Observe that by Lemma 7.3, since $\|\psi\| \le 2^{\frac{3}{4}+\gamma_1-\alpha}$,

$$\|W_{\frac{\delta}{m}}\nu\|_2 = \|dW_{\frac{\delta}{m}}\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\chi_\Box + W_{\frac{\delta}{m}}\psi\|_2 \ge \|dW_{\frac{\delta}{m}}\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\chi_\Box\|_2 - \|W_{\frac{\delta}{m}}\psi\|_2 \ge$$
$$\ge d(1-e^{-m^2/\delta^2}) - \sup|\pi_y(\Omega^2_{ij})| \cdot \sup|\pi_x(\Omega^1_{ij})| \cdot \frac{N_\delta}{\delta^2} m^2 \|\psi\|_1 \ge$$
$$\ge d(1-e^{-m^2/\delta^2}) - d\sup|\pi_y(\Omega^2_{ij})| \cdot \sup|\pi_x(\Omega^1_{ij})| \cdot 2^{(\frac{3}{4}+\gamma_1)m}\frac{m^2 N_\delta}{\delta}.$$

Summing up altogether

$$\|D_{\Omega^2}W_{\frac{\delta}{m}}\nu\|_2 \ge \|W_{\frac{\delta}{m}}\nu\|_2 - \|W_{\frac{\delta}{m}}\nu - D_{\Omega^2}W_{\frac{\delta}{m}}\nu\|_2 \ge d\left(1-2^{-m/4}-e^{-m^2/\delta^2}\right) -$$
$$- d\left(\sup|\pi_y(\Omega^2_{ij})| \cdot \sup|\pi_x(\Omega^1_{ij})| \cdot 2^{(\frac{3}{4}+\gamma_1)m} \cdot \frac{m^2N_\delta}{\delta} + 2^{(\frac{3}{4}+\gamma_1-\gamma_3-\alpha)m}\right).$$

We know that $\|\nu\| \le d(1+2^{(\gamma_1+\frac{3}{4}-\alpha)m})$. Hence

$$\|D_{\Omega^2}W_\delta\nu\| \ge (1-2^{-\gamma_4 m})\|\nu\|,$$

where $\gamma_4 > 0$ has been chosen such that

$$\sup|\pi_y(\Omega^2_{ij})| \cdot \sup|\pi_x(\Omega^1_{ij})| \cdot 2^{(\gamma_1+\frac{3}{4})m} \cdot \frac{m^2N_\delta}{\delta} + 2^{(\frac{3}{4}+\gamma_1-\gamma_3-\alpha)m} \le 2^{-\gamma_4 m}.$$

■

Remark 11. It follows from Lemma 4.2 and Remark 10 that we can choose the constant γ_4 to be $0 < \gamma_4 < \frac{1}{4} - \gamma_1 < \frac{1}{4}$.

Proposition 7.1. *Let Υ be a chain of partitions associated to the sequence $\eta \in \Sigma_\delta$. Let $\Omega^1 = \Upsilon^k$ and $\Omega^2 = \Upsilon^{k+1}$ be two consecutive partitions from the chain Υ. Let $\xi \stackrel{\text{def}}{=} \sigma^{2m(k-1)}\eta$ Consider a linear operator $\mathcal{A}\colon \mathfrak{X}_{\Omega_1} \to \mathfrak{X}_{\Omega_2}$, approximating the operator $P^2_{\xi*}$, defined according to (34). Let Ω^3 be another partition of the class $\mathcal{G}(m,\delta)$.*

$$D_{\Omega^3}W_{\frac{\delta}{m}}\mathcal{A}\colon \overline{\text{Cone}\,(1,\Omega^1)} \to \text{Cone}\left(2^{-\gamma_4 m},\Omega^3\right).$$

(See p. 8 for definition of a cone and the chain Υ.)

Proof. According to Theorem 3 p. 45, $\mathcal{A}\colon \text{Cone}\,(1,\Omega^1) \to \text{Cone}\left(2^{(\frac{3}{4}+\gamma_1-\alpha)m},\Omega^2\right)$. We may write then

$$\mathcal{A}\nu = 2^{2m}\left(\begin{smallmatrix}0\\d\end{smallmatrix}\right)\chi_\Box + \psi,\ \psi \in \mathfrak{X}_{\Omega_2},\ \|\psi\|_2 \le d2^{(2\frac{3}{4}+\gamma_1-\alpha)m},\ \sum_\Box \psi^{ij}_u = 0.$$

By straightforward calculation

$$D_{\Omega^3}W_{\frac{\delta}{m}}\mathcal{A}\nu = 2^{2m}D_{\Omega^3}W_{\frac{\delta}{m}}\left(\begin{smallmatrix}0\\d\end{smallmatrix}\right)\chi_\Box + D_{\Omega^3}W_{\frac{\delta}{m}}\psi.$$

Using Lemma 7.1

$$\|D_{\Omega^3}W_{\frac{\delta}{m}}\psi - W_{\frac{\delta}{m}}\psi\|_3 \leq 2^{-\gamma_3 m}\|\psi\|_2 \leq d2^{(2\frac{3}{4}+\gamma_1-\gamma_3-\alpha)m}.$$

Thus introducing γ_4 defined by Lemma 7.4 and using Lemma 7.3,

$$\|D_{\Omega^3}W_{\frac{\delta}{m}}\psi\|_3 \leq \|W_{\frac{\delta}{m}}\psi\|_3 + \|D_{\Omega^3}W_{\frac{\delta}{m}}\psi - W_{\frac{\delta}{m}}\psi\|_3 \leq$$
$$\leq d2^{(\gamma_1+2\frac{3}{4})m}\sup|\pi_y(\Omega^3_{ij})|\cdot\sup|\pi_x(\Omega^2_{ij})|\cdot m^2\frac{N_\delta}{\delta} + d2^{(2\frac{3}{4}+\gamma_1-\gamma_3-\alpha)m} \leq d2^{(2-\gamma_4)m}. \quad (111)$$

By Lemma 7.2 we deduce

$$\|D_{\Omega^3}W_{\frac{\delta}{m}}\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\chi_\square - \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\chi_\square\|_3 \leq 2^{-m/4}$$

Thus we may conclude

$$d2^{2m}D_{\Omega^3}W_{\frac{\delta}{m}}\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\chi_\square = d2^{2m}\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\chi_\square + \varphi \in \mathfrak{X}_{\Omega^3},$$

where $\|\varphi\|_3 \leq d2^{3m/2}$. Together with (111) we get the result. ■

Theorem 4. *Let Ω be a partition of $\mathbb{R}^2$ of the class $\mathcal{G}(m,\delta)$; and let $\|\xi\|_\infty \leq \delta$ be a sequence of real numbers. There exists $r_1(m) \ll r_2(m)$ and $\varepsilon_1(m) \ll \varepsilon_2(m)$ such that*

$$W_{\frac{\delta}{2m}}P^2_{\xi*}W_{\frac{\delta}{2m}} : \mathrm{Cone}\,(r_1,\varepsilon_1,\Omega) \to \mathrm{Cone}\,(r_2,\varepsilon_2,\Omega) \subsetneq \mathrm{Cone}\,(r_1,\varepsilon_1,\Omega)\,.$$
$$\left\|W_{\frac{\delta}{2m}}P^2_{\xi*}W_{\frac{\delta}{2m}}\,|_{\mathrm{Cone}(r_1,\varepsilon_1,\Omega)}\right\| \geq 2^{m-5}$$

(See p. 8 for definition of a cone in the space of vector fields).

Proof. Let Ω^1 be a canonical partition for the map P^2_ξ. First of all we shall find a number r_1 such that for any $\eta \in \mathrm{Cone}\,(r_1,\Omega)$ we have $D_{\Omega_1}W_{\frac{\delta}{2m}}\eta \in \mathrm{Cone}\,(1,\Omega^1)$. We may write $\eta = \left(\begin{smallmatrix}0\\d\end{smallmatrix}\right)\chi_\square + \psi$, with $\sum_\square \psi^{ij}_y = 0$ and $\|\psi\|_\Omega \leq dr_1$. Then

$$D_{\Omega^1}W_{\frac{\delta}{2m}}\eta = \left(\begin{smallmatrix}0\\d\end{smallmatrix}\right)D_{\Omega^1}W_{\frac{\delta}{2m}}\chi_\square + D_{\Omega^1}W_{\frac{\delta}{2m}}\psi;$$

and using Lemmas 7.1 and 7.3, we calculate

$$\|D_{\Omega^1}W_{\frac{\delta}{2m}}\psi\|_1 \leq \|W_\delta\psi\|_1 + \|D_{\Omega^1}W_{\frac{\delta}{2m}}\psi - W_{\frac{\delta}{2m}}\psi\|_1 \leq$$
$$\leq \left(2^{-\gamma_3 m} + 2^{2-2m}m^4\frac{N_\delta}{\delta^2}\right)\|\psi\|_\Omega \leq 5dr_1m^4 2^{-2m}\frac{N_\delta}{\delta^2}; \quad (112)$$

Using Lemma 7.2, we calculate

$$\|D_{\Omega^1}W_{\frac{\delta}{2m}}\chi_\square - \chi_\square\|_1 \leq 2^{1-m/4}, \quad (113)$$

which implies $D_{\Omega^1} W_\delta \binom{0}{d}\chi_\Box = \binom{0}{d}\chi_\Box + \psi_1$, where $\psi_1 \in \mathfrak{X}_{\Omega^1}$ and $\|\psi_1\|_1 \le 2^{1-m/4}$. Hence $D_{\Omega^1} W_{\frac{\delta}{2m}} \eta = \binom{0}{d}\chi_\Box + D_{\Omega^1} W_{\frac{\delta}{2m}} \chi_\Box + \psi_1$, where

$$\|D_{\Omega^1} W_{\frac{\delta}{2m}} \chi_\Box + \psi_1\|_1 \le d r_1 \Big(m^4 2^{-2m} \frac{N_\delta}{\delta^2} + 2^{1-m/4} \Big).$$

In order to guarantee $D_{\Omega_1} W_{\frac{\delta}{2m}} \eta \in \mathrm{Cone}\,(1, \Omega^1)$ it is sufficient to choose r_1 such that

$$m^4 2^{-2m} \frac{N_\delta}{\delta^2} \le \frac{1}{r_1}.$$

We set

$$r_1 \overset{\mathrm{def}}{=} \frac{2^{2m}\delta^2}{4m^4 N_\delta}. \tag{114}$$

We can also notice using Lemma 7.1 that

$$\|D_{\Omega^1} W_{\frac{\delta}{2m}} \eta - W_{\frac{\delta}{2m}} \eta\|_1 \le d r_1 2^{-\gamma_3 m}.$$

Taking into account $D_{\Omega^1} W_{\frac{\delta}{2m}} \eta \in \mathrm{Cone}\,(1, \Omega^1)$ we deduce $W_{\frac{\delta}{2m}} \eta \in \widehat{\mathrm{Cone}}\,(1, r_1 2^{-\gamma_3 m}, \Omega^1)$. We also observe that by Lemma 7.3 for any $v = \eta + g \in \widehat{\mathrm{Cone}}\,(r_1, \varepsilon_1, \Omega)$ we have

$$\big\|W_{\frac{\delta}{2m}} g\big\| \le 4\varepsilon_1 m^2 \frac{N_\delta}{2^{2m}\delta^2} = \frac{16\varepsilon_1}{m^2 r_1} =: \widetilde{\varepsilon_1}.$$

We will be assuming that $\widetilde{\varepsilon_1} \ge r_1 2^{-\gamma_3 m}$. Then without loss of generality

$$W_{\frac{\delta}{2m}} \colon \widehat{\mathrm{Cone}}\,(r_1, \varepsilon_1, \Omega) \to \widehat{\mathrm{Cone}}\,(1, \widetilde{\varepsilon_1}, \Omega)\,. \tag{115}$$

Let $\mathcal{A} \colon \mathfrak{X}_{\Omega^1} \to \mathfrak{X}_{\Omega^2}$ be a linear operator approximating $P^2_{\xi*}$ and defined by (34), p. 15. It follows from Theorem 3 p. 45, that $\mathcal{A} \colon \mathrm{Cone}\,(1, \Omega^1) \to \mathrm{Cone}\,\Big(2^{(\frac{3}{4}+\gamma_1-\alpha)m}, \Omega^1\Big) \subset \mathfrak{X}_{\Omega^2}$; moreover, the norm is growing exponentially with number of iterations $\|\mathcal{A}\,|_{\mathrm{Cone}(1,\Omega^1)}\,\| \ge 2^{2m-1}$. In particular, we see that for any vector field $\nu \in \mathrm{Cone}\,(1, \Omega^1)$,

$$\|\mathcal{A}\nu\|_2 = \|\mathcal{A}(\tbinom{0}{d}\chi_\Box + \psi)\|_2 \ge d\|\mathcal{A}\tbinom{0}{1}\chi_\Box\|_2 - \|\mathcal{A}\psi\|_2 \ge d2^{2m}\big(1 - 2^{(\gamma_1+\frac{3}{4}-\alpha)m}\big)$$

Consider a vector field $v = \nu + g \in \widehat{\mathrm{Cone}}\,(1, \widetilde{\varepsilon_1}, \Omega^1)$, where $\nu \in \mathrm{Cone}\,(1, \Omega^1) \subset \mathfrak{X}_{\Omega^1}$ is a piecewise constant part with the norm $\|\nu\|_1 \le d$ and $\|g\|_1 < \widetilde{\varepsilon_1} d$. Then by linearity $P^2_{\xi*} v = P^2_{\xi*}\nu + P^2_{\xi*} g$. By inequality (6.17) of Lemma 6.17,

$$\|P^2_{\xi*} g\|_\Omega \le m 2^{2m+2} \|g\|_1 \le m d \widetilde{\varepsilon_1} 2^{2m+2}. \tag{116}$$

By Proposition 6.3 for $\nu \in \mathrm{Cone}\,(1, \Omega^1) \subset \mathfrak{X}_{\Omega^1}$

$$\|W_{\frac{\delta}{2m}} (P^2_{\xi*} - \mathcal{A})\nu\|_\Omega \le 8 \frac{\sup \mathrm{diam}(\Omega_{ij})}{\delta} 2^{2m} \|\nu\|_1 \le d 2^{m+4}\delta. \tag{117}$$

We have decomposition

$$W_{\frac{\delta}{2m}} P^2_{\xi*} v = W_{\frac{\delta}{2m}} P^2_{\xi*}\nu + W_{\frac{\delta}{2m}} P^2_{\xi*} g = W_{\frac{\delta}{2m}} (P^2_{\xi*} - \mathcal{A})\nu + W_{\frac{\delta}{2m}} \mathcal{A}\nu + W_{\frac{\delta}{2m}} P^2_{\xi*} g. \tag{118}$$

We write $W_{\frac{\delta}{2m}}\mathcal{A}\nu$ and $W_{\frac{\delta}{2m}}P^2_{\xi*}u$ as a sum of piecewise-constant part and a remainder

$$W_{\frac{\delta}{2m}}\mathcal{A}\nu=\nu_1+g_1,\ \text{where } \nu_1=D_\Omega W_{\frac{\delta}{2m}}\mathcal{A}\nu\in\mathfrak{X}_\Omega,\ \text{and } g_1=W_{\frac{\delta}{2m}}\mathcal{A}\nu-D_\Omega W_{\frac{\delta}{2m}}\mathcal{A}\nu; \tag{119}$$

$$W_{\frac{\delta}{2m}}P^2_{\xi^*}g=\nu_2+g_2,\ \text{where } \nu_2=D_\Omega W_{\frac{\delta}{2m}}P^2_{\xi^*}g\in\mathfrak{X}_\Omega,\ \text{and } g_2=W_{\frac{\delta}{2m}}P^2_{\xi^*}g-D_\Omega W_{\frac{\delta}{2m}}P^2_{\xi^*}g. \tag{120}$$

We estimate all four terms separately.

Using Lemmas 7.1 and 6.18, since $\|\nu\|_1\le d$, we get

$$\|g_1\|_\Omega=\|W_{\frac{\delta}{2m}}\mathcal{A}\nu-D_\Omega W_{\frac{\delta}{2m}}\mathcal{A}\nu\|_\Omega\le 2^{-\gamma_3 m}\|\mathcal{A}\nu\|_\Omega\le d2^{(2-\gamma_3)m}. \tag{121}$$

By Lemmas 7.1 and 6.18, using $\|g\|_1\le d\widetilde{\varepsilon}_1$, and (116)

$$\|g_2\|_\Omega=\|W_{\frac{\delta}{2m}}P^2_{\xi^*}g-D_\Omega W_{\frac{\delta}{2m}}P^2_{\xi^*}g\|_\Omega\le 2^{-\gamma_3 m}\|P^2_{\xi^*}g\|_\Omega\le md\widetilde{\varepsilon}_1 2^{(2-\gamma_3)m+2}. \tag{122}$$

Finally, using (116) and (122),

$$\begin{aligned}\|\nu_2\|_\Omega=\|D_\Omega W_{\frac{\delta}{2m}}P^2_{\xi^*}g\|_\Omega\le\|P^2_{\xi^*}g\|_\Omega+\|W_{\frac{\delta}{2m}}P^2_{\xi^*}g-D_\Omega W_{\frac{\delta}{2m}}P^2_{\xi^*}g\|_\Omega\le\\ \le md\widetilde{\varepsilon}_1 2^{2m+2}\big(1+2^{-\gamma_3 m}\big).\end{aligned} \tag{123}$$

We now need a lower bound for the norm of ν_1 defined by (119). By Theorem 3 p. 45 we have $\mathcal{A}\nu\in\text{Cone}\left(2^{(\frac{3}{4}+\gamma_1-\alpha)m},\Omega^1\right)$, and Lemma 7.4 is applicable:

$$\|\nu_1\|_\Omega=\|D_\Omega W_{\frac{\delta}{2m}}\mathcal{A}\nu\|_\Omega\ge(1-2^{-\gamma_4 m})\cdot\|\mathcal{A}\nu\|_2\ge d2^{2m}\big(1-2^{(\frac{3}{4}+\gamma_1-\alpha)m}\big)(1-2^{-\gamma_4 m}). \tag{124}$$

We need to check that

$$\nu_1+\nu_2=D_\Omega W_{\frac{\delta}{2m}}\mathcal{A}\nu+D_\Omega W_{\frac{\delta}{2m}}P^2_{\xi^*}g\in\text{Cone}\,(r_2,\Omega)\,; \tag{125}$$

and to verify the inequality

$$\|g_1\|_\Omega+\|g_2\|_\Omega+\|W_{\frac{\delta}{2m}}(P^2_{\xi*}-\mathcal{A})\nu\|_\Omega\le\|\nu_1+\nu_2\|_\Omega\cdot\varepsilon_2. \tag{126}$$

Consider a vector field $\nu=\binom{0}{d}\chi_\square+\psi\in\text{Cone}\,(1,\Omega^1)$ with $\|\psi\|_1\le d$ and $\sum_\square\psi^{ij}_u=0$. Using Theorem 3 p. 45 we write $\mathcal{A}\nu=d2^{2m}\binom{0}{1}\chi_\square+\varphi$, where $\varphi\in\mathfrak{X}_{\Omega^2}$, and $\|\varphi\|_2\le 2^{(2\frac{3}{4}+\gamma_1-\alpha)m}$. For the first inclusion (125), we expand $D_\Omega W_{\frac{\delta}{2m}}\mathcal{A}\nu$ as following.

$$\begin{aligned}D_\Omega W_{\frac{\delta}{2m}}\mathcal{A}\nu&=D_\Omega W_{\frac{\delta}{2m}}(d2^{2m}\tbinom{0}{1}\chi_\square+\varphi)=d2^{2m}D_\Omega W_{\frac{\delta}{2m}}\tbinom{0}{1}\chi_\square+D_\Omega W_{\frac{\delta}{2m}}\varphi=\\&=d2^{2m}\tbinom{0}{1}\chi_\square+d2^{2m}\Big(D_\Omega W_{\frac{\delta}{2m}}\tbinom{0}{1}\chi_\square-W_{\frac{\delta}{2m}}\tbinom{0}{1}\chi_\square+W_{\frac{\delta}{2m}}\tbinom{0}{1}\chi_\square-\tbinom{0}{1}\chi_\square\Big)+\\&\qquad+(D_\Omega W_{\frac{\delta}{2m}}\varphi-W_{\frac{\delta}{2m}}\varphi)+W_{\frac{\delta}{2m}}\varphi.\end{aligned}$$

We see that by Lemma 7.2

$$d2^{2m}\|D_\Omega W_{\frac{\delta}{2m}}(\begin{smallmatrix}0\\1\end{smallmatrix})\chi_\square - W_{\frac{\delta}{2m}}(\begin{smallmatrix}0\\1\end{smallmatrix})\chi_\square\|_\Omega \le d2^{\frac{7}{4}m}; \tag{127}$$

$$d2^{2m}\|W_{\frac{\delta}{2m}}(\begin{smallmatrix}0\\1\end{smallmatrix})\chi_\square - (\begin{smallmatrix}0\\1\end{smallmatrix})\chi_\square\|_\Omega \le d2^{\frac{7}{4}m}. \tag{128}$$

By Lemma 7.1 again, since $\|\varphi\|_2 \le d2^{(2\frac{3}{4}+\gamma_1-\alpha)m}$

$$\|D_\Omega W_{\frac{\delta}{2m}}\varphi - W_{\frac{\delta}{2m}}\varphi\|_\Omega \le d2^{(2\frac{3}{4}+\gamma_1-\gamma_3-\alpha)m}. \tag{129}$$

Therefore we may write

$$D_\Omega W_{\frac{\delta}{2m}}\mathcal{A}\nu = d2^{2m}(\begin{smallmatrix}0\\1\end{smallmatrix})\chi_\square + \varphi \in \mathfrak{X}_\Omega, \tag{130}$$

where

$$\phi = d2^{2m}(D_\Omega W_{\frac{\delta}{2m}}(\begin{smallmatrix}0\\1\end{smallmatrix})\chi_\square - (\begin{smallmatrix}0\\1\end{smallmatrix})\chi_\square) + D_\Omega W_{\frac{\delta}{2m}}\varphi \in \mathfrak{X}_\Omega;$$

with the norm that can be bounded using (127), (128) and (129)

$$\begin{aligned}\|\phi\|_\Omega \le d2^{2m}\|D_\Omega W_{\frac{\delta}{2m}}(\begin{smallmatrix}0\\1\end{smallmatrix})\chi_\square - (\begin{smallmatrix}0\\1\end{smallmatrix})\chi_\square\|_\Omega + \|D_\Omega W_{\frac{\delta}{2m}}\varphi - W_{\frac{\delta}{2m}}\varphi\|_\Omega + \|W_{\frac{\delta}{2m}}\varphi\|_\Omega \le \\ \le d\Big(2^{\frac{7}{4}m+1} + 2^{(\gamma_1+2\frac{3}{4}-\alpha)m}\cdot\Big(2^{-\gamma_3 m} + \sup|\pi_y(\Omega^2_{kl})|\cdot\sup|\pi_x(\Omega^1_{ij})|\cdot m^2\frac{N_\delta}{\delta^2}\Big)\Big) \le \\ \le 4d\cdot 2^{(2-\gamma_4)m}.\end{aligned} \tag{131}$$

Thus using (130) and (119), (120), we write

$$D_\Omega W_{\frac{\delta}{2m}}\mathcal{A}\nu + D_\Omega W_{\frac{\delta}{2m}}P^{2m}_{\xi*}g = \nu_1+\nu_2 = d2^{2m}(\begin{smallmatrix}0\\1\end{smallmatrix})\chi_\square + \phi + \nu_2. \tag{132}$$

Then the condition (125): $\nu_1+\nu_2 \in \widehat{\mathrm{Cone}}\,(r_2,\varepsilon_2,\Omega)$ is equivalent to $\|\phi+\nu_2\|_\Omega \le dr_2 2^{2m}$. We see using (131) and (123) that

$$\begin{aligned}\|\phi+\nu_2\|_\Omega \le \|\phi\|_\Omega + \|\nu_2\|_\Omega \le 4d\cdot 2^{(2-\gamma_4)m} + 4dm\widetilde{\varepsilon}_1 2^{2m}\big(1+2^{-\gamma_3 m}\big) = \\ = 4d2^{2m}\big(2^{-\gamma_4 m} + m\widetilde{\varepsilon}_1\big(1+2^{-\gamma_3 m}\big)\big)\end{aligned} \tag{133}$$

Now recall the second inequality (126)

$$\|g_1\|_\Omega + \|g_2\|_\Omega + \|W_{\frac{\delta}{2m}}(P^2_{\xi*} - \mathcal{A})\nu\|_\Omega \le \varepsilon_2\|\nu_1+\nu_2\|_\Omega. \tag{134}$$

We know already from (117), (121) and (122),

$$\|g_1\|_\Omega + \|g_2\|_\Omega + \|W_{\frac{\delta}{2m}}(P^2_{\xi*} - \mathcal{A})\nu\|_\Omega \le d2^{2m}\Big(2^{-\gamma_3 m} + \widetilde{\varepsilon}_1 2^{-\gamma_3 m} + 2^{(\alpha-1)m+1}\Big) \le 3d\widetilde{\varepsilon}_1 2^{(2-\gamma_3)m}.$$

Using (124) and (123), we deduce, taking into account Remark 10 and Remark 11 $\gamma_3 < 1-\alpha$ and $\gamma_4 < \frac{1}{4} - \gamma_1$, and $\alpha = \frac{15}{16}$:

$$\|\nu_1 + \nu_2\| \geq d2^{2m}\big(1 - 2^{(\frac{3}{4}+\gamma_1-\alpha)m}\big)\big(1 - 2^{-\gamma_4 m}\big) - d2^{2m}\widetilde{\varepsilon}_1\big(1 + 2^{-\gamma_3 m}\big) \geq$$
$$\geq d2^{2m}\big(1 - 2^{(\frac{3}{4}+\gamma_1-\alpha)m} - 2^{-\gamma_4 m} - \widetilde{\varepsilon}_1 2^{-\gamma_3 m}\big) \geq d2^{2m}\big(1 - \widetilde{\varepsilon}_1 2^{-\frac{m}{24}}\big) \quad (135)$$

Therefore (125) and (126) would follow from

$$3\widetilde{\varepsilon}_1 2^{-\gamma_3 m} \leq \varepsilon_2\big(1 - \widetilde{\varepsilon}_1 2^{-\frac{m}{24}}\big) \quad (136)$$

$$2^{-\gamma_4 m} + \widetilde{\varepsilon}_1 + \widetilde{\varepsilon}_1 2^{-\gamma_3 m} < r_2. \quad (137)$$

Recall now that $\widetilde{\varepsilon}_1 = 4\varepsilon_1 m^2 \frac{N_\delta}{2^{2m}\delta^2}$. We may choose the following parameters for the cones $r_2 = 2^{-m\frac{1-\alpha}{4}} = 2^{-\frac{m\alpha}{64}}$, $\varepsilon_1 = 2^{-m\frac{1-\alpha}{2}} = 2^{-\frac{m\alpha}{32}}$, and $\varepsilon_2 = 2^{-2m\frac{1-\alpha}{2}} = 2^{-\frac{m\alpha}{16}}$. It is clear that $r_2 \ll r_1 = \frac{2^{2m}\delta^2}{4m^4 N_\delta}$ and the second condition on the norm follows immediately from (132), (133), and (134).

■

The proof of the existence of an invariant cone is complete. The fast dynamo theorem in dimension two follows as shown in Section 5.

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