FAST DYNAMO ON THE REAL LINE

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Abstract. In this paper we show that a piecewise expanding map on the interval, extended to the real line by a non-expanding map satisfying some mild hypthesis does induce a fast dynamo action on the functions on the real line in the sense that there exist a $\mathcal{L}_1$ function whose norm grows exponentially under induced action. This is the first step towards a solution of the kinematic fast dynamo problem.

1. Introduction

In this work we establish the fast dynamo theorem for the induced action on vector fields on the unstable manifold of the Poincaré map of the provisional flow. The unstable manifold is one dimensional and the settings are the following. Vector fields on a one-dimensional real manifold may be identified with functions $\mathbb{R} \to \mathbb{R}$; and an induced action on vector fields on $\mathbb{R}$ is given by a transfer operator $(f_* v)(y) = \sum_{x \in f^{-1}(y)} \mathrm{d}f(x) v(x)$.

Theorem 1 (Fast dynamo on $\mathbb{R}$). *There exist a measure-preserving piecewise-C^2 transformation $f\colon \mathbb{R} \to \mathbb{R}$ and an essentially bounded, absolutely integrable vector field v such that*

$$\lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n} \ln \left\| (\exp(\delta\Delta) f_*)^n v \right\|_{\mathcal{L}_1} > 0,$$

The map f may be realised as an induced action on the unstable manifold by the Poincaré map of the provisional fluid flow.

After introducing necessary tools, we deduce Theorem 1 in the Section 3 from two technical results: the noise Lemma 3.1 and the invariant cone Theorem 5. In the Section 4 we fix technical notation; and the last two Sections 5 and 6 are dedicated to the proof of Theorem 5.

2. Basic constructions

In this Section we introduce objects central for our investigations: small random perturbations of a dynamical system and a norm in the space of vector fields. We also specify the type of cones in the space vector fields we are interested in.

2.1. **Small random perturbations.** We construct a random dynamical system using skew-products. Let X be a real manifold and let $f\colon X \to X$ be a transformation. We consider its extension

$$\widehat{f}\colon X \times \mathbb{R}^n \to X \qquad \widehat{f}(x,\xi) \stackrel{\text{def}}{=} f(x) + \xi(1). \tag{1}$$

Let $\Sigma \subset \ell_\infty(\mathbb{R}^n)$ be a shift-invariant subset of two-sided bounded sequences of vectors in $\mathbb{R}^n$. We introduce a skew product over the Bernoulli shift

$$\sigma \times \widehat{f}\colon \Sigma \times X \to \Sigma \times X \qquad (\sigma \times \widehat{f})(\xi, z) \stackrel{\text{def}}{=} (\sigma(\xi), \widehat{f}(z, \xi(1))). \tag{2}$$

The induced transformation on fibers we denote by

$$f_\xi\colon X \to X, \qquad f_\xi(z) \stackrel{\text{def}}{=} \widehat{f}(z, \xi(1)). \tag{3}$$

Its iterations are given by

$$f_\xi^k(z) \stackrel{\text{def}}{=} \widehat{f}(f_\xi^{k-1}(z), \xi(k)). \tag{4}$$

Remark 1. The following identities follow from the definition of the map f_ξ.

$$f_\xi^k = f_{\xi(k)} \circ f_{\xi(k-1)} \circ \ldots \circ f_{\xi(1)}; \tag{5}$$

$$f_\xi^{-k} = (f_\xi^k)^{-1} = f_{\xi(1)}^{-1} \circ f_{\xi(2)}^{-1} \circ \ldots \circ f_{\xi(k)}^{-1}; \tag{6}$$

$$f_\xi^{n-k} = f_\xi^n \circ f_\xi^{-k} = f_\xi^{-k} \circ f_\xi^n = \begin{cases} f_{\sigma^n(\xi)}^{n-k}, & \text{if } n < k; \\ f_{\sigma^k(\xi)}^{n-k}, & \text{if } n > k. \end{cases} \tag{7}$$

Definition 1. We call the map f_ξ *a random perturbation* of the map f associated to the sequence $\xi \in \Sigma$.

2.2. **Norm in the space of vector fields.** Piecewise constant vector fields are proved to be very useful to us. We define a norm in the space of essentially bounded and absolutely integrable vector fields Φ, using partitions.

Definition 2. A norm in the space of essentially bounded and absolutely integrable functions, associated to a partition $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$ of $\mathbb{R}$ is given by

$$\|f\|_\Omega = \max\Big(\sum_{j\in\mathbb{Z}} \frac{2^{-m}}{|\Omega_j|} \int_{\Omega_j} |f(x)| \mathrm{d}x, 2^{-m/2} \sup |f|\Big). \tag{8}$$

The first term we refer to as the weighted $\mathcal{L}_1$-norm and write

$$\|f\|_{\Omega,\mathcal{L}_1} := \sum_{j\in\mathbb{Z}} \frac{2^{-m}}{|\Omega_j|} \int_{\Omega_j} |f(x)| \mathrm{d}x,$$

it depends, of course, on the partition chosen.

The subspace of Φ, consisting of piecewise constant vector fields associated to the partition Ω we denote by Φ_Ω. Observe that for any step function $\phi = \sum\limits_{j\in\mathbb{Z}} c_j \chi_{\Omega_j} \in \Phi_\Omega$ we have that

$$\|\phi\|_\Omega = \max\Big(2^{-m}\sum_{j\in\mathbb{Z}}|c_j|, 2^{-m/2}\sup|c_j|\Big). \tag{9}$$

2.3. **Cones in vector fields on $\mathbb{R}$.** We reserve a notation for a cone of radius r with the main axis $\chi_{[-1,1]}$ in the space Φ_Ω of piecewise constant functions, associated to a partition Ω:

$$\mathrm{Cone}\,(r,\Omega) \stackrel{\mathrm{def}}{=} \Big\{\eta = d\chi_{[-1,1]} + \varphi \mid \varphi = \sum_{j\in\mathbb{Z}} c_j\chi_{\Omega_j};\ \sum_{j=N_l}^{N_r} c_j = 0;\ \|\varphi\|_\Omega \le dr\Big\}. \tag{10}$$

We extend the cone $\mathrm{Cone}\,(r,\Omega)$ to include general functions from the main space:

$$\widehat{\mathrm{Cone}}\,(r,\varepsilon,\Omega) \stackrel{\mathrm{def}}{=} \Big\{f = \eta + g, \mid \eta\in\mathrm{Cone}\,(r,\Omega)\,,\ \|g\|_\Omega \le \varepsilon\|\eta\|_\Omega\Big\}. \tag{11}$$

We say that the cone $\widehat{\mathrm{Cone}}\,(r_1,\varepsilon_1,\Omega^1)$ is smaller than the cone $\widehat{\mathrm{Cone}}\,(r_2,\varepsilon_2,\Omega^2)$ and write $\widehat{\mathrm{Cone}}\,(r_1,\varepsilon_1,\Omega^1) \ll \widehat{\mathrm{Cone}}\,(r_2,\varepsilon_2,\Omega^2)$, if $r_1 > r_2$ and $\varepsilon_1 > \varepsilon_2$; we do not assume here that $\widehat{\mathrm{Cone}}\,(r_1,\varepsilon_1,\Omega^1) \cap \widehat{\mathrm{Cone}}\,(r_2,\varepsilon_2,\Omega^2) \ne \varnothing$.

3. Fast dynamo theorem in dimension one

In this Section we give a proof of our main result, Theorem 1. The argument has three steps. The first step is the noise Lemma 3.1, which suggests to replace the operator $(\exp(\delta\Delta)f_*)^n$ in our considerations with the operator $\exp(\delta\Delta)f_{t*}^n$ for some sequence t. The second step is to choose a large $m \gg 1$ and to construct explicitly an invariant cone for the operator $\exp(\frac{\delta}{2m}\Delta)f_{t*}^m\exp(\frac{\delta}{2m}\Delta)$ with $t\in\ell_\infty(\mathbb{R})$, $\|t\|\le\delta$. The third step is to deduce the fast dynamo theorem from the existence of an invariant cone.

An instant proof of the noise Lemma 3.1 is given in this Subsection. The invariant cone Theorem 5 for a specific map is proved in the Section 6 after the preparatory Section 5.

We begin with a simple observation that the exponent of the Laplacian operator[1], is the convolution with the Gaussian kernel, in particular

$$\exp(\delta\Delta)v = w_\delta * v,\ \text{where } w_\delta(x) = \frac{1}{\sqrt{2\pi}\delta}\exp\big(-\frac{x^2}{2\delta^2}\big).$$

[1] $\Delta\colon v\to \mathrm{d}^2 v$ in the case of the real line

The latter operator is also known as *the Weiertstrass transform* $W_\delta(v) \stackrel{\text{def}}{=} (w_\delta * v)$; this notation we use throughout.

The following statement is generally known, but we give a proof for completeness.

Lemma 3.1 (Noise Lemma)**.** *For any map $f\colon \mathbb{R} \to \mathbb{R}$ and for any function v we have*

$$W_{\frac{\delta}{2}} f_*(W_\delta f_*)^{m-1} v(x) = \int_{\mathbb{R}^{m-1}} w_\delta(t_1) w_\delta(t_2) \dots w_\delta(t_{m-1}) (W_{\frac{\delta}{2}} f^m_{\overline{0t}*} v)(x) \mathrm{d}t_1 \mathrm{d}t_2 \dots \mathrm{d}t_{m-1}, \tag{12}$$

where $\overline{0t} = (0, t_1, t_2, \dots, t_{m-1}) \in \mathbb{R}^m$.

Proof. Observe that $f^{-1}(x-t) = f_t^{-1}(x)$, because $f_t(x) = f(x) + t$. By straightforward calculation,

$$\begin{aligned} W_{\frac{\delta}{2}} f_*(W_\delta f_*)^{m-1} v(x) &= W_{\frac{\delta}{2}} f_*(W_\delta f_*)^{m-2} W_\delta f_* v(x) = \\ &= W_{\frac{\delta}{2}} f_*(W_\delta f_*)^{m-2} \int_{\mathbb{R}} w_\delta(t)(f_* v)(x-t) \mathrm{d}t = \\ &= W_{\frac{\delta}{2}} f_*(W_\delta f_*)^{m-2} \int_{\mathbb{R}} w_\delta(t_1)(f_{t_1 *} v)(x) \mathrm{d}t_1 = \dots = \\ &= W_{\frac{\delta}{2}} \int_{\mathbb{R}^{m-1}} w_\delta(t_1) \dots w_\delta(t_{m-1})(f_* f_{t_1 *} \dots f_{t_{m-1}*} v)(x) \mathrm{d}t_1 \dots \mathrm{d}t_{m-1} = \\ &= \int_{\mathbb{R}^{m-1}} w_\delta(t_1) \dots w_\delta(t_{m-1})(W_{\frac{\delta}{2}} f^m_{\overline{0t}*} v)(x) \mathrm{d}t_1 \dots \mathrm{d}t_{m-1}. \end{aligned}$$

■

Let $s_2 \le 2 \le s_1$, be two real numbers such that $\log \frac{s_1}{s_2} = \varkappa \ll 1$; and let $\delta = 2^{-m\alpha}$ be a small real number with $\frac{15}{16} \le \alpha < 1$. Consider the map

$$\ell(x) = \begin{cases} s_1 x + s_1 - 1, & \text{if } -1 < x < \frac{2}{s_1} - 1; \\ s_2 x + 1 - s_2, & \text{if } \frac{2}{s_1} - 1 < x < 1; \\ -x, & \text{otherwise}; \end{cases} \tag{13}$$

and define its extension $\widehat{\ell}\colon \mathbb{R}^2 \to \mathbb{R}$ by $\widehat{\ell}(x,y) = \ell(x) + y$. We then associate a small perturbation ℓ^m_ξ to any sequence $\xi \in \ell_\infty(\mathbb{R})$ and $\|\xi\|_\infty \le \delta$.

The existence of an invariant cone for the operator $W_{\frac{\delta}{2m}} \ell^m_{\xi *} W_{\frac{\delta}{2m}}$ is established in the end of Section 6 in the following

Theorem 5. (*Invariant cone.*) For any sequence $\xi \in \ell_\infty(\mathbb{R})$ with $\|\xi\| \le \delta$ there exists an $m \gg 1$, a partition $\Omega(m)$, and four numbers $r_2(m) \lll r_1(m)$; $\varepsilon_2(m) \lll \varepsilon_1(m) \lll 1$

such that

$$W_{\frac{\delta}{2m}} \ell_{\xi*}^m W_{\frac{\delta}{2m}} : \overline{\widehat{\mathrm{Cone}}\,(r_1, \varepsilon_1, \Omega)} \to \widehat{\mathrm{Cone}}\,(r_2, \varepsilon_2, \Omega) \subset \widehat{\mathrm{Cone}}\,(r_1, \varepsilon_1, \Omega)\,. \tag{106}$$

$$\forall f \in \overline{\widehat{\mathrm{Cone}}\,(r_1, \varepsilon_1, \Omega)} \colon \|W_{\frac{\delta}{2m}} \ell_{\xi*}^m W_{\frac{\delta}{2m}} f\|_\Omega \geq 2^{m-2} \|f\|_\Omega. \tag{107}$$

We choose $\delta = 2^{-m\alpha}$, the partition $\Omega_m = \bigcup_j \big[2^{-m}j, 2^{-m}(j+1)\big]$, and fix four dimension parameters of two cones $r_1 = \frac{\delta s_2^m}{4mN_\delta}$, $r_2 = \delta^{\frac{1}{64}}$, $\varepsilon_1 = \delta^{\frac{1}{32}}$ and $\varepsilon_2 = \delta^{\frac{1}{24}}$ such that the Theorem holds true.

Lemma 3.2. *In the notations introduced above, for any* $f \in \overline{\widehat{\mathrm{Cone}}\,(r_1, \varepsilon_1, \Omega_m)}$

$$\int_{[-\delta,\delta]^{m-1}} w_{\frac{\delta}{m}}(t_1) \dots w_{\frac{\delta}{m}}(t_{m-1}) (W_{\frac{\delta}{2m}} \ell_{t*}^m W_{\frac{\delta}{2m}} f) \mathrm{d}t_1 \dots \mathrm{d}t_{m-1} \in \widehat{\mathrm{Cone}}\,\left(e^2 r_2, e^2 \varepsilon_2, \Omega_m\right). \tag{14}$$

Proof. By Theorem 5 we know that for any $|t| \in [-\delta, \delta]^m$ and any $f \in \widehat{\mathrm{Cone}}\,(r_1, \varepsilon_1, \Omega_m)$

$$W_{\frac{\delta}{m}} \ell_{t*}^m W_{\frac{\delta}{2m}} f = d\chi_{[-1,1]} + \psi_t + g_t \in \widehat{\mathrm{Cone}}\,(r_2, \varepsilon_2, \Omega_m)\,,$$

where $\psi_t \in \Phi_{\Omega_m}$, $\|\psi_t\|_{\Omega_m} \leq dr_2$ and $\|g_t\|_{\Omega_m} \leq d\varepsilon_2$. Observe that Ω_m is independent on t. Therefore,

$$\int_{[-\delta,\delta]^{m-1}} w_{\frac{\delta}{m}}(t_1) \dots w_{\frac{\delta}{m}}(t_{m-1}) \chi_{[-1,1]} \mathrm{d}t_1 \dots \mathrm{d}t_{m-1} =$$

$$= \chi_{[-1,1]} \Big(\int_{-\delta}^{\delta} w_{\frac{\delta}{m}}(t) \mathrm{d}t \Big)^{m-1} = \chi_{[-1,1]} \Big(1 - \frac{2}{m}\Big)^{m-1} \geq e^{-2} \chi_{[-1,1]}, \tag{15}$$

for m large enough. Since $\psi_t \in \Phi_{\Omega_m}$ for any $t \in [-\delta, \delta]^m$,

$$\int_{[-\delta,\delta]^m} w_{\frac{\delta}{m}}(t_1) \dots w_{\frac{\delta}{m}}(t_{m-1}) \psi_t \mathrm{d}t_1 \dots \mathrm{d}t_{m-1} \in \Phi_{\Omega_m},$$

and we calculate Ω_m-norm.

$$\Big\| \int_{[-\delta,\delta]^{m-1}} w_{\frac{\delta}{m}}(t_1) \dots w_{\frac{\delta}{m}}(t_{m-1}) \psi_t \mathrm{d}t_1 \dots \mathrm{d}t_{m-1} \Big\|_{\Omega_m} \leq$$

$$\leq \sum_{j \in \mathbb{Z}} \frac{2^{-m}}{|\Omega_{mj}|} \int_{[-\delta,\delta]^{m-1}} w_{\frac{\delta}{m}}(t_1) \dots w_{\frac{\delta}{m}}(t_{m-1}) \Big(\int_{\Omega_{mj}} |\psi_t(x)| \mathrm{d}x \Big) \mathrm{d}t_1 \dots \mathrm{d}t_{m-1} \leq$$

$$\leq \sup_t \|\psi_t\|_{\Omega_m} \leq dr_2.$$

Similarly,

$$\Big\| \int_{[-\delta,\delta]^{m-1}} w_{\frac{\delta}{m}}(t_1) \dots w_{\frac{\delta}{m}}(t_{m-1}) g_t \mathrm{d}t_1 \dots \mathrm{d}t_{m-1} \Big\|_{\Omega_m} \leq d\varepsilon_2.$$

Observe that

$$\int_{-1}^{1}\int_{[-\delta,\delta]^{m-1}} w_{\frac{\delta}{m}}(t_1)\dots w_{\frac{\delta}{m}}(t_{m-1})\psi_t(x)\mathrm{d}t_1\dots\mathrm{d}t_{m-1}\mathrm{d}x =$$

$$= \int_{[-\delta,\delta]^{m-1}} w_{\frac{\delta}{m}}(t_1)\dots w_{\frac{\delta}{m}}(t_{m-1})\mathrm{d}t_1\dots\mathrm{d}t_{m-1}\cdot\int_{-1}^{1}\psi_t(x)\mathrm{d}x = 0.$$

Summing up, for any $f\in\overline{\widehat{\mathrm{Cone}}\,(\varepsilon_1,r_1,\Omega_m)}$

$$\int_{[-\delta,\delta]^{m-1}} w_{\frac{\delta}{m}}(t_1)\dots w_{\frac{\delta}{m}}(t_{m-1})W_{\frac{\delta}{2m}}\ell_{t*}^m W_{\frac{\delta}{2m}} f\mathrm{d}t_1\dots\mathrm{d}t_{m-1}\in\widehat{\mathrm{Cone}}\left(e^2r_2,e^2\varepsilon_2,\Omega_m\right).$$

■

Lemma 3.3. *In the notations introduced above,*

$$W_{\frac{\delta}{2m}}\ell_*(W_\delta\ell_*)^{m-1}W_{\frac{\delta}{2m}}:\overline{\widehat{\mathrm{Cone}}\,(r_1,\varepsilon_1,\Omega_m)}\to\widehat{\mathrm{Cone}}\,(r_2,\varepsilon_2,\Omega_m)\subsetneq\widehat{\mathrm{Cone}}\,(r_1,\varepsilon_1,\Omega_m);$$

$$\|W_{\frac{\delta}{2m}}\ell_*(W_\delta\ell_*)^{m-1}W_{\frac{\delta}{2m}}\;|_{\widehat{\mathrm{Cone}}(r_1,\varepsilon_1,\Omega_m)}\|\geq 2^{m-5}$$

Proof. By Lemma 3.1 for any $f\in\mathrm{Cone}\,(r_1,\Omega_m)$

$$W_{\frac{\delta}{2m}}\ell_*(W_{\frac{\delta}{m}}\ell_*)^{m-1}W_{\frac{\delta}{2m}}f =$$

$$= \int_{\mathbb{R}^{m-1}} w_{\frac{\delta}{m}}(t_1)\dots w_{\frac{\delta}{m}}(t_{m-1})(W_{\frac{\delta}{2m}}\ell_{t*}^m W_{\frac{\delta}{2m}}f)\mathrm{d}t_1\mathrm{d}t_2\dots\mathrm{d}t_{m-1} =$$

$$= \Big(\int_{\mathbb{R}^{m-1}\setminus[-\delta,\delta]^{m-1}}+\int_{[-\delta,\delta]^{m-1}}\Big)\prod_{j=1}^{m-1} w_{\frac{\delta}{m}}(t_j)(W_{\frac{\delta}{2m}}\ell_{t*}^m W_{\frac{\delta}{2m}}f)\mathrm{d}\bar{t}. \quad (16)$$

By Lemma 3.2 we know that for any $f\in\widehat{\mathrm{Cone}}\,(r_1,\varepsilon_1,\Omega_m)$

$$\int_{[-\delta,\delta]^{m-1}}\prod_{j=1}^{m-1} w_{\frac{\delta}{m}}(t_j)(W_{\frac{\delta}{2m}}\ell_{t*}^m W_{\frac{\delta}{2m}}f)\mathrm{d}\bar{t}\in\widehat{\mathrm{Cone}}\left(e^2r_2,e^2\varepsilon_2,\Omega_m\right).$$

We estimate the first term

$$\Big\|\int_{\mathbb{R}^{m-1}\setminus[-\delta,\delta]^{m-1}}\prod_{j=1}^{m-1} w_{\frac{\delta}{m}}(t_j)(W_{\frac{\delta}{2m}}\ell_{t*}^m W_{\frac{\delta}{2m}}f)\mathrm{d}\bar{t}\Big\|_{\Omega_m}\leq$$

$$\leq\sum_{j\in\mathbb{Z}}\frac{2^{-m}}{|\Omega_{m_j}^{\,3}|}\int_{\mathbb{R}^{m-1}\setminus[-\delta,\delta]^{m-1}}\prod_{j=1}^{m-1} w_{\frac{\delta}{m}}(t_j)\Big(\int_{\Omega_{m_j}^{\,3}}\Big|W_{\frac{\delta}{2m}}\ell_{t*}^m W_{\frac{\delta}{2m}}f(x)\Big|\mathrm{d}x\Big)\mathrm{d}\bar{t}\leq$$

$$\leq\sup_t\|W_{\frac{\delta}{2m}}\ell_{t*}^m W_{\frac{\delta}{2m}}f\|_{\Omega_m}\int_{\mathbb{R}^{m-1}\setminus[-\delta,\delta]^{m-1}}\prod_{j=1}^{m-1} w_{\frac{\delta}{m}}(t_j)\mathrm{d}\bar{t}.$$

We shall find an upper bound for the integral:

$$\int_{\mathbb{R}^{m-1}\setminus[-\delta,\delta]^{m-1}} \prod_{j=1}^{m-1} w_{\frac{\delta}{m}}(t_j)\mathrm{d}\bar{t} \leq$$

$$\leq 2^m \Big(\int_{\delta}^{+\infty} w_{\frac{\delta}{m}}(t)\mathrm{d}t\Big)^m + 2m\int_{\delta}^{+\infty}\int_{[-\delta,\delta]^{m-1}} w_{\frac{\delta}{m}}(t_1)\dots w_{\frac{\delta}{m}}(t_{m-1})\mathrm{d}t_1\dots\mathrm{d}t_{m-1} \leq$$

$$\leq 2^m e^{-m^2} + 2me^{-m}.$$

We may also observe that for any $f \in \widehat{\mathrm{Cone}}\,(r_1, \varepsilon_1, \Omega_m)$,

$$\sup_t \|W_{\frac{\delta}{2m}} \ell_{t*}^m W_{\frac{\delta}{2m}} f\|_{\Omega_m} \leq \Big(\frac{N_\delta}{\delta s_2^m}\Big)^2 \|\ell_{t*}^m f\|_{\Omega_m} \leq$$

$$\leq \Big(\frac{N_\delta}{\delta s_2^m}\Big)^2 \frac{2^{-m}}{\inf|\Omega_{mj}|}\|\ell_{t*}^m f\|_{\mathcal{L}_1} \leq \Big(\frac{N_\delta}{\delta s_2^m}\Big)^2 2^m \cdot \frac{s_1^m}{s_2^m}\|f\|_{\Omega_m} \leq \frac{2^m s_1^m N_\delta^2}{s_2^{3m}\delta^2}\|f\|_{\Omega_m}.$$

We put the last two together and we see that

$$\sup_t \|W_{\frac{\delta}{2m}} \ell_{t*}^m W_{\frac{\delta}{2m}} f\|_{\Omega_m} \int_{\mathbb{R}^{m-1}\setminus[-\delta,\delta]^{m-1}} \prod_{j=1}^{m-1} w_{\frac{\delta}{m}}(t_j)\mathrm{d}\bar{t} \leq$$

$$\leq \Big(\frac{N_\delta}{\delta s_2^m}\Big)^2 2^m \cdot \frac{s_1^m}{s_2^m}\|f\|_{\Omega_m} \leq \frac{2^m s_1^m N_\delta^2}{s_2^{3m}\delta^2} \cdot 2me^{-m}\|f\|_{\Omega_m}.$$

We need to verify

$$\frac{2^m s_1^m N_\delta^2}{s_2^{3m}\delta^2} \cdot 2me^{-m} \ll 2^m\varepsilon_2, \text{ where } \varepsilon_2 = \delta^{\frac{1}{24}}.$$

It is equivalent to

$$\frac{s_1 2^{2(1+\alpha-\alpha\log_{s_1}2)}}{s_2^3 e} < 2^{-\frac{\alpha}{24}};$$

and holds true for $\varkappa = \log\frac{s_1}{s_2}$ sufficiently small.

For the second inequality we recall Corollary 1 of Theorem 5 again

$$\forall f \in \mathrm{Cone}\,(r_1, \varepsilon_1, \Omega_m) : \|W_{\frac{\delta}{m}} \ell_{\xi*}^m W_{\frac{\delta}{m}} f\|_{\Omega_m} \geq 2^{m-2}\|f\|_{\Omega_m}.$$

Then

$$\Big\|\int_{[-\delta,\delta]^{m-1}} \prod_{j=1}^{m-1} w_{\frac{\delta}{m}}(t_j)(W_{\frac{\delta}{2m}} \ell_{t*}^m W_{\frac{\delta}{2m}} f)\mathrm{d}\bar{t}\Big\|_{\Omega_m} \geq$$

$$\geq \inf_{t\in[-\delta,\delta]^m} \|W_{\frac{\delta}{2m}} \ell_{t*}^m W_{\frac{\delta}{2m}} f\|_{\Omega_m} \cdot \int_{[-\delta,\delta]^{m-1}} \prod_{j=1}^{m-1} w_{\frac{\delta}{m}}(t_j) \geq 2^{m-2}e^{-2}\|f\|_{\Omega_m}. \quad (17)$$

Taking into account

$$\Big\|\int_{\mathbb{R}^{m-1}\setminus[-\delta,\delta]^{m-1}}\prod_{j=1}^{m-1} w_{\frac{\delta}{m}}(t_j)(W_{\frac{\delta}{2m}}\ell_{t*}^m W_{\frac{\delta}{2m}} f)\mathrm{d}\bar{t}\Big\|_{\Omega_m} \le 2^m\varepsilon_2\|f\|_{\Omega_m},$$

we get the result. ■

Theorem 1.[Fast dynamo on $\mathbb{R}$] There exist a measure-preserving piecewise-C^2 transformation $f\colon \mathbb{R}\to\mathbb{R}$ and an essentially bounded, absolutely integrable vector field v such that

$$\lim_{\delta\to 0}\lim_{n\to\infty}\frac{1}{n}\ln\big\|(\exp(\delta\Delta)f_*)^n v\big\|_{\mathcal{L}_1} > 0,$$

The map f may be realised as an induced action of the Poincaré map of the provisional fluid flow on the unstable manifold.

Proof. Let us choose $f=\ell$, where $\ell\colon\mathbb{R}\to\mathbb{R}$ is given by (13). It follows by straightforward calculation that $W_\delta W_\delta = 2W_{2\delta}$ for any number $\delta>0$. The Theorem follows from Lemma 3.3 with $v = W_{\frac{\delta}{2m}}\chi_{[-1,1]}$. ■

The remaining part of the paper is dedicated to the proof of existence of an invariant cone for the operator $W_{\frac{\delta}{2m}}\ell_{t*}^m W_{\frac{\delta}{2m}}$ for arbitrary $\|t\|\le\delta$. Therefore, we assume that a large $m\gg 1$ is fixed.

4. Notation

In this Section we fix notation we use throught the proof of Invariant Cone Theorem 5.

The following letters are reserved for constants: α, β, γ, γ_1, $\varkappa$, s_1, s_2. The admissible range of values will be specified later.

Given a subset $I\subset\mathbb{R}^n$ we denote by $|I|$ its Lebesgue measure. We say that two sets I_1 and I_2 are δ-close and write $|I_1-I_2|<\delta$ if I_1 belongs to the δ-neighbourhood of I_2 or I_2 belongs to the δ-neighbourhood of I_1. Otherwise, we write $|I_1-I_2|>\delta$. The indicator function of a set I we denote by χ_I.

Let δ_{ij} be the Dirac delta function:

$$\delta_{ij}=\begin{cases}1, & \text{if } i=j\\ 0, & \text{otherwise.}\end{cases}$$

The supremum norm of a sequence of real numbers $\xi\in\ell_\infty(\mathbb{R})$ we denote by $\|\xi\| = \sup\limits_{k\in\mathbb{N}}|\xi_k|$. Whenever supremum or infimum are taken along the whole range of values, we omit the range.

Let $\delta = 2^{-m\alpha}$ be a small real number with $\frac{15}{16}<\alpha<1$.

We write $x \ll y$ when x is *exponentially small* compared to y, namely, there exist a small number $0 < \varepsilon < 1$ such that $x < 2^{-\varepsilon m} y$.

Definition 3. We say that a collection of intervals $\Omega = \{\Omega_j\}_{j \in \mathbb{Z}}$ makes *a partition of the class* $\mathcal{G}(m, \delta, s_1, s_2)$, if $\bigcup \overline{\Omega_j} = \mathbb{R}$, $\Omega_i \cap \Omega_j = \varnothing$ if $i \neq j$, and the following conditions hold true.

(1) The interval $[-1, 1]$ contains at least 2^{m-1} and at most 2^m intervals of the partition, and $\{\pm 1\}$ are the end points of some intervals of the partition.
(2) The length of intervals Ω_j is bounded away from zero and from infinity
$$\frac{1}{m s_1^m} \le |\Omega_j| \le 2\Big(\frac{1}{s_1^m} + \frac{1}{s_2^m}\Big).$$
(3) Any interval $I \subset \mathbb{R}$ of the length $|I| = \delta$ contains not more than
$$N_\delta = 2^{m+1} \delta^{\log_{s_1} 2} = 2^{m(1 - \alpha \log_{s_1} 2) + 1}$$
intervals of the partition.
(4) Any interval of the partition $\Omega_j \subset \mathbb{R} \setminus [-1 - m\delta, 1 + m\delta]$ has length $|\Omega_j| = 2^{-m}$.

We write $\mathcal{G}(m, \delta, s_1, s_2)$ to indicate dependence on m, δ, s_1, and s_2; we will abuse notations and omit m, δ, s_1, or s_2, when it leads to no confusion and the dependence is of no importance.

We number intervals of a partition Ω in the natural order, starting from $\Omega_0 \ni 0$. We set Ω_{N_l} to be the most left interval of Ω inside $[-1, 1]$, and Ω_{N_r} to be the most right interval of Ω inside $[-1, 1]$.

Here we deal with essentially bounded absolutely integrable functions on the real line. We refer to the space $\Phi \stackrel{\text{def}}{=} \mathcal{L}_1(\mathbb{R}) \cap \mathcal{L}_\infty(\mathbb{R})$ as the main space. "Any function" refers to a function from the main space always.

Given a partition $\Omega = \{\Omega_j\}_{j \in \mathbb{Z}}$ of the class $\mathcal{G}$, we denote the associated space of step functions by Φ_Ω and address the basis $\{\chi_{\Omega_j}\}_{j \in \mathbb{Z}}$ as the canonical basis of Φ_Ω.

Definition 4. We associate *a weighted transfer operator* f_*, acting on the main space, to a map f on the real line by[1]
$$(f_* \phi)(x) \colon = \sum_{y \in f^{-1}(x)} \operatorname{sgn} \mathrm{d} f(y) \phi(y). \tag{18}$$

[1]Transfer operator is a bounded linear operator. In this case, it is chosen to be one dimensional analogue of induced action on vector fields by area-preserving transformations. Transfer operators with negative coefficients have been considered, for instance, in [15].

5. Transfer operator as a dynamo operator

In this section we first show that any generalised toy dynamo operator has an invariant cone. Afterwards, we prove that there exists a map $\ell\colon \mathbb{R} \to \mathbb{R}$ such that for any small perturbation ℓ^m_ξ with $\|\xi\|_\infty \le \delta$ we can find a linear operator $\mathcal{A}\colon \Phi \to \Phi$ and two partitions Ω^1 and Ω^2 of $\mathbb{R}$ such that $\mathcal{A}\colon \Phi_{\Omega^1} \to \Phi_{\Omega^2}$ and $\|(\ell^m_{\xi*}-\mathcal{A})W_\delta\| \le 2^{-\gamma m}(\|\ell^m_{\xi*}\|+\|\mathcal{A}\|)$ for some $\gamma > 0$. Moreover, the matrix of $\mathcal{A}\mid_{\Phi_{\Omega^1}}$ satisfies conditions (D1)–(D4). In other words, for any sequence $\|\xi\|_\infty \le \delta$, the operator $\ell^m_{\xi*}$ may be approximated by a generalised toy dynamo.

5.1. The model matrix.

The plan is to choose suitable subspaces of Φ and approximate the operator $\ell^m_{\xi*}$ by a simple matrix. In this Subsection we describe the matrix we would like to obtain and show that for any matrix $\mathcal{A}$, satisfying these conditions, there exists a pair of cones $C_1, C_2 \subset \Phi$ such that $C_2 \ll C_1$ and $\mathcal{A}(\overline{C_1}) \subset C_2$ (but $C_2 \not\subset C_1$).

Let $\mathcal{A}$ be a linear operator acting on the main space. Assume that there exists two partitions Ω^1, Ω^2 of the class $\mathcal{G}$ such that $\mathcal{A}\colon \Phi_{\Omega^1} \to \Phi_{\Omega^2}$. Here and below we denote by N^1_l and N^1_r the indices of the first and the last intervals of the partition Ω^1 inside $[-1,1]$, respectively; and let N^2_l and N^2_r be the indices of the first and the last intervals of the partition Ω^2 inside $[-1,1]$, respectively. In other words, the sets $\Omega^2_i \times \Omega^1_j$ with $N^2_l \le i \le N^2_r$, and $N^1_l \le j \le N^1_r$ make a partition of the unit square.

We define several sets of indices in order to describe the properties of the operator $\mathcal{A}$ important to us. Let a_{ij} be coefficients of the matrix of $\mathcal{A}$ in the canonical bases of the subspaces Φ_{Ω^1} and Φ_{Ω^2}.

Accelerator:

$$\mathrm{Ar}\colon = \bigl\{ j \in \{N^1_l, \dots, N^1_r\} \mid \#\{i \in \{N^2_l, \dots, N^2_r\} \mid a_{ij} = 1\} \ge 2^m - N_\delta \bigr\}. \tag{19}$$

Inflow diffusion:

$$\mathrm{D_{in}}\colon = \bigl\{ (i,j) \in \{N^2_l, \dots, N^2_r\} \times \{N^1_l, \dots, N^1_r\} \mid a_{ij} \ne 1 \bigr\}. \tag{20}$$

Outflow diffusion:

$$\mathrm{D_{out}}\colon = \{N^2_l - mN_\delta, \dots, N^2_r + mN_\delta\} \times \{N^1_l - mN_\delta, \dots, N^1_r + mN_\delta\} - \\ - \{N^2_l, \dots, N^2_r\} \times \{N^1_l, \dots, N^1_r\}. \tag{21}$$

Indifferent subspace:

$$\mathrm{Sp}\colon = \mathbb{Z}^2 \setminus \{N^2_l - mN_\delta, \dots, N^2_r + mN_\delta\} \times \{N^1_l - mN_\delta, \dots, N^1_r + mN_\delta\}. \tag{22}$$

We are interested in linear operators $\mathcal{A}$ such that the following conditions hold true for the matrix coefficients in the canonical bases.

(D1) $\max|a_{ij}|+1 \le m^2\big(\frac{s_1}{s_2}\big)^m$;
(D2) $\#\mathrm{D}_{\mathrm{in}} \le m\delta s_1^{2m}$;
(D3) for any pair $(i,j)\in \mathrm{Sp}$ we have $a_{ij}=0$ whenever $|i-j|>mN_\delta$;
(D4) $\#\mathrm{Ar} \ge 2^{m-2}$.

Definition 5. We say that a linear operator $\mathcal{A}\colon \mathcal{L}_1(\mathbb{R})\cap\mathcal{L}_\infty(\mathbb{R}) \to \mathcal{L}_1(\mathbb{R})\cap\mathcal{L}_\infty(\mathbb{R})$ is a *generalised toy dynamo* if there exist two partitions Ω^1 and Ω^2 of the class $\mathcal{G}$ such that $\mathcal{A}(\Phi_{\Omega^1})\subset\Phi_{\Omega^2}$ and the conditions (D1)–(D4) hold true in the settings introduced above.

Remark 2. All theorems and the main result hold true for an operator $\mathcal{A}$ that satisfies conditions (D1)–(D4) with right parts of the inequalities multiplied by polynomials in m.

When we have several partitions, e.g. Ω^1, Ω^2, and Ω^3 of the class $\mathcal{G}$ we refer to the norms associated to the partitions by $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_3$, respectively.

We will need the following fact.

Remark 3. For any $s_1\le 2\le s_2$, satisfying $\big(\log s_1-\log s_2\big)\ll 1$, and $\delta=2^{-\alpha m}$ there exists a number $0<\gamma_1=2(1-\alpha)<1/4$ such that

$$m^2\delta\cdot\frac{s_1^{3m}}{2^m s_2^m} < 2^{m\gamma_1}. \tag{23}$$

for m large enough.

Lemma 5.1. *Let $\mathcal{A}\colon\Phi_{\Omega^1}\to\Phi_{\Omega^2}$ be a generalised toy dynamo and let $\phi=\sum\limits_{j\in\mathbb{Z}}c_j\chi_{\Omega^1_j}$ be a step function. Then*

$$\sum_{i=N_l^2}^{N_r^2}\sum_{j=N_l^1}^{N_r^1}|c_j|\cdot|1-a_{ij}| \le 2^{m(3/2+\gamma_1)}\|\phi\|_1.$$

Proof. By straightforward calculation,

$$\sum_{i=N_l^2}^{N_r^2}\sum_{j=N_l^1}^{N_r^1}|c_j|\cdot|1-a_{ij}| = \sum_{(i,j)\in\mathrm{D}_{\mathrm{in}}}|c_j|\cdot|1-a_{ij}| \le \sup|1-a_{ij}|\cdot\#\mathrm{D}_{\mathrm{in}}\cdot\sup|c_j| \le$$

$$\le\frac{s_1^m}{s_2^m}\cdot m^2\delta s_1^{2m}\cdot 2^{m/2}\|\phi\|_1 \le 2^{m(3/2+\gamma_1)}\|\phi\|_1.$$

■

Definition 6. Let Ω^1, Ω^2 be two partitions of the class $\mathcal{G}(m)$. We define *the kernel* of $\mathcal{A}^*\chi_{[-1,1]}\colon \Phi_{\Omega^1} \to \Phi_{\Omega^2}$ to be the set

$$\operatorname{Ker}\mathcal{A}^*\chi_{[-1,1]}\colon = \Big\{\phi \in \Phi_{\Omega^1} \mid \int_{-1}^{1} \mathcal{A}\phi(x)\mathrm{d}x = 0\Big\} =$$

$$= \Big\{\phi = \sum_{j\in\mathbb{Z}} c_j|\Omega_j^1|\colon \sum_{i=N_l^2}^{N_r^2}\sum_{j\in\mathbb{Z}} a_{ij}c_j|\Omega_i^2| = 0\Big\}. \quad (24)$$

Proposition 5.1. *Let* $2^{\gamma_1} < s_2 < 2$. *Then for any two partitions of the class* $\mathcal{G}$ *and a generalised toy dynamo* $\mathcal{A}\colon \Phi_{\Omega^1} \to \Phi_{\Omega^2}$

$$\Phi_{\Omega^1} = \chi_{[-1,1]} \oplus \operatorname{Ker}\mathcal{A}^*\chi_{[-1,1]}.$$

In other words, for any $\phi \in \Phi_{\Omega^1}$ *there exist* $\psi \in \operatorname{Ker}\mathcal{A}^*\chi_{[-1,1]}$ *and* $d \in \mathbb{R}$ *such that*

$$\phi = d\chi_{[-1,1]} + \psi. \quad (25)$$

Proof. Let $\chi_{[-1,1]} = \sum_{j\in\mathbb{Z}} u_j\chi_{\Omega_j^1}$, where $u_j = 1$ for $N_l^1 \le j \le N_r^1$ and $u_j = 0$ otherwise. Let $\phi = \sum_{j\in\mathbb{Z}} c_j\chi_{\Omega_j^1} \in \Phi_{\Omega^1}$ be a step function. We want to find a function $\psi \in \Phi_{\Omega^1}$ such that $\psi \in \operatorname{Ker}\mathcal{A}^*$. By definition of the kernel 6, using (25) we write

$$\int_{-1}^{1} \mathcal{A}\psi(x)\mathrm{d}x = \int_{-1}^{1} \mathcal{A}(\varphi - d\chi_{[-1,1]})(x)\mathrm{d}x = \int_{-1}^{1} \sum_{i,j\in\mathbb{Z}} a_{ij}(c_j - du_j)\chi_{\Omega_i^2}(x)\mathrm{d}x =$$

$$= \sum_{i=N_l^2}^{N_r^2}\sum_{j\in\mathbb{Z}} a_{ij}(c_j - du_j)|\Omega_i^2| = 0.$$

We want to solve the last equality for d. It is sufficient to show that for any generalised toy dynamo $\mathcal{A}$ we have that

$$\sum_{i=N_l^2}^{N_r^2}\sum_{j\in\mathbb{Z}} u_ja_{ij}|\Omega_i^2| \neq 0.$$

By straightforward calculation,

$$\sum_{i=N_l^2}^{N_r^2}\sum_{j\in\mathbb{Z}} u_ja_{ij}|\Omega_i^2| = \sum_{j=N_l^1}^{N_r^1}\sum_{i=N_l^2}^{N_r^2} a_{ij}|\Omega_i^2| = 2(N_r^1 - N_l^1) + \sum_{j=N_l^1}^{N_r^1}\sum_{i=N_l^2}^{N_r^2}(a_{ij} - 1)|\Omega_i^2|.$$

Using conditions (D1)–(D4), we estimate the last term as follows

$$\sum_{j=N_l^1}^{N_r^1} \sum_{i=N_l^2}^{N_r^2} (a_{ij}-1)|\Omega_i^2| = \sum_{\mathrm{D_{in}}} (a_{ij}-1)|\Omega_i^2| \le \#\mathrm{D_{in}} \cdot \sup|a_{ij}-1| \cdot \sup|\Omega_i^2| \le$$

$$\le s_1^{2m} m^2 \delta \cdot \frac{s_1^m}{s_2^m} \cdot 2(s_1^{-m} + s_2^{-m}).$$

We see that $\sum\limits_{i=N_l^2}^{N_r^2} \sum\limits_{j\in\mathbb{Z}} u_j a_{ij} |\Omega_i| \ne 0$ under condition that

$$s_1^{2m} m^2 \delta \cdot \frac{s_1^m}{s_2^m} \cdot 2(s_1^{-m} + s_2^{-m}) < 2(N_r^1 - N_l^1). \tag{26}$$

Recall that, since Ω^1 is of the class $\mathcal{G}$, we have $N_r^1 - N_l^1 > 2^{m-1}$. We also know from (23) that there exists $\gamma_1 < 1/4$ such that

$$m^2\delta \cdot \frac{s_1^{3m}}{2^m s_2^m} < 2^{m\gamma_1}.$$

Therefore (26) holds true under condition that $2^{\gamma_1} < s_2 < 2$. ■

Lemma 5.2. *Let* $\eta = d\chi_{[-1,1]} + \psi \in \Phi_\Omega$ *be a step function such that* $\|\psi\| \le dr$ *for some* $r \ll 1$. *Then* $\eta \in \mathrm{Cone}\left(\frac{2r}{1-2r}, \Omega\right)$.

Proof. We would like to write $\psi = \beta\chi_{[-1,1]} + \widetilde{\psi}$, where $\widetilde{\psi} = \sum\limits_{j\in\mathbb{Z}} \widetilde{c_j}\chi_{\Omega_j}$ and $\sum\limits_{j=N_l}^{N_r} \widetilde{c_j} = 0$. Let us assume that $\psi = \sum\limits_{j\in\mathbb{Z}} c_j\chi_{\Omega_j}$, then

$$\sum_{j=N_l}^{N_r} c_j\chi_{\Omega_j} = \beta\sum_{j=N_l}^{N_r} \chi_{\Omega_j} + \sum_{j=N_l}^{N_r} \widetilde{c_j}\chi_{\Omega_j}.$$

implies $\widetilde{c_j} = c_j - \beta$ and consequently $\sum\limits_{j=N_l}^{N_r} (c_j - \beta) = 0$. Thus we have an upper bound for $|\beta|$:

$$|\beta| = \left|\frac{1}{N_r - N_l} \sum_{j=N_l}^{N_r} c_j\right| \le \frac{1}{2^{m-1}} \sum_{j\in\mathbb{Z}} |c_j| = 2\|\psi\| \le 2dr.$$

Therefore we deduce that

$$\eta = (d+\beta)\chi_{[-1,1]} + \widetilde{\psi} \in C\Big(\frac{|\beta|}{|d|-|\beta|}, \Omega\Big) \subset C\Big(\frac{2r}{1-2r}, \Omega\Big).$$

■

Definition 7. Given Ω^1 and Ω^2, two partitions of the class $\mathcal{G}$, we define *a linear operator* $\mathcal{E}\colon \Phi_{\Omega^1} \to \Phi_{\Omega^2}$ by the matrix

$$E_{ij} = \begin{cases} 1, & \text{if } N_l^2 \le i \le N_r^2 \text{ and } N_l^1 \le j \le N_r^1, \\ \delta_{ij}, & \text{otherwise.} \end{cases} \tag{27}$$

Remark 4. The operator $\mathcal{E}$ is a generalised toy dynamo.

Lemma 5.3. *Consider a function* $\varphi \in \operatorname{Ker} \mathcal{E}^* \chi_{[-1,1]}$. *Then* $\|\mathcal{E}\varphi\|_2 \le \|\varphi\|_1$.

Proof. Let $\varphi \in \Phi_{\Omega^1}$ be a step function. We may write $\varphi = \sum\limits_{j\in\mathbb{Z}} c_j \chi_{\Omega_j^1}$, then

$$\mathcal{E}\varphi = \sum_{j=N_l^1}^{N_r^1} c_j \chi_{[-1,1]} + \Big(\sum_{j<N_l^1} + \sum_{j>N_r^1}\Big) c_j \chi_{\Omega_j^2};$$

and the condition $\varphi \in \operatorname{Ker} \mathcal{E}^* \chi_{[-1,1]}$ implies $\sum\limits_{j=N_l^1}^{N_r^1} c_j = 0$. Therefore

$$\|\mathcal{E}\varphi\|_2 = \Big\| \Big(\sum_{j<N_l^1} + \sum_{j>N_r^1}\Big) c_j \chi_{\Omega_j^2} \Big\|_2 = \Big(\sum_{j<N_l^1} + \sum_{j>N_r^1}\Big) |c_j| \le \|\varphi\|_1.$$

■

Proposition 5.2. *Let* s_1 *be small enough so that* $\log_2 s_1 \le 64/63$. *Let* Ω^1 *and* Ω^2 *be partitions of the class* $\mathcal{G}$. *Consider a generalised toy dynamo operator* $\mathcal{A}\colon \Phi_{\Omega^1} \to \Phi_{\Omega^2}$. *Then for any* $\phi \in \Phi_{\Omega^1}$

$$\|(\mathcal{A} - \mathcal{E})\phi\|_2 \le 2^{m(1/2+\gamma_1)} \|\phi\|_1,$$

where γ_1 *satisfies the inequality* (23).

Proof. Let $\phi = \sum_{j\in\mathbb{Z}} c_j \chi_{\Omega_j^1} \in \Phi_{\Omega^1}$ be a step function with the unit norm

$$\|\phi\|_1 = \max\big(2^{-m} \sum_{j\in\mathbb{Z}} |c_j|, 2^{-m/2}, \sup |c_j|\big) = 1,$$

which implies $\sum_{j\in\mathbb{Z}} |c_j| \le 2^m$ and $\sup |c_j| \le 2^{m/2}$. By straightforward calculation,

$$\begin{aligned}(\mathcal{A} - \mathcal{E})\phi &= \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} c_j (a_{ij} - E_{ij}) \chi_{\Omega_i^2} = \\ &= \sum_{i=N_l^2}^{N_r^2} \sum_{j=N_l^1}^{N_r^1} c_j (a_{ij} - 1) \chi_{\Omega_i^2} + \sum_{\mathrm{D_{out}}} c_j (a_{ij} - \delta_{ij}) \chi_{\Omega_i^2} + \sum_{\mathrm{Sp}} c_j (a_{ij} - \delta_{ij}) \chi_{\Omega_i^2}.\end{aligned}$$

Observe that

$$\#\mathrm{D_{out}} \le \big(4m^2N_\delta^2 + 2mN_\delta(N_r^2 - N_l^2) + 2mN_\delta(N_r^1 - N_l^1)\big) = \\ = 2mN_\delta(2mN_\delta + N_r^2 - N_l^2 + N_r^1 - N_l^1).$$

Therefore, using $\|\phi\|_1 \le 1$, Lemma 5.1, definition of the set $\mathrm{D_{out}}$, and condition (D3),

$$\begin{aligned}
&\|(\mathcal{A} - \mathcal{E})\phi\|_{\mathcal{L}_1,\Omega^2} \le \\
&\quad \le 2^{-m}\Big(\sum_{i=N_l^2}^{N_r^2}\sum_{j=N_l^1}^{N_r^1} |c_j|\cdot|a_{ij}-1| + \sum_{\mathrm{D_{out}}} |c_j|\cdot|a_{ij}-\delta_{ij}| + \sum_{\mathrm{Sp}} |c_j|\cdot|a_{ij}-\delta_{ij}|\Big) \le \\
&\quad \le 2^{-m}\big(2^{m(3/2+\gamma_1)} + \sup|c_j|\cdot\sup|a_{ij}|\cdot\#\mathrm{D_{out}} + \sum_{\mathbb{Z}} |c_j|\cdot mN_\delta\cdot\sup|a_{ij}|\big) \le \\
&\quad \le 2^{m(1/2+\gamma_1)} + 2^{-m/2}\frac{s_1^m}{s_2^m}\cdot 2mN_\delta(2mN_\delta + N_r^2 - N_l^2 + N_r^1 - N_l^1) + mN_\delta\frac{s_1^m}{s_2^m}.
\end{aligned}$$

By straightforward calculation we see that for s_1 small enough so that $\log_2 s_1 \le 64/63$

$$\frac{s_1^m}{s_2^m}\cdot mN_\delta < \frac{s_1^m}{s_2^m}\cdot m2^{m(1-\alpha\log_{s_1}2)} \le \frac{s_1^m}{s_2^m}\cdot\frac{\delta s_1^{2m}}{2^m} = 2^{m\gamma_1}.$$

Therefore, under the same condition, since $N_\delta \ll 2^m$,

$$2^{-m/2}\cdot\frac{s_1^m}{s_2^m}\cdot m^2N_\delta^2 < 2^{m\gamma_1}\cdot mN_\delta\cdot 2^{-m/2} < 2^{m(1/2+\gamma_1)}.$$

Finally,

$$2^{1-m/2}mN_\delta\cdot\frac{s_1^m}{s_2^m}\cdot\big(N_r^2 - N_l^2 + N_r^1 - N_l^1\big) \le \frac{s_1^m}{s_2^m}\cdot 2^{m/2}\cdot 4mN_\delta < 2^{m(1/2+\gamma_1)}.$$

Summing up,

$$\|(\mathcal{A} - \mathcal{E})\phi\|_2 \le 3\cdot 2^{m(1/2+\gamma_1)}.$$

Now, for the maximum norm, we have that

$$\|(\mathcal{A} - \mathcal{E})\phi\|_\infty \le \max_{x\in\mathbb{R}}\sum_{j\in\mathbb{Z}}\sum_{i\in\mathbb{Z}} |c_j|\cdot|a_{ij} - E_{ij}|\chi_{\Omega_i^2}(x) \le \sum_{j\in\mathbb{Z}} |c_j|\frac{s_1^m}{s_2^m} \le 2^m\frac{s_1^m}{s_2^m}.$$

Thus $2^{-m/2}\|(\mathcal{A} - \mathcal{E})\phi\|_\infty \le 2^{m(1/2+\gamma_1)}$. ■

Lemma 5.4. *Let Ω^1 and Ω^2 be two partitions of the class $\mathcal{G}$. Let $\mathcal{E}\colon \Phi_{\Omega^1} \to \Phi_{\Omega^2}$ be a linear operator with the matrix defined by* (27) *in the canonical basis. Then for any function $\phi \in \Phi_{\Omega^1}$*

$$\|\mathcal{E}\phi\|_2 \le 2^m\|\phi\|_1 \quad \text{and} \quad \|\mathcal{E}\chi_{[-1,1]}\|_2 \ge 2^{m-2}.$$

Proof. Let $\phi = \sum_{j\in\mathbb{Z}} c_j\chi_{\Omega^1_j} \in \Phi_{\Omega^1}$ be a step function of the unit norm. Then, by straightforward calculation,

$$\mathcal{E}\phi = \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} c_j E_{ij}\chi_{\Omega^2_i} = \sum_{i=N^2_l}^{N^2_r}\sum_{j=N^1_l}^{N^1_r} c_j\chi_{\Omega^2_i} + \sum_{\mathrm{D_{out}}\cup\mathrm{D_{in}}} c_j\delta_{ij}\chi_{\Omega_i} =$$

$$= \sum_{j=N^1_l}^{N^1_r} c_j\chi_{[-1,1]} + \Big(\sum_{j<N^1_l} + \sum_{j>N^1_r}\Big)c_j\chi_{\Omega_j};$$

so the weighted $\mathcal{L}_1$-norm is

$$\|\mathcal{E}\phi\|_{2,\mathcal{L}_1} = 2^{-m}\cdot\Big|\sum_{j=N^1_l}^{N^1_r} c_j\Big|\cdot(N^2_r - N^2_l) + 2^{-m}\Big(\sum_{j<N^1_l} + \sum_{j>N^1_r}\Big)|c_j| \le 2^m + 1.$$

The upper estimate for the supremum norm is easy:

$$\|\mathcal{E}\phi\|_\infty = \max_{x\in\mathbb{R}}\Big(\sum_{j=N^1_l}^{N^1_r} c_j\chi_{[-1,1]}(x) + \Big(\sum_{j<N^1_l} + \sum_{j>N^1_r}\Big)c_j\chi_{\Omega_j}(x)\Big) \le 2^m.$$

Hence $\|\mathcal{E}\phi\|_2 \le 2^m\|\phi\|$. Obviously,

$$\|\mathcal{E}\chi_{[-1,1]}\|_2 \ge \|\mathcal{E}\phi\|_{2,\mathcal{L}_1} = 2^{-m}(N^1_r - N^1_l)(N^2_r - N^2_l) \ge 2^{m-2}.$$

■

Let us consider two cones $\mathrm{Cone}\,(1,\Omega^1) \subset \Phi_{\Omega^1}$ and $\mathrm{Cone}\,\big(2^{(\gamma_1-1/2)m},\Omega^2\big) \subset \Phi_{\Omega^2}$ in correspondence with general definition p. 3:

$$\mathrm{Cone}\,\big(1,\Omega^1\big) \stackrel{\mathrm{def}}{=} \Big\{\phi = d\chi_{[-1,1]} + \psi \mid \psi = \sum_{j\in\mathbb{Z}} c_j\chi_{\Omega^1_j};\ \sum_{j=N_l}^{N_r} c_j = 0;\ \|\psi\|_1 \le d\Big\}; \tag{28}$$

$$\mathrm{Cone}\,\big(2^{(\gamma_1-1/2)m},\Omega^2\big) \stackrel{\mathrm{def}}{=}$$

$$\Big\{\phi = d\chi_{[-1,1]} + \psi \mid \psi = \sum_{j\in\mathbb{Z}} c_j\chi_{\Omega^2_j}; \sum_{j=N_l}^{N_r} c_j = 0; \|\psi\|_2 \le d2^{m(\gamma_1-1/2)}\Big\}. \tag{29}$$

Theorem 2. *Assume that m is large enough so that the inequality* (23) *holds true for some $0 < \gamma_1 < 1/4$ and all sufficiently small $\varkappa$. Additionally, assume that $\log_2 s_1 \le 64/63$. Let Ω^1 and Ω^2 be two partitions of the class $\mathcal{G}$. Then for any generalised toy dynamo $\mathcal{A}\colon \Phi_{\Omega^1} \to \Phi_{\Omega^2}$ we have $\mathcal{A}\colon \overline{\mathrm{Cone}\,(1,\Omega^1)} \to \mathrm{Cone}\,\big(2^{m(\gamma_1-1/2)},\Omega^2\big)$; Moreover, for any $\eta \in \mathrm{Cone}\,(1,\Omega^1)$ we have $\|\mathcal{A}\eta\|_2 \ge (N^2_r - N^1_l)\|\eta\| \ge 2^{m-1}\|\eta\|$.*

Proof. Let $\phi \in \mathrm{Cone}\,(1, \Omega^1)$ be a step function, $\phi = d\chi_{[-1,1]} + \psi$, where $\psi = \sum_{j\in\mathbb{Z}} c_j \chi_{\Omega^1_j}$, with $\|\psi\|_1 \le d$ and $\sum_{j=N_l^1}^{N_r^1} c_j = 0$. We may write

$$\mathcal{A}\phi = (\mathcal{A}-\mathcal{E})\phi + \mathcal{E}\phi = d\mathcal{E}\chi_{[-1,1]} + (\mathcal{A}-\mathcal{E})\phi + \mathcal{E}\psi.$$

Obviously, $\|\phi\|_1 \le 2d$, thus by Proposition 5.2

$$\|(\mathcal{A}-\mathcal{E})\phi\|_2 \le \|\mathcal{A}-\mathcal{E}\| \cdot \|\phi\|_1 \le d2^{m(1/2+\gamma_1)+1}.$$

By Lemma 5.3, $\|\mathcal{E}\psi\|_2 \le \|\psi\|_1 = d$. Therefore $\|(\mathcal{A}-\mathcal{E})\phi + \mathcal{E}\psi\|_2 \le d2^{m(1/2+\gamma_1)+1} + d$, so we conclude

$$\mathcal{A}\phi = \tilde{d}\chi_{[-1,1]} + d(\mathcal{A}-\mathcal{E})\chi_{[-1,1]} + (\mathcal{A}-\mathcal{E})\psi + \mathcal{E}\psi,$$

where $\tilde{d} \ge d2^{m-2}$ and $\|d(\mathcal{A}-\mathcal{E})\chi_{[-1,1]} + (\mathcal{A}-\mathcal{E})\psi + \mathcal{E}\psi\|_2 \le d(2^{m(1/2+\gamma_1)+1}+1)$. Theorem now follows from Lemma 5.2.

■

5.2. **A dynamo map.** Recall the map we considered is Section 3 Let $s_2 \le 2 \le s_1$, be two real numbers such that $\log \frac{s_1}{s_2} = \varkappa \ll 1$. and let $\delta = 2^{-m\alpha}$ be a small real number with $\frac{15}{16} \le \alpha < 1$.

We extend an expanding map on the interval $[-1,1]$ to the line $\mathbb{R}^2$ as follows (13)

$$\ell(x) = \begin{cases} s_1 x + s_1 - 1, & \text{if } -1 < x < \frac{2}{s_1} - 1; \\ s_2 x + 1 - s_2, & \text{if } \frac{2}{s_1} - 1 < x < 1; \\ -x, & \text{otherwise.} \end{cases} \tag{30}$$

and define its extension $\widehat{\ell}\colon \mathbb{R}^2 \to \mathbb{R}$ by $\widehat{\ell}(x,y) = \ell(x) + y$. We associate a small perturbation ℓ_ξ to any sequence $\xi \in \ell_\infty(\mathbb{R})$ and $\|\xi\|_\infty \le \delta$.

We associate a transfer operator to a map ℓ_ξ^m by

$$(\ell^m_{\xi *}\phi)(x) \colon = \sum_{y \in \ell_\xi^{-m}(x)} \operatorname{sgn} \mathrm{d}\ell_\xi^m(y)\phi(y). \tag{31}$$

The map ℓ outside the unit interval is not important to us and we chose a simple map that changes direction of the vector field, to make it non-trivial. The exact form is not relevant here.

First of all we show that with any sequence ξ we can associate a (canonical) partition of the class $\mathcal{G}$. Then we approximate the operator $\ell^m_{\xi *}$ by a generalised toy dynamo (Theorem 3 on p. 31).

5.3. **Canonical partition for the perturbation ℓ_ξ^m.** In this section we construct a partition of the class $\mathcal{G}(m)$ associated to the sequence ξ. Later we will refer to it as the canonical partition of the map ℓ_ξ^m.

Recall Definition 3 of the partition $\mathcal{G}$:

Definition 3. We say that a collection of intervals $\Omega = \{\Omega_j\}_{j\in\mathbb{Z}}$ makes *a partition of the class* $\mathcal{G}(m,\delta,s_1,s_2)$, if $\bigcup \overline{\Omega_j} = \mathbb{R}$, $\Omega_i \cap \Omega_j = \varnothing$ if $i \neq j$, and the following conditions hold true.

(1) The interval $[-1,1]$ contains at least 2^{m-1} and at most 2^m intervals of the partition, and $\{\pm 1\}$ are the end points of some intervals of the partition.
(2) The length of intervals Ω_j is bounded away from zero and from infinity
$$\frac{1}{ms_1^m} \leq |\Omega_j| \leq 2\Big(\frac{1}{s_1^m} + \frac{1}{s_2^m}\Big).$$
(3) Any interval $I \subset \mathbb{R}$ of the length $|I| = \delta$ contains not more than
$$N_\delta = 2^{m+1}\delta^{\log_{s_1} 2} = 2^{m(1-\alpha \log_{s_1} 2)+1}$$
intervals of the partition.
(4) Any interval of the partition $\Omega_j \subset \mathbb{R} \setminus [-1-m\delta, 1+m\delta]$ has length $|\Omega_j| = 2^{-m}$.

We fix s_1 and s_2 in the definition of the map ℓ (13) and a sequence $\xi \in \ell_\infty(\mathbb{R})$ with a norm $\|\ell\| \leq \delta = 2^{-m\alpha}$.

The map ℓ_ξ is piecewise linear and so any iteration is a such. Let
$$-\infty = a_0^{(k)} < a_1^{(k)} < \ldots < a_{N_k+1}^{(k)} = +\infty \tag{32}$$
be all the points of discontinuity of the map ℓ_ξ^k. Define the corresponding partition of the real line $\mathfrak{a}^{(k)} = \bigcup_{j=0}^{N_k} \overline{\mathfrak{a}_j^{(k)}}$, where $\mathfrak{a}_j^{(k)} = (a_j^{(k)}, a_{j+1}^{(k)})$ are the partition intervals. Observe that for any k we have that $\{\pm 1\}$ are the endpoints of some intervals of the partition. Let $a_{l_k}^{(k)} = -1$ and $a_{r_k}^{(k)} = 1$.

We shall modify the partition $\mathfrak{a}^{(m)}$ and obtain the canonical partition for the map ℓ_ξ^m.

Definition 8. We call a branch $\ell_\xi^n(\mathfrak{a}_j^{(n)})$ of the map ℓ_ξ^n *main*, if for any $0 < k < n$ we have that $\ell_\xi^k(\mathfrak{a}_j^{(n)}) \subset [-1,1]$.

Definition 9. We call a main branch $\ell_\xi^k(\mathfrak{a}_j^{(k)})$ of the map ℓ_ξ^k *long*, if
$$\big|\ell_\xi^k(\mathfrak{a}_j^{(k)}) \cap [-1,1]\big| > \frac{2}{s_2}.$$

Lemma 5.5. *The map ℓ_ξ^m has at most $2^{m(1-\alpha_1)+1}$ main branches that are not long, where $\alpha_1 < \frac{\alpha}{\log_2 s_1}$ is chosen such that $s_1^{\alpha_1} < 2^\alpha$.*

Proof. Let $\mathfrak{a}_j^{(m)}$ be a domain of a main branch which is not long, that is $|\ell_\xi^m(\mathfrak{a}_j^{(m)}) \cap [-1,1]| < \frac{2}{s_2}$. Since $\ell_\xi^{(m)}(\mathfrak{a}_j^{(m)})$ is an interval, a connected subset of $\mathbb{R}$, we conclude $|\ell_\xi^m(a_j{}^{(m)})+1| > 1-\frac{1}{s_2}$ or $|\ell_\xi^m(a_{j+1}{}^{(m)}) - 1| > 1 - \frac{1}{s_2}$. Without loss of generality we may assume that the first holds true. By definition, $a_j^{(m)}$ is a point of discontinuity. Therefore, for some $k < m$ we have that $\ell_\xi^k(a_j^{(m)}) = -1+\xi(k)$; hence we deduce that $\ell_\xi^{(m)}(a_j^{(m)}) = \ell_\xi^{m-k}(-1+\xi(k))$. So we conclude $|\ell_\xi^{m-k}(-1+\xi(k))+1| > 1 - \frac{1}{s_2}$, and, consequently, $k < m(1-\alpha_1)+1$. Indeed, if $k > m(1-\alpha_1)+1$, then $m-k < m\alpha_1 - 1$, and it follows that

$$|\ell_\xi^{m-k}(-1+\xi(k))+1| < s_1^{m\alpha_1-1}\delta < \frac{2}{s_1} = 2 - \frac{2}{s_2}.$$

Since the map ℓ_ξ^k has at most 2^k main branches, we conclude that there are at most $2^{m(1-\alpha_1)+1}$ points $a_j^{(m)}$ such that $\ell_\xi^k(a_j^{(m)}) = -1+\xi(k)$.

Summing up, the map ℓ_ξ^m has at most $2^{m(1-\alpha_1)+1}$ main branches that are not long. ∎

Lemma 5.6. *Let $1 \le k \le m\alpha \log_{s_1} 2$ and let (a,b) be the domain of a main branch of the map ℓ_ξ^k. Then*

$$|\ell_\xi^k(a)+1| < \delta\frac{s_1^k-1}{s_1-1} < \frac{2}{s_1} - \delta;$$
$$|\ell_\xi^k(b)-1| < \delta\frac{s_2^k-1}{s_2-1} < \frac{2}{s_1} - \delta.$$

Proof. By induction in k. The case $k=1$ is obvious. Recall that $\mathfrak{a}^{(k)} \subset \mathfrak{a}^{(k+1)}$ and

$$\mathfrak{a}^{(k+1)} \setminus \mathfrak{a}^{(k)} = \{\ell_\xi^{-k}(-1), \ell_\xi^{-k}(1), \ell_\xi^{-k}(2/s_1-1)\}.$$

Therefore for $x \in \mathfrak{a}^{(k)}$ we have

$$\ell_\xi^{k+1}(x) = \ell_{\sigma^k(\xi)}\ell_\xi^k(x) = s_1\ell_\xi^k(x) + s_1 - 1 - \xi(k+1), \quad \text{if } |\ell_\xi^k(x)+1| < 2/s_1 - \delta$$
$$\ell_\xi^{k+1}(x) = \ell_{\sigma^k(\xi)}\ell_\xi^k(x) = s_2\ell_\xi^k(x) - s_2 + 1 - \xi(k+1), \quad \text{if } |\ell_\xi^k(x)-1| < 2/s_1 - \delta$$

In the first case we know that, by induction assumption,

$$|\ell_\xi^{k+1}(x)+1| \le s_1|\ell_\xi^k(x)+1| + |\xi(k+1)| + 1 \le \frac{s_1^{k+1}-1}{s_1-1}\delta.$$

In the second case,

$$|\ell_\xi^{k+1}(x)-1| \le s_2|\ell_\xi^k(x)-1| + |\xi(k+1)| + 1 \le \frac{s_2^{k+1}-1}{s_2-1}\delta.$$

■

Corollary 1. *Let $1 \le k \le m\alpha \log_{s_1} 2$. Then for any domain (a,b) of a main branch of the map ℓ_ξ^k we have that $\frac{2}{s_1} - 1 = 1 - \frac{2}{s_2} \in \ell_\xi^k(a,b)$.*

Proof. By assumption, $\ell_\xi^k(a) < \ell_\xi^k(b)$ and from Lemma 5.6 it follows that

$$\ell_\xi^k(a) < \frac{2}{s_1} - \delta - 1 < \frac{2}{s_1} - 1 < 1 - \frac{2}{s_2} + \delta < \ell_\xi^k(b).$$

■

Corollary 2. *Let $1 \le n \le m\alpha \log_{s_1} 2$. Then any main branch of the map ℓ_ξ^n is δ-close to one of the ends of the interval $[-1,1]$: in other words, either $1 - \delta \in \ell_\xi^n(a,b)$ or $\delta - 1 \in \ell_\xi^n(a,b)$, or both.*

Proof. By induction in n. The case $n = 1$ is obvious. Observe that (a,b) cannot be an interval of continuity of the map ℓ_ξ^n for any $k < n$. Therefore ℓ_ξ^{n-1} is either continuous at a, or at b, or at both end points. In any case (a,b) belongs to an interval of continuity of ℓ_ξ^{n-1} satisfying conditions of Corollary 1 of Lemma 5.6. By definition of ℓ_ξ, we see that either $\ell_\xi^{n-1}(a) = \frac{2}{s_1} - 1$ or $\ell_\xi^{n-1}(b) = \frac{2}{s_1} - 1$. Without loss of generality assume that $\ell_\xi^{n-1}(b) = \frac{2}{s_1} - 1$. Then we see that $\ell_\xi^n(a,b) \supset (\xi(n), 1 + \xi(n)) \ni 1 - \delta$. Similarly, $\ell_\xi^{n-1}(a) = \frac{2}{s_1} - 1$ implies that $\ell_\xi^n(a,b) \supset (-1 + \xi(n), \xi(n)) \ni \delta - 1$. ■

Lemma 5.7. *The map ℓ_ξ^k for any $1 \le k \le m\alpha \log_{s_1} 2$ has exactly 2^k long branches.*

Proof. By induction in k. The case $k = 1$ is trivial. It follows from Lemma 5.6 and Corollary 1 of Lemma 5.6 that any long branch of the map ℓ_ξ^k contains at least two long branches of the map ℓ_ξ^{k-1}. ■

Corollary 1. *The map ℓ_ξ^m has at least 2^{m-2} long branches, provided $2\alpha \log_{s_1} 2 > 1$.*

Proof. If $2\alpha \log_{s_1} 2 > 1$, then $m - m\alpha \log_{s_1} 2 < m\alpha \log_{s_1} 2$ and therefore the map $P_\eta^{m - m\alpha \log_{s_1} 2}$ has at least $2^{m - m\alpha \log_{s_1} 2}$ long branches for any $\eta \in \ell_\infty(\mathbb{R})$ with $\|\eta\| \le \delta$. Let $\eta = \sigma^{m\alpha \log_{s_1} 2}\xi$. Then we can decompose $\ell_\xi^m = \ell_\eta^{m(1-\alpha \log_{s_1} 2)} \ell_\xi^{m\alpha \log_{s_1} 2}$. According to Lemma 5.7 the map $\ell_\xi^{m\alpha \log_{s_1} 2}$ has at $2^{m\alpha \log_{s_1} 2}$ long branches. By definition of a long branch, its image is at least $\frac{2}{s_2}$ long; using Corollaries 1 and 2 of Lemma 5.6, we deduce that for any domain (a,b) of a long branch we have that either $(-1+\delta, -1+\frac{2}{s_2}) \subset \ell_\xi^{m \log_{s_1} 2}(a,b)$ or $(1 - \frac{2}{s_2}; 1 - \delta) \subset \ell_\xi^{m \log_{s_1} 2}(a,b)$. Moreover, any of two intervals $(-1, \frac{2}{s_1} - 1)$ and $(\frac{2}{s_1} - 1, 1)$ contains exactly $2^{m(1-\alpha \log_{s_1} 2) - 1}$ long branches of the map $\ell_\xi^{m(1-\alpha \log_{s_1} 2)}$.

We can find an upper bound for the length of a domain of a long branch of the map $\ell_\eta^{m(1-\alpha\log_{s_1}2)}$: it easy to show by induction in number of iterations that any long branch (a,b) has a domain of the length at least

$$|b-a| = (2 - s_1^{m(1-\alpha)\log_{s_1}2}\delta)s_1^{-m(1-\alpha\log_{s_1}2)} = 2s_1^{-m(1-\alpha\log_{s_1}2)} - \delta \geq s_1^{-m(1-\alpha\log_{s_1}2)}$$

Therefore any of the intervals $(-1+\delta, \frac{2}{s_1}-1)$ and $(\frac{2}{s_1}-1, 1-\delta)$ contains at least

$$2^{m(1-\alpha\log_{s_1}2)-1} - s_1^{m(1-\alpha\log_{s_1}2)}\delta = 2^{m(1-\alpha\log_{s_1}2)-1} - 2^{m(\log_2 s_1 - 2\alpha)} \geq 2^{m(1-\alpha\log_{s_1}2)-1} - 2$$

long branches of the map $\ell_\xi^{m(1-\alpha\log_{s_1}2)}$.

Therefore, the composition has at least $2^{m\alpha\log_{s_1}2}(2^{m(1-\alpha\log_{s_1}2)-1}-2)$ long branches, which comes as $2^{m-1} - 2^{m\alpha\log_{s_1}2} > 2^{m-2}$, as promised. ■

Canonical partition construction. Let us consider the set of end points of domains of long branches

$$D_l\colon = \{x \mid x \text{ is an endpoint of a domain of a long branch of the map } \ell_\xi^m\} \cup \{\pm 1\} = \\ = \{-1 = d_1 < d_2 < \ldots < d_N = 1\},$$

and define a partition $\Omega = \bigcup_{j=1}^{N} \Omega_j$ of the interval $[-1,1]$ by $\Omega_j = (d_j, d_{j+1})$; $j = 1, \ldots, N$.

Let us denote by $U_\varepsilon(\Omega_j)$ a neighbourhood of Ω_j of the size ε.

We shall set $\varepsilon = (2s_1^m)^{-1}$. If for some $\Omega_j = (d_j, d_{j+1})$, containing a long branch of the map ℓ_ξ^m, there exist points of discontinuity of the map ℓ_ξ^m in a neighbourhood $U_\varepsilon(\Omega_j) \cap [-1,1]$, then we extend the interval Ω_j to include all these points.

Let $\Omega'_j = (d'_j, d'_{j+1})$, $j = 1, \ldots, N$ be a new collection of intervals. If there exist two intervals (d'_j, d'_{j+1}) and (d'_{j+2}, d'_{j+3}) containing long branches of the map ℓ_ξ^m, and such that $d_{j+2} - d_{j+1} < (ms_1^m)^{-1}$, then we replace the interval (d_j, d_{j+1}) in Ω' with the interval (d_j, d_{j+2}).

Now the length of any interval of the partition Ω', containing a long branch, is not more than $2(s_1^{-m} + s_2^{-m})$. Assume that there exist two intervals (d'_j, d'_{j+1}) and (d'_{j+2}, d'_{j+3}) containing long branches of the map ℓ_ξ^m, such that $d'_{j+2} - d'_{j+1} > s_2^{-m}$, then we split the interval (d'_{j+1}, d'_{j+2}) into intervals of the size s_2^{-m}, allowing one of them to be longer, or smaller, if necessary. More precisely, let $\Omega'_j = (d'_j, d'_{j+1})$ be an interval of Ω' that doesn't contain a long branch. Let $n := \left[s_2^m(d'_{j+1} - d'_j)\right]$ be the number of "whole" intervals of the length s_2^{-m} that could fit inside (d'_j, d'_{j+1}). If $(d'_{j+1} - d'_j) - ns_2^m < s_1^{-m}$, we split the interval (d'_j, d'_{j+1}) into n intervals; adding the intervals $(d'_j + ks_2^{-m}, d'_j + (k+1)s_2^{-m})$, $0 \leq k < n$ to the partition Ω'. Otherwise, we split the interval (d'_j, d'_{j+1}) into $n+1$ intervals, adding $(d'_j + ks_2^{-m}, d'_j + (k+1)s_2^{-m})$, $0 \le k \le n$ to Ω'.

The intervals $(a_0^{(m)}, -1)$ and $(1, a_{N_m}^{(m)})$, do not contain any long branches, and we define the partition there as described above. Finally, we define the partition on $(-\infty, a_1^{(m)})$ and $(a_{N_m}^{(m)}, +\infty)$ splitting them into equal intervals of the length 2^{-m}.

We have obtained a partition of the real line, that satisfies Conditions (D2) and (D4) of Definition 3. We have to check other conditions of Definition 3.

Lemma 5.8. *The partition constructed satisfies Condition (D3). Any interval $I \subset \mathbb{R}$ of the length δ contains at most $N_\delta < 2^{m(1-\alpha \log_{s_1} 2)+1}$ intervals of the partition.*

Proof. The statement holds true for any interval $I \subset \mathbb{R} \setminus [a_1^{(m)}, a_{N_m}^{(m)}]$ of the length δ. Assume that $I \subset [a_1^{(m)}, a_{N_m}^{(m)}]$, and $|I| = \delta$. Then there are two possibilities:

(1) the interval I contains a long branch;
(2) the interval I doesn't contain a long branch.

Consider the first case. Observe that for any $k_0 \le m$ and for any interval $I_0 \subset [-1, 1]$ of the length $|I_0| < s_1^{-k_0}$ such that $\ell_\xi^k(I_0) \subset [-1, 1]$ for all $k < k_0$, the map $\ell_\xi^{k_0}$ is one-to-one on I_0. (Easy to check by induction). Since for any long branch $\mathfrak{a}_j^{(m)}$ we have $1 - \frac{2}{s_1} \in \ell_\xi(\mathfrak{a}_j^{(m)})$, we conclude that $k_0 \colon = [-\log_2 \delta \log_{s_1} 2] = [\alpha m \log_{s_1} 2]$ and then we see that the map $\ell_\xi^{k_0}$ is one-to-one on any interval I_0 of the length less than δ such that $\ell_\xi^k(I_0) \subset [-1, 1]$ for all $k < k_0$. Thus any interval of the length δ contains at most 2^{m-k_0} long branches of the map ℓ_ξ^m. Consequently, any interval I with $|I| \le \delta$ contains at most $2^{m-k_0} < N_\delta$ intervals of the partition with a long branch inside.

Assume now that the interval I of the length $|I| = \delta$ contains some intervals of the partition that do not contain a long branch inside. Let $I_0 \subset I$ be a maximal by inclusion subinterval not containing a long branch. Then by construction of the partition, it contains at most one interval of the partition Ω of the length less than s_2^{-m}. Since the interval I contains at most 2^{m-k_0} long branches, it may contain not more than $2^{m-k_0} + 2$ intervals I_0 without a long branch inside. Therefore, the interval I contains not more than $\delta s_2^m + 2^{m-k_0+1} < N_\delta$ intervals of the partition.

In the second case, an argument similar to the one above shows that an interval I of the length $|I| = \delta$ and without a long branch inside contains not more than $\delta s_2^m + 1 < N_\delta$ intervals of the partition.

∎

Lemma 5.9. *The partition constructed satisfies Condition (D1) of Definition 3. The interval* $[-1,1]$ *contains at least* 2^{m-1} *and at most* $2^m - 2^{m\alpha\log_{s_1}2} + m\delta s_1^m$ *intervals of the partition.*

Proof. By Corollary 1 of Lemma 5.7, the map ℓ_ξ^m has at least 2^{m-2} long branches, provided s_1 is chosen such that $2\alpha\log_{s_1}2 > 1$. Every long branch belongs to exactly one of intervals of the partition, and the escaping set of measure $m\delta$ contains at most $m\delta s_1^m$ intervals. ∎

Summing up, we conclude that the construction leads to a partition of the class $\mathcal{G}$, as desired.

We shall refer to the resulting partition Ω as the canonical partition of the map ℓ_ξ^m.

Lemma 5.10. *Any interval of a canonical partition* Ω *has at most two main branches of the map* ℓ_ξ^m.

Proof. If an interval Ω_j of the partition contains more than one main branch, one of the main branches is not long. Let it be $\mathfrak{a}_k^{(m)}$. Then by Definition 9 $\left|\ell_\xi^m(\mathfrak{a}_k^{(m)})\cap[-1,1]\right| < \frac{2}{s_2}$.

Now we repeat the calculation of Lemma 5.5. The end points of the interval $\mathfrak{a}_k^{(m)}$ are the points of discontinuity of the map ℓ_ξ^m. Then there exists two numbers $n_1 < m$ and $n_2 < m$ such that $\ell_\xi^{n_1}(a_k^{(m)}) = -1 + \xi(n_1)$ and $\ell_\xi^{n_2}(a_{k+1}^{(m)}) = 1 - \xi(n_2)$. Therefore,

$$|\ell_\xi^m(a_k^{(m)}) + 1| = |\ell_{\sigma^{n_1}\xi}^{m-n_1}(-1+\xi(n_1)) + 1| \le s_1^{m-n_1}\delta;$$
$$|\ell_\xi^m(a_{k+1}^{(m)}) - 1| = |\ell_{\sigma^{n_2}\xi}^{m-n_2}(1-\xi(n_2)) - 1| \le s_2^{m-n_2}\delta.$$

Since by assumption $\left|\ell_\xi^m(\mathfrak{a}_k^{(m)})\right| < \frac{2}{s_2}$, we deduce $2 - \delta(s_2^{m-n_2} + s_1^{m-n_1}) \le \frac{2}{s_2}$. The latter is equivalent to $\delta(s_2^{m-n_2} + s_1^{m-n_1}) \ge \frac{2}{s_1}$, which implies that either $\delta s_2^{m-n_2} \ge \frac{1}{s_1}$, or $\delta s_1^{m-n_1} \ge \frac{1}{s_1}$, or both. Hence we get an upper bound on n_1 or n_2, respectively:

$$n_{1,2} < m_0 \colon = m\Big(1 - \frac{\alpha}{\log_2 s_1}\Big) + 10.$$

Therefore one of the end points of $\mathfrak{a}_k^{(m)}$ is an end point of the main branch of the map ℓ_ξ^n with $n < m_0$. Observe that all main branches of the map ℓ_ξ^m are long. Any interval of the length $s_1^{-m_0}$ contains at not more than one main branch of the map $\ell_\xi^{m_0}$. Therefore the distance between short main branches of the map ℓ_ξ^m is at least $s_1^{-m_0} \gg 2(s_1^{-m} + s_2^{-m}) = \sup|\Omega_j|$, and any interval of the partition contains not more than one short main branch of the map ℓ_ξ^m. Therefore, any interval of the partition contains at most two main branches. ∎

5.4. **Approximating $\ell_{\xi*}^m$ by a generalised toy dynamo operator.** Here we prove the main result of this Section, Theorem 3, which establishes the existence of a generalised toy dynamo operator, a close approximation of $\ell_{\xi*}^m$ for arbitrary $\|\xi\|_\infty \le \delta$.
Construction. Let a partition Ω^2 of the class $\mathcal{G}$ be given. Let ℓ_ξ^m be as above, and let $\mathfrak{a}^{(m)}$ be a partition of the real line by its points of discontinuity and let Ω^1 be the canonical partition of the map ℓ_ξ^m. Introduce the joint partition: $\mathfrak{a}^{(m)} \cup \Omega^1 = \{d_j\}_{j\in\mathbb{Z}}$. We assume the natural numbering: $[d_0; d_1] \ni 0$ and $d_j < d_{j+1}$ for any $j \in \mathbb{Z}$. Define the image of the joint partition by

$$\left\{b_j^\pm \colon = \lim_{y\to d_j\pm 0} \ell_\xi^m(y)\right\}_{j\in\mathbb{Z}}.$$

Then on the interval (d_j, d_{j+1}) the map ℓ_ξ^m is given by

$$\ell_\xi^m(x)\colon = \frac{b_{j+1}^- - b_j^+}{d_{j+1}-d_j}x + \frac{b_j^+ d_{j+1} - b_{j+1}^- d_j}{d_{j+1}-d_j}, \quad d_j < x < d_{j+1}.$$

We define an approximating map $\widehat{\ell_\xi^m}$ to be

$$\widehat{\ell_\xi^m}(x)\colon = \frac{\lfloor b_{j+1}^-\rfloor - \lceil b_j^+\rceil}{d_{j+1}-d_j}x + \frac{\lceil b_j^+\rceil d_{j+1} - \lfloor b_{j+1}^-\rfloor d_j}{d_{j+1}-d_j}, \quad d_j < x < d_{j+1};$$

where $\lfloor x\rfloor$ stands for the closest to x point of the partition Ω^2, which is smaller than x; and $\lceil x\rceil$ stands for the closest to x point of the partition Ω^2, which is larger than x. In particular, branches of the map $\widehat{\ell_\xi^m}$ are not longer than branches of the map ℓ_ξ^m.

We define an operator $\mathcal{T}\colon \Phi_{\Omega^1} \to \Phi_{\Omega^2}$ by

$$(\mathcal{T}\phi)(x)\colon = \sum_{y\in\widehat{\ell_\xi}^{-m}(x)} \operatorname{sgn} \mathrm{d}\widehat{\ell_\xi^m}(y)\phi(y). \tag{33}$$

Lemma 5.11. *The operator $\mathcal{T}$ is a linear operator between two subspaces of step functions associated to the partitions Ω^1 and Ω^2 (see p. 2 for definition): $\mathcal{T}\colon \Phi_{\Omega^1} \to \Phi_{\Omega^2}$.*

Proof. Linearity is obvious. It is sufficient to show that for any interval $\Omega_j^1 \in \Omega^1$ of the first partition,

$$(\mathcal{T}\chi_{\Omega_j^1})(x)\colon = \sum_{y\in\widehat{\ell_\xi}^{-m}(x)} \operatorname{sgn} \mathrm{d}\widehat{\ell_\xi^m}(y)\chi_{\Omega_j^1}(y) \in \Phi_{\Omega^2}.$$

By definition of $\widehat{\ell_\xi^m}$, we see $\lim_{y\to d_j-0} \widehat{\ell_\xi^m}(y) = [b_j^-]$ and $\lim_{y\to d_j+0} \widehat{\ell_\xi^m}(y) = [b_j^+]$, therefore all points of $\Omega_k^2 \subset \Omega^2$ have the same number of preimages with respect to ℓ_ξ^m for any interval Ω_k^2. Moreover, $\widehat{\ell_\xi}^{-m}(\Omega_k^2)$ does not contain any point of Ω^1 inside, as it is piecewise monotone on a subpartition $\Omega^1 \cup \mathfrak{a}^{(m)}$. ∎

Definition 10. We introduce *the k-escaping set*

$$E_k \colon = \{x \in [-1,1] \mid \exists n < k \ \ell_\xi^n(x) \notin [-1,1]\}. \tag{34}$$

Lemma 5.12. *In the canonical bases of Φ_{Ω^1} and Φ_{Ω^2}*

$$\sup_{y\in\Omega_i^2} \#\{x \in \Omega_j^1 \mid \ell_\xi^m(x) = y\} \le m^2 \frac{s_1^m}{s_2^m}.$$

Proof. Observe that the map is one-to-one on any interval $I \subset [-1,1] \setminus E_m$ of the length $|I| \le 2s_1^{-m}$.

Given an element Ω_j^1, consider a maximal by inclusion interval $I \subset E_m \cap \Omega_j^1$, such that $|I| \le s_1^{-m}$. We shall show that

$$\max_{y\in\mathbb{R}} \#\big\{x \in I \mid \ell_\xi^m(x) = y\big\} \le 3ms_1^3. \tag{35}$$

There are two possibilities:

(1) the map ℓ_ξ^m is continuous on $I \subset E_m \cap \Omega_j^1$;
(2) the map ℓ_ξ^m is not continuous on $I \subset E_m \cap \Omega_j^1$.

In the first case the map $\ell_\xi|_I$ is a bijection and (35) holds true.

Now consider the second case: the map ℓ_ξ^m is not continuous on $I \subset E_m \cap \Omega_j^1$. We may find the smallest k_0 such that $\ell_\xi^{k_0}(I) \not\subset [-1,1]$. Then

$$\ell_\xi^{k_0}(I) \cap [-1,1] \subset (-1, -1 + s_1^{k_0-m}) \sqcup (1 - s_1^{k_0-m}, 1).$$

Let $m_0 \stackrel{\text{def}}{=} \frac{1+m\alpha}{\log_2 s_1} - 2$. It follows by induction in k that for any $k_0 \le k < m_0$ the image $\ell_\xi^k(I) \cap [-1,1]$ may be covered by two disjoint intervals in particular,

$$\ell_\xi^k(I) \cap [-1,1] \subset (-1, -1 + \delta_k^1 + s_1^{k-m}) \sqcup (1 + \delta_k^2 - s_1^{k-m}, 1),$$

where $\delta_k^1 = \sum\limits_{j=k_0}^{k} s_1^{k-j}\xi_j$ and $\delta_k^2 = \sum\limits_{j=k_0}^{k} s_2^{k-j}\xi_j$ with $|\delta_k^{1,2}| \le s_1^{k-k_0+1}\delta$, and $-1 + \frac{2}{s_1} \notin \ell_\xi^k(I)$ for all $k_0 \le k < m_0$. In particular, for any $x_1, x_2 \in I$ such that $\ell_\xi^{k_0}(x_1) \in (-1, -1+s_1^{k_0-m})$ and $\ell_\xi^{k_0}(x_2) \in (1 - s_1^{k_0-m}, 1)$ we have for all $k < m_0$:

$$\big|\ell_\xi^k(x_1) - \ell_\xi^k(x_2)\big| \ge (1 + \delta_k^2 - s_1^{k-m}) - (-1 + \delta_k^1 + s_1^{k-m}) = 2 - 2s_1^{k-m} + (\delta_k^2 - \delta_k^1) \ge 1.$$

The map $\ell_{\sigma^{k_0}\xi}^{m_0-k_0}$ is a bijection on any of the intervals $(-1, -1+s_1^{k_0-m})$ and $(1-s_1^{k_0-m}, 1)$.

Therefore, we deduce that the map $\ell_\xi^{m_0}$ is a bijection on I. It follows that the image $\ell_\xi^{m_0}(I)$ consists of not more than $3m_0$ intervals each of which is not longer than $s_1^{m_0-m}$. Let $\eta = \sigma^{m_0}\xi$ and consider the map $\ell_\eta^{m-m_0}$. We claim that it is a bijection on any interval $I \subset \mathbb{R}$ of the length $|I| \le s_1^{m_0-m-3}$. Indeed, if $\ell_\eta^{m-m_0}$ is continuous on I, then it is a bijection. Assume that for some $k_0 \le m - m_0$ the map $\ell_\eta^{k_0}$ is not continuous on I. Then

$$\ell_\eta^{k_0}(I) \cap [-1,1] \subset (-1; -1 + s_1^{m_0+k_0-m-3} + \delta) \sqcup (1 - s_1^{m_0+k_0-m-3} - \delta; 1),$$

and for any $k_0 < k \le m - m_0$

$$\ell_\eta^{k} \cap [-1,1] \subset (-1; -1 + s_1^{k+1}\delta + s_1^{m_0+k-m-3}) \sqcup (1 - s_1^{k+1} - s_1^{m_0+k-m-3}; 1).$$

By straightforward calculation we see that provided $s_1 \le 2^{2\alpha}$

$$s_1^{m_0+k-m-3} + s_1^{k+1}\delta \le \frac{1}{s_1}.$$

Therefore for any interval I of the length $|I| \le s_1^{m_0-m-3}$ and for any two points $x_1, x_2 \in I$ with $x_1 \ne x_2$ we have that $\ell_\eta^k(x_1) \ne \ell_\eta^k(x_2)$ for all $1 \le k \le m - m_0$. We see that the image $\ell_\xi^{m_0}$ may be covered by not more than $3m_0 s_1^3$ intervals of the length $s_1^{m_0-m-3}$. Hence we conclude that for any interval $|I| \le s_1^{-m}$

$$\sup_{y\in\mathbb{R}} \#\big\{x \in I \mid \ell_\xi^m(x) = y\big\} \le 3m_0 s_1^3 < 3m s_1^3 + 3.$$

Since by Lemma 5.10 any interval of the partition contains at most two main branches, the set $\Omega \cap E_m$ is a union of not more than two intervals, which may be covered by $2 + 2\frac{s_1^m}{s_2^m}$ disjoint intervals of the length s_1^{-m}. Therefore

$$\max_{y\in\mathbb{R}} \#\big\{x \in \Omega_i^1 \mid \ell_\xi^m(x) = y\big\} \le 3m s_1^3 \Big(\frac{s_1}{s_2}\Big)^m < m^2 \Big(\frac{s_1}{s_2}\Big)^m.$$

■

Corollary 1. *In the canonical bases of Φ_{Ω^1} and Φ_{Ω^2} the matrix of the operator $\mathcal{T}$ satisfies condition* (D1)*:*

$$\max |\tau_{ij}| + 1 \le m^2 \Big(\frac{s_1}{s_2}\Big)^m.$$

Proof. Recall the definition of the operator $\mathcal{T}$:

$$(\mathcal{T}\phi)(x) \colon = \sum_{y\in\widehat{\ell}_\xi^{-m}(x)} \operatorname{sgn} \mathrm{d}\widehat{\ell}_\xi^m(y)\phi(y). \tag{33}$$

Then for $\phi = \chi_{\Omega_j^1}$ we have

$$(\mathcal{T}\phi)(x) \colon = \sum_{i\in\mathbb{Z}} \tau_{ij}\chi_{\Omega_i^2}(x) = \sum_{i\in\mathbb{Z}} \sum_{y\in\widehat{\ell}_\xi^{-m}(x)} \operatorname{sgn} \mathrm{d}\widehat{\ell}_\xi^m(y)\chi_{\Omega_j^1}(y)\chi_{\Omega_i^2}(x) =$$

$$= \sum_{i\in\mathbb{Z}} \sum_{y\in\widehat{\ell}_\xi^{-m}(x)\cap\Omega_j^1} \operatorname{sgn} \mathrm{d}\widehat{\ell}_\xi^m(y)\chi_{\Omega_i^2}(x); \tag{36}$$

therefore

$$\tau_{ij} = \sum_{y \in \widehat{\ell}_\xi^{-m}(\Omega_i^2) \cap \Omega_j^1} \operatorname{sgn} \mathrm{d}\widehat{\ell}_\xi^m(y).$$

The definition of the map $\widehat{\ell}_\xi$ guarantees that τ_{ij} are well-defined; in particular

$$|\tau_{ij}| \le \#\{x \in \Omega_j^1 \mid \widehat{\ell}_\xi^m(x) = y \in \Omega_i^2\},$$

and the right hand side is independent on the choice of y. Obviously,

$$\sup_x \#\{x \in \Omega_j^1 \mid \widehat{\ell}_\xi^m(x) = y \in \Omega_i^2\} \le \sup_x \#\{x \in \Omega_j^1 \mid \ell_\xi^m(x) = y \in \Omega_i^2\},$$

■

Corollary 2. *We have the following upper bound for a total number of preimages of a point $x \in \mathbb{R}$:*

$$\sup_{x \in \mathbb{R}} \big\{ y \in \mathbb{R} \mid \ell_\xi^m(y) = x \big\} \le 2m^2 \Big(\frac{2s_1}{s_2}\Big)^m; \tag{37}$$

$$\sup_{x \in \mathbb{R}} \big\{ y \in \mathbb{R} \mid \widehat{\ell}_\xi^m(y) = x \big\} \le 2m^2 \Big(\frac{2s_1}{s_2}\Big)^m. \tag{38}$$

Proof. By definition of a partition of the class $\mathcal{G}$, the interval $[-1,1]$ contains not more than 2^m intervals of the partition; and intervals $[-1-m\delta;-1]$ and $[1;1+m\delta]$ contain not more than mN_δ intervals of the partition each. Finally, both maps are bijections on the complement to $[-1-m\delta, 1+m\delta]$. ■

Lemma 5.13. *Let Ω^1 be the canonical partition of the map ℓ_ξ^m. Let Ω^2 be another partition of the class $\mathcal{G}$. Then*

$$\#\big\{(i,j) \in \Box \mid \Omega_i^2 \subset [-1,1] \cap \ell_\xi^m(\Omega_j^1 \cap E_m)\big\} \le m^2 \delta s_1^{2m}$$

Proof. We shall prove that

$$\sum_{\mathrm{D_{in}}^2} |\Omega_i^2| \le \sum_{\mathfrak{a}_j^{(m)} \subset E_m} |\ell_\xi^m(\mathfrak{a}_j^{(m)})| \le s_1^m \delta;$$

then the Lemma will from from the lower bound on the size of the elements of partition.

Indeed, by induction one can show that

$$\sum_{\mathfrak{a}_j^{(k)} \subset E_k} |\ell_\xi^k(\mathfrak{a}_j^{(k)})| \le \frac{s_1^k - 2^k}{s_1 - 2}\delta.$$

The case $k = 1$ is trivial. Then we proceed

$$\sum_{\mathfrak{a}_j^{(k)} \subset E_k} |\ell_\xi^k(\mathfrak{a}_j^{(k)})| \le \sum_{\mathfrak{a}_j^{(k-1)} \subset E_{k-1}} |\ell_\xi^k(\mathfrak{a}_j^{(k-1)})| + \sum_{\mathfrak{a}_j^{(k)} \subset E_k \setminus E_{k-1}} |\ell_\xi^k(\mathfrak{a}_j^{(k)})| \le$$
$$\le s_1 \delta \cdot \frac{s_1^{k-1} - 2^{k-1}}{s_1 - 2} + 2^k \delta = \frac{s_1^k - 2^k}{s_1 - 2} \delta.$$

■

Corollary 1. *Let Ω^1 be the canonical partition of the map ℓ_ξ^m. Let Ω^2 be another partition of the class $\mathcal{G}$; and let $\widehat{\ell_\xi^m}$ be a map defined as above on p. 24. Then*

$$\#\{(i,j) \in \square \mid \Omega_i^2 \subset [-1,1] \cap \widehat{\ell_\xi^m}(\Omega_j^1 \cap E_m)\} \le m^2 \delta s_1^{2m}$$

Proof. The inequality for the map $\widehat{\ell_\xi^m}$ follows from the fact that images of all branches under adjusted map $\widehat{\ell_\xi^m}$ are shorter than the images of the same branches under the original map ℓ_ξ^m. ■

Proposition 5.3. *In the canonical bases of Φ_{Ω^1} and Φ_{Ω^2} the operator $\mathcal{T}$ defined by* (33) *is a generalised toy dynamo.*

Proof. We have checked the condition (D1) already. We should verify the following conditions.

- (D2) $\#\mathrm{D_{in}} \le 3m^2 \delta s_1^{2m}$;
- (D3) for any pair $(i,j) \in \mathrm{Sp}$ we have that $\tau_{ij} = 0$ whenever $|i - j| > mN_\delta$;
- (D4) $\#\mathrm{Ar} \ge 2^{m-2}$.

where

$$\mathrm{Ar}\colon = \big\{ j \in \{N_l^1 \dots N_r^1\} \mid \#\{i \in \{N_l^2 \dots N_r^2\} \mid \tau_{ij} = 1\} \ge 2^m - N_\delta \big\};$$
$$\mathrm{D_{in}}\colon = \big\{ (i,j) \in \{N_l^2 \dots N_r^2\} \times \{N_l^1 \dots N_r^1\} \mid \tau_{ij} \ne 1 \big\}.$$

To verify the condition (D2): $\#\mathrm{D_{in}} \le 3s_1^{2m} m^2 \delta$ we shall show that $\sum_{\mathrm{D_{in}}} |\Omega_i^2| \le 3s_1^m m \delta$ and then taking into account $|\Omega_i^2| \ge \frac{s_1^{-m}}{m}$ we get the result. Let E_m be the m-escaping set as defined by (34) above.

We introduce three subsets of the set $\mathrm{D_{in}}$.

$$\mathrm{D_{in}}^1\colon = \{(i,j) \in \mathrm{D_{in}} \mid \Omega_i^2 \subset [-1,1] \setminus \widehat{\ell_\xi^m}(\Omega_j^1 \setminus E_m)\}$$

— complement to the images of the main branches;

$$\mathrm{D_{in}}^2\colon = \{(i,j) \in \mathrm{D_{in}} \mid \Omega_i^2 \subset [-1,1] \cap \widehat{\ell_\xi^m}(\Omega_j^1 \cap E_m)\}$$

— image of the points that were mapped outside $[-1,1]$ and back;

$$\mathrm{D_{in}}^3 \colon = \{(i,j) \in \mathrm{D_{in}} \mid \Omega_i^2 \subset [-1,1] \cap \widehat{\ell}_\xi^m(\Omega_j^1 \setminus E_m)\}$$

— image of the points that were inside $[-1,1]$ in first m iterations.

We claim that $\mathrm{D_{in}} = \mathrm{D_{in}}^1 \cup \mathrm{D_{in}}^2 \cup \mathrm{D_{in}}^3$: indeed, for any pair of indices $(i,j) \in \mathrm{D_{in}}$ we have that $\Omega_j^1 \cap E_m \neq \varnothing$. We shall show that $\sum_{\mathrm{D_{in}}^1} |\Omega_i^2| \leq s_1^m m\delta$, $\sum_{\mathrm{D_{in}}^2} |\Omega_i^2| \leq s_1^m m\delta$, and $\#\mathrm{D_{in}}^3 \leq s_1^{2m} m^2 \delta$.

We start with $\mathrm{D_{in}}^1$ and recall the original partition $\mathfrak{a}^{(m)}$ by the points of discontinuity of the map ℓ_ξ^m. Let $J(\mathrm{D_{in}}^1)$ be the union of intervals with indices corresponding to $\mathrm{D_{in}}^1$:

$$J(\mathrm{D_{in}}^1) \colon = \bigcup_{j \colon (i,j) \in \mathrm{D_{in}}^1} (\Omega_j^1 \setminus E_m).$$

We may write then

$$\sum_{\mathrm{D_{in}}^1} |\Omega_i^2| \leq 2 \cdot \#\{\mathfrak{a}_j^{(m)} \subset J(\mathrm{D_{in}}^1)\} - \sum_{\mathfrak{a}_j^{(m)} \subset J(\mathrm{D_{in}}^1)} |\widehat{\ell}_\xi^m(\mathfrak{a}_j^{(m)})| \leq 2^{m+1} - \sum_{\mathfrak{a}_j^{(m)} \subset [-1,1] \setminus E_m} |\widehat{\ell}_\xi^m(\mathfrak{a}_j^{(m)})|;$$

and we shall show by induction in k that

$$2^{k+1} - \sum_{\mathfrak{a}_j^{(k)} \subset [-1,1] \setminus E_k} |\ell_\xi^k(\mathfrak{a}_j^{(k)})| < s_1^k \cdot k 2^{-k\alpha}, \text{ where } \|\xi\|_\infty \leq 2^{-k\alpha}.$$

The case $k = 1$ is trivial. Let $\mathfrak{b}^{(k)}$ be the canonical partition of the map ℓ_0^k. This partition has 2^k elements in $[-1,1]$. There exists a correspondence between the sets of indices $\tau \colon \{i \in \mathbb{Z} \mid \mathfrak{a}_i^{(k)} \subset (-1,1)\} \to \{-2^k, \dots, 2^k - 1\}$ that satisfies $d\ell_\xi^k \mid_{\mathfrak{a}_j^{(k)}} = d\ell_0^k \mid_{\mathfrak{b}_{\tau(j)}^{(k)}}$ and $\tau(j_1) \neq \tau(j_2)$ for all $j_1 \neq j_2$. In particular, $\operatorname{sgn} d\widehat{\ell}_\xi^k \mid_{\mathfrak{a}_j^{(k)}} = \operatorname{sgn} d\widehat{\ell}_0^k \mid_{\mathfrak{b}_{\tau(j)}^{(k)}}$.

We split the intervals $\mathfrak{a}_j^{(k)}$ into two groups:

$$B_1^k \colon = \{j \in \{-2^k, \dots, 2^k - 1\} \mid j = \tau(i) \text{ for some } i \in \mathbb{Z}\};$$
$$B_2^k \colon = \{-2^k, \dots, 2^k - 1\} \setminus B_1^k.$$

We also see that $\ell_0^k(\mathfrak{b}_j^{(k)}) = [-1,1]$ for any interval of the partition $\mathfrak{b}^{(k)}$.

$$\begin{aligned}
2^{k+1} - \sum_{\mathfrak{a}_j^{(k)}\subset[-1,1]\setminus E_k} |\ell_\xi^k(\mathfrak{a}_j^{(k)})| &= \Big(\sum_{j\in B_1^k} + \sum_{j\in B_2^k}\Big)|\ell_0^k(\mathfrak{b}_{\tau(j)}^{(k)})| - \sum_{\mathfrak{a}_j^{(k)}\subset[-1,1]\setminus E_k} |\ell_\xi^k(\mathfrak{a}_j^{(k)})| = \\
&= \sum_{j\in B_1^k} |\ell_0^k(\mathfrak{b}_j^{(k)}) \setminus \ell_\xi^k(\mathfrak{a}_j^k)| + \sum_{j\in B_2^k} |\ell_0^k(\mathfrak{b}_j^{(k)})| \le \\
&\le s_1 \sum_{j\in B_1^{k-1}} |\ell_0^{k-1}(\mathfrak{b}_j^{(k-1)}) \setminus \ell_\xi^{k-1}(\mathfrak{a}_{\tau(j)}^{k-1})| + 2^k\delta + 2\sum_{j\in B_2^{k-1}} |\ell_0^{k-1}(\mathfrak{b}_j^{(k-1)})| \le \\
&\le s_1\Big(2^{k-1} - \sum_{B_1^{k-1}} |\ell_\xi^{k-1}(\mathfrak{a}_{\tau(j)}^{k-1})|\Big) + 2^k\delta \le \\
&\le s_1^k(k-1)\delta + 2^k\delta \le s_1^k k\delta.
\end{aligned}$$

Therefore we deduce that

$$2^{m+1} - \sum_{\mathfrak{a}_j^{(k)}\subset[-1,1]\setminus E_k} |\ell_\xi^k(\mathfrak{a}_j^{(k)})| < s_1^m \cdot m 2^{-m\alpha}, \text{ where } \|\xi\|_\infty \le 2^{-k\alpha}.$$

Since there are not more than 2^m main branches, and the length of intervals of the partition Ω^2 is bounded $|\Omega_i^2| \le 2(s_2^{-m} + s_1^{-m})$, we get

$$2^{m+1} - \sum_{\mathfrak{a}_j^{(k)}\subset[-1,1]\setminus E_k} |\widehat{\ell_\xi^k}(\mathfrak{a}_j^{(k)})| < s_1^m \cdot m2^{-m\alpha} + 2^m(s_2^{-m} + s_1^{-m}) \le 2m\delta s_1^m;$$

provided $s_2 < 2 < s_1$ are chosen such that $s_1 s_2 > 2^{1+\alpha}$, which is possible.

The inequality for the set ${\mathrm{D_{in}}}^2$ follows from 1 of Lemma 5.13.

Finally, for the set ${\mathrm{D_{in}}}^3$ we observe that $(i,j) \in {\mathrm{D_{in}}}^3$ if and only if there exist two main branches $\mathfrak{a}_{j_1}^{(m)}, \mathfrak{a}_{j_2}^{(m)} \subset \Omega_j^1$ such that for any $k < m$ we have $\widehat{\ell_\xi^k}(\mathfrak{a}_{j_1}^{(m)}) \subset [-1,1]$ and $\widehat{\ell_\xi^k}(\mathfrak{a}_{j_2}^{(m)}) \subset [-1,1]$, and $\widehat{\ell_\xi^m}(\mathfrak{a}_{j_1}^{(m)}) \cap \widehat{\ell_\xi^m}(\mathfrak{a}_{j_2}^{(m)}) \cap \Omega_i^2 \neq \varnothing$. Since both $\mathfrak{a}_{j_1}^{(m)}$ and $\mathfrak{a}_{j_2}^{(m)}$ are belong to the same element of the partition we conclude that either $|\ell_\xi^m(\mathfrak{a}_{j_1}^{(m)})| \le \frac{2}{s_2}$ or $|\ell_\xi^m(\mathfrak{a}_{j_2}^{(m)})| \le \frac{2}{s_2}$. By Lemma 5.5 there are at most $2^{m(1-\alpha_1)}$ main branches with this property. Without loss of generality we assume the latter. Then by definition of $\widehat{\ell_\xi^m}$ we have $|\widehat{\ell_\xi^m}(\mathfrak{a}_{j_2}^{(m)})| \le \frac{2}{s_2} + 2(s_2^{-m} + s_1^{-m})$. Hence $\#{\mathrm{D_{in}}}^3 \le 2^{m(1-\alpha_1)}\frac{2N_\delta}{\delta}$. It follows that

$$\#\mathrm{D_{in}} = \#{\mathrm{D_{in}}}^1 + \#{\mathrm{D_{in}}}^2 + \#{\mathrm{D_{in}}}^3 \le 2s_1^{2m}m\delta + 2^{m(1-\alpha)}\frac{2N_\delta}{\delta} \le 3s_1^{2m}m\delta,$$

as required.

The condition (D3) follows from the fact that the map $\widehat{\ell_\xi^m}$ is linear and on the compliment $\mathbb{R}\setminus[-1-m\delta, 1+m\delta]$ (in other words, the complement consists of two pieces of continuity), and, moreover, it is given by $\widehat{\ell_\xi^m}(x) = x + b$ on these set. Therefore,

$\tau_{ij} = 0$ whenever $|i-j| > b \cdot N_\delta \delta^{-1}$. Obviously, $|b| \le m\delta$, so we get $\tau_{ij} = 0$ whenever $|i-j| > mN_\delta$.

■

Now it only remains to show that the generalised toy dynamo, constructed from the map $\widehat{\ell_\xi^m}$, is a good approximation to the operator $\ell_{\xi*}^m$.

Theorem 3. *Let Ω^2 be a partition of the class $\mathcal{G}$. Consider a sequence $\xi \in \ell_\infty(\mathbb{R})$ with $\|\xi\|_\infty \le 2^{-m\alpha}$ and let Ω^1 be the canonical partition of the map ℓ_ξ^m. Then for the operator $\mathcal{T} = \widehat{\ell_{\xi*}^m}\colon \Phi \to \Phi$ defined by (33) and for any essentially bounded integrable function $g \in \mathcal{L}_1(\mathbb{R})$ we have*

$$\|(\ell_{\xi*}^m - \mathcal{T})W_\delta g\|_2 \le \left(\frac{s_1^3}{2^{1/2+\alpha}s_2}\right)^m \cdot m\|g\|_1.$$

Proof. Let $\|g\|_{\Omega_1} = 1$ and let $f = W_\delta g$. Then $\|f\|_\infty \le \|g\|_\infty \le 2^{m/2}$, since

$$\|g\|_1 = \max\Big(2^{-m}\sum_{j\in\mathbb{Z}} \frac{1}{|\Omega_j^1|}\int_{\Omega_j^1} |g(x)|\mathrm{d}x, 2^{-m/2}\sup_{x\in\mathbb{R}}|g(x)|\Big) \le 1.$$

By definition, we write

$$\mathcal{T}f(x) = \sum_{y\in\widehat{\ell_\xi}^{-m}(x)} \operatorname{sgn}\big(\widehat{\ell_\xi^m}\big)'(y)f(y); \tag{39}$$

$$\ell_{\xi*}^m f(x) = \sum_{y\in\ell_\xi^{-m}(x)} \operatorname{sgn}\big(\ell_\xi^m\big)'(y)f(y). \tag{40}$$

We begin with weighted $\mathcal{L}_1$-norm.

$$\|\ell_{\xi*}^m f - \widehat{\ell_{\xi*}^m} f\|_2 = \sum_{j\in\mathbb{Z}} \frac{2^{-m}}{|\Omega_j^2|}\int_{\Omega_j^2}\big|\mathcal{T}f(x) - \ell_{\xi*}^m f(x)\big|\mathrm{d}x \le$$

$$\le \Big(\int_{-\infty}^{-1-m\delta} + \int_{1+m\delta}^{+\infty}\Big)\big|\mathcal{T}f(x) - \ell_{\xi*}^m f(x)\big|\mathrm{d}x + \tag{41}$$

$$+ \Big(\frac{s_1}{2}\Big)^m\Big(\int_{-1-m\delta}^{-1} + \int_1^{1+m\delta}\Big)\big|\mathcal{T}f(x) - \ell_{\xi*}^m f(x)\big|\mathrm{d}x + \tag{42}$$

$$+ 2^{-m}\sum_{j=N_l^2}^{N_r^2} \frac{1}{|\Omega_j^2|}\int_{\Omega_j^2}\big|\mathcal{T}f(x) - \ell_{\xi*}^m f(x)\big|\mathrm{d}x. \tag{43}$$

We estimate all three terms separately. By the very definition, on the infinite intervals $(1+m\delta, +\infty)$ and $(-\infty, -1-m\delta)$ the map ℓ_ξ^m is given by $\ell_\xi^m(x) = (-1)^m\big(x + \sum_{j=1}^m \xi(j)\big)$.

Therefore, the map $\widehat{\ell}_\xi^m$ is one to one on each of the intervals $(-\infty, -1-m\delta)$ and $(1+\delta, +\infty)$; moreover,

$$\ell_\xi^{-m}((-\infty, -1-m\delta) \cup (1+m\delta, +\infty)) \subset (-\infty, -1) \cup (1, +\infty).$$

Observe that for the last point $a_N \in \mathbb{R}$ of the last point of discontinuity of the map ℓ_ξ^m we have, using Lemma 6.3:

$$\int_{a_N}^{+\infty} |f(x)| \mathrm{d}x = \int_{a_N}^{+\infty} |(W_\delta g)(x)| \mathrm{d}x = \sum_{j=N_2}^{+\infty} \frac{2^{-m}}{|\Omega_j^2|} \int_{\Omega_j^2} |W_\delta g(x)| \mathrm{d}x \le \|W_\delta g\|_1 \le \frac{mN_\delta}{s_2^m \delta}.$$

The first difference we estimate by the sum of absolute values.

$$\begin{aligned}
&\int_{1+m\delta}^{+\infty} \Big| \sum_{y \in \widehat{\ell}_\xi^{-m}(x)} \operatorname{sgn}\big(\widehat{\ell}_\xi^m\big)'(y) f(y) - \sum_{y \in \ell_\xi^{-m}(x)} \operatorname{sgn}\big(\ell_\xi^m\big)'(y) f(y) \Big| \mathrm{d}x = \\
&\qquad = \int_{1+m\delta}^{+\infty} \Big| f(\widehat{\ell}_\xi^{-m}(x)) - f(\ell_\xi^{-m}(x)) \Big| \mathrm{d}x \le 2\Big(\int_{-\infty}^{-1} + \int_1^{+\infty} \Big) |f(x)| \mathrm{d}x = \\
&\qquad = 2\Big(\int_{-\infty}^{a_1} + \int_{a_1}^{-1} + \int_1^{a_N} + \int_{a_N}^{+\infty} \Big) |f(x)| \mathrm{d}x,
\end{aligned}$$

where a_1 and a_N are the first and the last points of discontinuity of the map ℓ_ξ^m. Summing up,

$$\int_{1+m\delta}^{+\infty} |\ell_{\xi*}^m f(x) - \mathcal{T} f(x)| \mathrm{d}x \le 4m\delta \sup |f| + 4\|f\|_1 \le 4\Big(m\delta 2^{m/2} + \frac{mN_\delta}{s_2^m \delta} \Big) \|g\|_1 \le 8 \frac{mN_\delta}{s_2^m \delta}. \tag{44}$$

Similarly,

$$\int_{-\infty}^{-1-m\delta} |\ell_{\xi*}^m f(x) - \mathcal{T} f(x)| \mathrm{d}x \le 8 \frac{mN_\delta}{s_2^m \delta}. \tag{45}$$

Summing up (44) and (45), and taking into account that $\|f\|_1 = 1$, we get an upper bound for the first term (41):

$$\Big(\int_{-\infty}^{-1-m\delta} + \int_{1+m\delta}^{+\infty} \Big) |\ell_{\xi*}^m f(x) - \mathcal{T} f(x)| \mathrm{d}x \le 16 \frac{mN_\delta}{s_2^m \delta}. \tag{46}$$

Now we use a rough upper bound to estimate the second term. Since by Corollary 2 of Lemma 5.12 any point has at most $2m^2\big(\frac{2s_1}{s_2}\big)^m$ preimages with respect to $\widehat{\ell}_\xi^m$ or ℓ_ξ^m;

and taking into account $\|f\|_\infty \le 2^{m/2}$.

$$\int_1^{1+m\delta} |(\ell^m_{\xi*} - \mathcal{T})f(x)| \mathrm{d}x \le \int_1^{1+m\delta} |\ell^m_{\xi*} f(x)| + |\mathcal{T}f(x)| \mathrm{d}x \le$$
$$\le \sup_x \big(|\ell^m_{\xi*} f(x)| + |\mathcal{T}f(x)|\big) m\delta \le \Big(\frac{s_1}{s_2}\Big)^m m^3 2^{m+1}\delta \|f\|_\infty \le$$
$$\le m 2^{m(3/2-\alpha)} \Big(\frac{s_1}{s_2}\Big)^m .$$

Therefore we get an upper bound for the second term (42):

$$\Big(\frac{s_1}{2}\Big)^m \Big(\int_1^{1+m\delta} + \int_{-1-m\delta}^{-1}\Big) \big|(\ell^m_{\xi*} - \mathcal{T})f(x)\big| \mathrm{d}x \le m 2^{m(1/2-\alpha)} \Big(\frac{s_1^2}{s_2}\Big)^m . \tag{47}$$

The third term (43) is a little more complicated. We split the sum into two terms: long branches and all other intervals. Let $\mathfrak{a}^{(m)}$ be a partition of $\mathbb{R}$ by the points of discontinuity (cf. (32)) and let $\mathfrak{a}^{(m)}_n = (a^{(m)}_n, a^{(m)}_{n+1})$ be its intervals. Let $\mathfrak{a}_{n_l} = (-1, a^{(m)}_{n_l+1})$ and $\mathfrak{a}_{n_r} = (a^{(m)}_{n_r-1}, 1)$ be the most left and the most right intervals of the partition inside the interval $[-1,1]$. Let E_m be the m-escaping set as defined by (34) above. By definition of ℓ_{ξ^*} and $\mathcal{T}$,

$$2^{-m} \sum_{i=N_l^2}^{N_r^2} \frac{1}{|\Omega^2_i|} \int_{\Omega^2_i} |\ell^m_{\xi*} f(x) - \mathcal{T}f(x)| \mathrm{d}x =$$
$$= \sum_{i=N_l^2}^{N_r^2} \frac{2^{-m}}{|\Omega^2_i|} \int_{\Omega^2_i} \Big| \sum_{y \in \ell_\xi^{-m}(x)} \operatorname{sgn}\big(\ell^m_\xi\big)'(y) f(y) - \sum_{\hat y \in \widehat{\ell}_\xi^{-m}(x)} \operatorname{sgn}\big(\widehat{\ell}^m_\xi\big)'(\hat y) f(\hat y) \Big| \mathrm{d}x.$$

Let us introduce two functions

$$h(j,x)\colon \mathbb{Z}\times\mathbb{R} \to \mathbb{R}; \qquad h(j,x) = \sum_{y\in \ell_\xi^{-m}(x)} \operatorname{sgn}\big(\ell^m_\xi\big)'(y) \chi_{\mathfrak{a}^{(m)}_j}(y) f(y);$$

and

$$\widehat{h}(j,x)\colon \mathbb{Z}\times\mathbb{R} \to \mathbb{R}; \qquad \widehat{h}(j,x) = \sum_{\hat y\in \widehat{\ell}_\xi^{-m}(x)} \operatorname{sgn}\big(\widehat{\ell}^m_\xi\big)'(\hat y) \chi_{\mathfrak{a}^{(m)}_j}(\hat y) f(\hat y).$$

Then we see that

$$\sum_{j\in\mathbb{Z}} h(j,x) = \sum_{y\in \ell_\xi^{-m}(x)} \operatorname{sgn}\big(\ell^m_\xi\big)'(y) f(y);$$

and

$$\sum_{j\in\mathbb{Z}}\widehat{h}(j,x)=\sum_{\hat{y}\in\widehat{\ell}_\xi^{-m}(x)}\operatorname{sgn}\bigl(\widehat{\ell_\xi^m}\bigr)'(\hat{y})f(\hat{y});$$

both sums are well-defined, because they have finite number of non-zero terms, since by Corollary 2 of Lemma 5.12 the total number of preimages of a point is not more than $m^3 2^{m+1}s_1^m s_2^{-m}$. Therefore we may write

$$2^{-m}\sum_{i=N_l^2}^{N_r^2}\frac{1}{|\Omega_i^2|}\int_{\Omega_i^2}|\ell_{\xi*}^m f(x)-\mathcal{T}f(x)|\mathrm{d}x=\sum_{i=N_l^2}^{N_r^2}\frac{2^{-m}}{|\Omega_i^2|}\int_{\Omega_i^2}\Bigl|\sum_{j\in\mathbb{Z}}\bigl(h(j,x)-\widehat{h}(j,x)\bigr)\Bigr|\mathrm{d}x=$$

$$=\Bigl(\sum_{j<n_l^{(m)}}+\sum_{j=n_l^{(m)}}^{n_r^{(m)}}+\sum_{j>n_r^{(m)}}\Bigr)\sum_{i=N_l^2}^{N_r^2}\frac{2^{-m}}{|\Omega_i^2|}\int_{\Omega_i^2}\Bigl|h(j,x)-\widehat{h}(j,x)\Bigr|\mathrm{d}x=$$

$$=\Bigl(\sum_{j<n_l^{(m)}}+\sum_{\mathfrak{a}_j^{(m)}\subset E_m}+\sum_{\mathfrak{a}_j^{(m)}\not\subset E_m}+\sum_{j>n_r^{(m)}}\Bigr)\sum_{i=N_l^2}^{N_r^2}\frac{2^{-m}}{|\Omega_i^2|}\int_{\Omega_i^2}\Bigl|h(j,x)-\widehat{h}(j,x)\Bigr|\mathrm{d}x. \quad (48)$$

First we estimate the finite sums:

$$\Bigl(\sum_{j<n_l^{(m)}}+\sum_{j>n_r^{(m)}}\Bigr)\sum_{i=N_l^2}^{N_r^2}\frac{2^{-m}}{|\Omega_i^2|}\int_{\Omega_i^2}\Bigl|h(j,x)-\widehat{h}(j,x)\Bigr|\mathrm{d}x\le$$

$$\le\Bigl(\sum_{k=N_l^1-mN_\delta}^{N_l^1}+\sum_{k=N_r^1}^{N_r^1+mN_\delta}\Bigr)\sum_{\mathfrak{a}_j^{(m)}\subset\Omega_k^1}\sum_{i=N_l^2}^{N_r^2}\frac{2^{-m}}{|\Omega_i^2|}\int_{\Omega_i^2}|h(j,x)|+\bigl|\widehat{h}(j,x)\bigr|\mathrm{d}x\le$$

$$\le 2mN_\delta\cdot\sup|\tau_{ij}|\cdot\sup|f|\le 2mN_\delta\Bigl(\frac{s_1}{s_2}\Bigr)^m\|g\|_\infty\le 2m\Bigl(\frac{s_1^3}{2^{1/2+\alpha}s_2}\Bigr)^m; \quad (49)$$

for all $s_1\ge 2$.

Observe that for any domain of a main branch $\mathfrak{a}_j^{(m)}\not\subset E_m$ and $y,\hat{y}\in\mathfrak{a}_j^{(m)}$, such that $\ell_\xi^m(y)=\widehat{\ell_\xi^m}(\hat{y})$ we have that $\operatorname{sgn}(\ell_\xi^m)'(y)=\operatorname{sgn}(\widehat{\ell_\xi^m})'(\hat{y})=1$ and $(\ell_\xi^m)'(y)>s_2^m$. As before, let $\mathfrak{a}_j^{(m)}=(a_j^{(m)},a_{j+1}^{(m)})$. Then

$$|\hat{y}-y|\le\frac{1}{\inf|(\ell_\xi^m)'|}\max\bigl(\widehat{\ell_\xi^m}(a_j^{(m)})-\ell_\xi^m(a_j^{(m)}),\ell_\xi^m(a_{j+1}^{(m)})-\widehat{\ell_\xi^m}(a_{j+1}^{(m)})\bigr)\le\frac{1}{s_2^{2m}}.$$

Hence for any $f\in W_\delta(\mathcal{L}_1(\mathbb{R}))$ we see that

$$|f(\hat{y})-f(y)|\le\frac{1}{s_2^{2m}}\sup(W_\delta g)'\le\frac{\|g\|_\infty}{s_2^{2m}\delta}\le\frac{2^{m/2}}{\delta s_2^{2m}}.$$

Summing up, since the total number of main branches is not more than 2^m, we get for the first term of (48):

$$\sum_{\mathfrak{a}_j^{(m)}\not\subset E_m}\sum_{i=N_l^2}^{N_r^2}\frac{2^{-m}}{|\Omega_i^2|}\int_{\Omega_i^2}\big|h(j,x)-\widehat{h}(j,x)\big|\mathrm{d}x \le \frac{2^{-m}}{\inf|\Omega_i^2|}\int_{-1}^{1}\frac{2^{3m/2}}{\delta s_2^{2m}}\mathrm{d}x \le 2\Big(\frac{s_1 2^{1/2+\alpha}}{s_2^2}\Big)^m. \tag{50}$$

To estimate the last term, we introduce two sets of indices

$$D \stackrel{\text{def}}{=} \big\{(s,t)\in\Box \mid \Omega_s^2\subset[-1,1]\cap\ell_\xi^m(\Omega_t^1\cap E_m)\big\};$$
$$\widehat{D} \stackrel{\text{def}}{=} \big\{(s,t)\in\Box \mid \Omega_s^2\subset[-1,1]\cap\widehat{\ell}_\xi^m(\Omega_t^1\cap E_m)\big\}.$$

By Lemma 5.13 and its Corollary 1, we see $\#D\le m^2\delta s_1^{2m}$ and $\#\widehat{D}\le m^2\delta s_1^{2m}$. Observe that

$$\bigcup_{i,j}\big\{(\mathfrak{a}_j^{(m)}\times\Omega_i^2)\mid \mathfrak{a}_j^{(m)}\subset E_m,\ \Omega_i^2\subset\widehat{\ell}_\xi^m(\mathfrak{a}_j^{(m)}\cap[-1,1])\big\}\subset$$
$$\big\{(\Omega_t^1\times\Omega_s^2)\mid (s,t)\in\Box,\ \Omega_s^2\subset[-1,1]\cap\widehat{\ell}_\xi^m(\Omega_t^1\cap E_m)\big\}.$$

along with

$$\bigcup_{i,j}\big\{(\mathfrak{a}_j^{(m)}\times\Omega_i^2)\mid \mathfrak{a}_j^{(m)}\subset E_m,\ \Omega_i^2\subset\ell_\xi^m(\mathfrak{a}_j^{(m)}\cap[-1,1])\big\}\subset$$
$$\big\{(\Omega_t^1\times\Omega_s^2)\mid (s,t)\in\Box,\ \Omega_s^2\subset[-1,1]\cap\ell_\xi^m(\Omega_t^1\cap E_m)\big\}.$$

. Hence we calculate an upper bound for the second term of (48):

$$2^{-m}\sum_{\mathfrak{a}_j^{(m)}\subset E_m}\sum_{i=N_l^2}^{N_r^2}\frac{1}{|\Omega_i^2|}\int_{\Omega_i^2}\big|h(j,x)-\widehat{h}(j,x)\big|\mathrm{d}x\le$$
$$\le 2^{-m}\sup|f|\sum_{\mathfrak{a}_j^{(m)}\subset E_m}\sum_{i=N_l^2}^{N_r^2}\frac{1}{|\Omega_i^2|}\int_{\Omega_i^2}\Big(\sum_{y\in\ell_\xi^{-m}(x)}\chi_{\mathfrak{a}_j^{(m)}}(y)+\sum_{\hat{y}\in\widehat{\ell}_\xi^{-m}(x)}\chi_{\mathfrak{a}_j^{(m)}}(\hat{y})\Big)\mathrm{d}x\le$$
$$\le 2^{-m}\sup|g|\sum_{(i,j)\in D\cup\widehat{D}}\frac{1}{|\Omega_i^2|}\int_{\Omega_i^2}|\tau_{ij}|\mathrm{d}x\le 2^{-m}\sup|g|\cdot\sup|\tau_{ij}|\cdot\#(D\cup\widehat{D})\le$$
$$\le\frac{1}{2^m}\cdot\|g\|_\infty\cdot\Big(\frac{s_1}{s_2}\Big)^m\cdot m^2\delta s_1^{2m}\le m^2\cdot\Big(\frac{s_1^3}{2^{1/2+\alpha}s_2}\Big)^m. \tag{51}$$

Now we collect the four estimates (46), (47), (49), (50), and (51) together and get for any function g with $\|g\|_1 = 1$:

$$\|\ell_{\xi*}^m W_\delta g - \widehat{\ell}_{\xi*}^m W_\delta g\|_{\mathcal{L}1,\Omega^2} \leq \leq 16\frac{mN_\delta}{s_2^m \delta} + m\Big(\frac{2^{(1/2-\alpha)}s_1^2}{s_2}\Big)^m + 2\Big(\frac{s_1 2^{1/2+\alpha}}{s_2^2}\Big)^m + 2m^2\Big(\frac{s_1^3}{2^{1/2+\alpha}s_2}\Big)^m \leq \leq 3m^2\Big(\frac{s_1^3}{2^{1/2+\alpha}s_2}\Big)^m. \quad (52)$$

for m large enough and $s_2 < 2 < s_1$ chosen such that $s_1 s_2^2 \geq 2^{1/2+2\alpha}$.

Now we turn our attention to the supremum norm. We may write

$$\sup_x\big|\ell_{\xi*}^m f(x) - \widehat{\ell}_{\xi*}^m f(x)\big| = \sup_x\Big|\sum_{y\in\widehat{\ell}_\xi^{-m}(x)} \operatorname{sgn}\big(\widehat{\ell}_\xi^m\big)'(y)f(y) - \sum_{y\in\ell_\xi^{-m}(x)} \operatorname{sgn}\big(\ell_\xi^m\big)'(y)f(y)\Big| \leq$$
$$\leq \sup_x\Big|\sum_{i\in\mathbb{Z}}\Big(\sum_{y\in\widehat{\ell}_\xi^{-m}(x)\cap\Omega_i^1} \operatorname{sgn}\big(\widehat{\ell}_\xi^m\big)'(y)f(y) - \sum_{y\in\ell_\xi^{-m}(x)\cap\Omega_i^1} \operatorname{sgn}\big(\ell_\xi^m\big)'(y)f(y)\Big)\Big| \leq$$
$$\leq \sup_x\Big|\Big(\sum_{-\infty}^{N_l^1-mN_\delta} + \sum_{N_l^1-mN_\delta}^{+\infty}\Big)\Big(\sum_{y\in\widehat{\ell}_\xi^{-m}(x)\cap\Omega_i^1} \operatorname{sgn}\big(\widehat{\ell}_\xi^m\big)'(y)f(y) - \sum_{y\in\ell_\xi^{-m}(x)\cap\Omega_i^1} \operatorname{sgn}\big(\ell_\xi^m\big)'(y)f(y)\Big)\Big| +$$
$$+ \sup_x\Big|\sum_{N_l^1-mN_\delta}^{N_r^1+mN_\delta}\Big(\sum_{y\in\widehat{\ell}_\xi^{-m}(x)\cap\Omega_i^1} \operatorname{sgn}\big(\widehat{\ell}_\xi^m\big)'(y)f(y) - \sum_{y\in\ell_\xi^{-m}(x)\cap\Omega_i^1} \operatorname{sgn}\big(\ell_\xi^m\big)'(y)f(y)\Big)\Big|. \quad (53)$$

Observe that

$$\sup_x\Big|\sum_{N_l^1-mN_\delta}^{N_r^1+mN_\delta}\Big(\sum_{y\in\widehat{\ell}_\xi^{-m}(x)\cap\Omega_i^1} \operatorname{sgn}\big(\widehat{\ell}_\xi^m\big)'(y)f(y) - \sum_{y\in\ell_\xi^{-m}(x)\cap\Omega_i^1} \operatorname{sgn}\big(\ell_\xi^m\big)'(y)f(y)\Big)\Big| \leq$$
$$\leq 2\sup_x \sum_{N_l^1-mN_\delta}^{N_r^1+mN_\delta} \#\{y\in\ell_\xi^{-m}(x)\cap\Omega_i^1\}\sup_{\Omega_i}|f(y)| \leq 2\sup_x \sum_{N_l^1-mN_\delta}^{N_r^1+mN_\delta}|\tau_{ij}|\sup_{\Omega_i}|f(y)| \leq$$
$$\leq \sup|\tau_{ij}|\sum_{N_l^1-mN_\delta}^{N_r^1+mN_\delta}\sup_{\Omega_i^1}|f(y)|. \quad (54)$$

Our goal is to estimate the last sum from above via weighted $\mathcal{L}_1$-norm. Recall that $f = W_\delta g$. By definition of the weighted $\mathcal{L}_1$ norm, we see

$$\|W_\delta g\|_1 \geq \sum_{N_l^1 - mN_\delta}^{N_r^1 + mN_\delta} \frac{2^{-m}}{|\Omega_i^1|} \int_{\Omega_i^1} |W_\delta g| =$$

$$= 2^{-m} \sum_{N_l^1 - mN_\delta}^{N_r^1 + mN_\delta} \Big(\frac{1}{|\Omega_i^1|} \int_{\Omega_i^1} |W_\delta g| - \sup_{\Omega_i^1} |W_\delta g| \Big) + \sum_{N_l^1 - mN_\delta}^{N_r^1 + mN_\delta} 2^{-m} \sup_{\Omega_i^1} |W_\delta g|;$$

in particular,

$$2^{-m} \sum_{N_l^1 - mN_\delta}^{N_r^1 + mN_\delta} \sup_{\Omega_i^1} |W_\delta g| \leq \|W_\delta g\|_1 + 2^{-m} \sum_{N_l^1 - mN_\delta}^{N_r^1 + mN_\delta} \Big(\sup_{\Omega_i^1} |W_\delta g| - \frac{1}{|\Omega_i^1|} \int_{\Omega_i^1} |W_\delta g| \Big). \quad (55)$$

We know that for any bounded, continuous, absolutely integrable, and piecewise differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ and any finite interval I

$$\Big| \sup_I f - \frac{1}{|I|} \int_I f \Big| \leq \int_I |f'|.$$

Therefore

$$\sum_{N_l^1 - mN_\delta}^{N_r^1 + mN_\delta} \Big| \sup_{\Omega_i^1} |W_\delta g| - \frac{1}{|\Omega_i^1|} \int_{\Omega_i^1} |W_\delta g| \Big| \leq \sum_{N_l^1 - mN_\delta}^{N_r^1 + mN_\delta} \int_{\Omega_i^1} \Big| \frac{\mathrm{d}}{\mathrm{d}x} |W_\delta g(x)| \Big| \mathrm{d}x <$$

$$< \int_{-2}^{2} \Big| \frac{\mathrm{d}}{\mathrm{d}x} |W_\delta g(x)| \Big| \leq \int_{\mathbb{R}} \Big| \frac{\mathrm{d}}{\mathrm{d}x} \int_{\mathbb{R}} w_\delta(x-t) |g(t)| \mathrm{d}t \Big| \mathrm{d}x \leq \int_{\mathbb{R}} \int_{\mathbb{R}} \Big| \frac{\mathrm{d}w_\delta(x-t)}{\mathrm{d}x} \Big| \cdot |g(t)| \mathrm{d}t \mathrm{d}x =$$

$$= \int_{\mathbb{R}} |g(t)| \int_{\mathbb{R}} \Big| \frac{\mathrm{d}w_\delta(x-t)}{\mathrm{d}x} \Big| \mathrm{d}x \mathrm{d}t \leq \frac{1}{\sqrt{2\pi}\delta} \sum_{j \in \mathbb{Z}} \int_{\Omega_j^1} |g(t)| \mathrm{d}t \leq \frac{2^m}{\delta} \sup |\Omega_j^1| \cdot \|g\|_{\mathcal{L}_1, \Omega^1}. \quad (56)$$

Hence, substituting (56) to (55), and using Lemma 6.3

$$\sum_{N_l^1 - mN_\delta}^{N_r^1 + mN_\delta} \sup_{\Omega_i^1} |W_\delta g| \leq 2^m \|W_\delta g\|_1 + \frac{2^m \sup |\Omega_i^1|}{\delta} \|g\|_1 \leq \Big(\frac{2^m N_\delta}{s_2^m \delta} + \frac{2^m}{s_2^m \delta} \Big) \|g\|_1 \leq$$

$$\leq \frac{2^{m+1} N_\delta}{s_2^m \delta} \|g\|_1. \quad (57)$$

Finally, taking into account $\|g\|_{\Omega_1} = 1$, we substitute (57) to (54) and get for the second term of (53)

$$2^{-m/2}\sup_x\Big|\sum_{N_l^1-mN_\delta}^{N_r^1+mN_\delta}\Big(\sum_{y\in\widehat{\ell}_\xi^{-m}(x)\cap\Omega_i^1}\operatorname{sgn}\big(\widehat{\ell}_\xi^m\big)'(y)f(y)-\sum_{y\in\ell_\xi^{-m}(x)\cap\Omega_i^1}\operatorname{sgn}\big(\ell_\xi^m\big)'(y)f(y)\Big)\Big|\le$$
$$\le\sup|\tau_{ij}|\frac{2^{m/2+1}N_\delta}{s_2^m\delta}\le\frac{2m^2N_\delta}{\delta}\Big(\frac{2^{1/2}s_1}{s_2^2}\Big)^m. \quad (58)$$

Let us define $A(x)\overset{\text{def}}{=}(x-s_2^{-m},x+s_2^{-m})$. We have the following upper bound for the first sum in (53):

$$\sup_x\Big|\Big(\sum_{-\infty}^{N_l^1-mN_\delta}+\sum_{N_l^1-mN_\delta}^{+\infty}\Big)\Big(\sum_{y\in\widehat{\ell}_\xi^{-m}(x)\cap\Omega_i^1}\operatorname{sgn}\big(\widehat{\ell}_\xi^m\big)'(y)f(y)-\sum_{y\in\ell_\xi^{-m}(x)\cap\Omega_i^1}\operatorname{sgn}\big(\ell_\xi^m\big)'(y)f(y)\Big)\Big|\le$$
$$\le\sup_{x\in\mathbb{R}}\sup_{y_1,y_2\in A(x)}|f(y_1)-f(y_2)|\le\sup_{|y_1-y_2|\le 2s_2^{-m}}|f(y_1)-f(y_2)|\le\frac{\sup|f|}{2\delta s_2^m}\le\frac{2^{m/2}}{2\delta s_2^m}. \quad (59)$$

Summing up (58) and (59), we get in (53)

$$2^{-m/2}\sup_x\big|\ell_{\xi*}^m f(x)-\widehat{\ell}_{\xi*}^m f(x)\big|\le\frac{2m^2N_\delta}{\delta}\Big(\frac{2^{1/2}s_1}{s_2^2}\Big)^m+\frac{1}{2\delta s_2^m}\le 3m\Big(\frac{s_1^3}{2^{1/2+\alpha}s_2}\Big)^m, \quad (60)$$

(by straightforward calculation).

■

6. Invariant cone in $\mathbf{\Phi}$.

In this section we construct an invariant cone in the space of essentially bounded and absolutely integrable functions Φ for the operator $W_{\frac{\delta}{2m}}\ell_{\xi*}^m W_{\frac{\delta}{2m}}$, which is independent of the choice of $\|\xi\|\le\delta$. We exploit the properties of the Weierstrass transform that we prove below.

6.1. **Discretization and the Weierstrass transform toolbox.**

Here we prove a few estimates showing that the image of the Weierstrass transform with Gaussian kernel of a large variance compared to the size of elements of a partition may be very well approximated by a step function on the partition.

Definition 11. Given a partition Ω of the class $\mathcal{G}$ we define *a linear discretization operator* D_Ω:

$$D_\Omega\colon \mathcal{L}_1(\mathbb{R})\cap\mathcal{L}_\infty(\mathbb{R})\to\Phi_\Omega\cap\mathcal{L}_1(\mathbb{R})\cap\mathcal{L}_\infty(\mathbb{R});$$
$$D_\Omega\colon f\mapsto\sum_{j\in\mathbb{Z}} d_j\chi_{\Omega_j},\quad d_j=\frac{1}{2}\big(\max_{\Omega_j} f(x)+\min_{\Omega_j} f(x)\big). \tag{61}$$

Definition 12. *The Weierstrass transform W_δ* is a convolution with the Gaussian kernel with variance δ^2

$$W_\delta\colon f\mapsto w_\delta * f,\text{ where } w_\delta(x)=\frac{1}{\sqrt{2\pi}\delta}e^{-\frac{x^2}{2\delta^2}}. \tag{62}$$

Lemma 6.1. *Let $f\colon\mathbb{R}\to\mathbb{R}$ be a differentiable function. Then*

$$\|f-D_\Omega f\|_{\Omega,\mathcal{L}_1}\le 2^{-m-1}\int_{\mathbb{R}}\Big|\frac{\mathrm{d}f(x)}{\mathrm{d}x}\Big|\mathrm{d}x. \tag{63}$$

Proof. Indeed, by straightforward calculation,

$$\begin{aligned}\|f-D_\Omega f\|_{\Omega,\mathcal{L}_1}&=\sum_{k\in\mathbb{Z}}\frac{2^{-m}}{|\Omega_k|}\int_{\Omega_k}|f(x)-D_\Omega f(x)|\mathrm{d}x\le\\&\le\sum_{k\in\mathbb{Z}}\frac{2^{-m}}{|\Omega_k|}\int_{\Omega_k}|\max_{\Omega_k} f(x)-\min_{\Omega_k} f(x)|\mathrm{d}x=\\&\le 2^{-m}\sum_{k\in\mathbb{Z}}|\max_{\Omega_k} f(x)-\min_{\Omega_k} f(x)|\le 2^{-m}\sum_{k\in\mathbb{Z}}\int_{\Omega_k}\Big|\frac{\mathrm{d}f(x)}{\mathrm{d}x}\Big|\mathrm{d}x=\\&=2^{-m}\int_{\mathbb{R}}\Big|\frac{\mathrm{d}f(x)}{\mathrm{d}x}\Big|\mathrm{d}x.\end{aligned}$$

∎

Lemma 6.2. *Let Ω^1 and Ω^2 be two partitions of the class $\mathcal{G}(m,\delta,s_1,s_2)$. Let D_{Ω^1} be a discretization operator and let W_δ be the Weierstrass transform defined above. Then for any bounded integrable function f*

$$\|D_{\Omega^1}W_\delta f-W_\delta f\|_1\le\frac{\max(\sup|\Omega^1_j|,\sup|\Omega^2_j|)}{\delta}\|f\|_2\le\frac{1}{s_2^m\delta}\|f\|_2.$$

Remark 5. The dispersion δ in the Gaussian kernel is the same δ as in the definition of a partition of the class $\mathcal{G}$.

Proof. We begin with estimation of the $\mathcal{L}_\infty$-norm. Let $D_{\Omega^1}W_\delta f = \sum_{j\in\mathbb{Z}} d_j\chi_{\Omega^1_j}$. Then

$$\begin{aligned}
\|D_{\Omega^1}W_\delta f - W_\delta f\|_\infty &= \sup_{x\in\mathbb{R}}\Big|\int_{\mathbb{R}} w_\delta(x-t)f(t)\mathrm{d}t - \sum_{j\in\mathbb{Z}} d_j\chi_{\Omega^1_j}(x)\Big| = \\
&= \frac{1}{2}\sup_{k\in\mathbb{Z}}\Big|\max_{\Omega^1_k}\int_{\mathbb{R}} w_\delta(x-t)f(t)\mathrm{d}t - \min_{\Omega^1_k}\int_{\mathbb{R}} w_\delta(x-t)f(t)\mathrm{d}t\Big| \leq \\
&\leq \frac{1}{2}\sup_{k\in\mathbb{Z}}\Big(|\Omega^1_k|\cdot\max_{\Omega^1_k}\Big|\frac{\mathrm{d}}{\mathrm{d}x}\int_{\mathbb{R}} w_\delta(x-t)f(t)\mathrm{d}t\Big|\Big) \leq \\
&\leq \sup|\Omega^1_k|\sup_{x\in\mathbb{R}}\Big|\int_{\mathbb{R}}\frac{\mathrm{d}}{\mathrm{d}x}w_\delta(x-t)f(t)\mathrm{d}t\Big| \leq \\
&\leq \sup|\Omega^1_k|\sup|f(x)|\sup_{x\in\mathbb{R}}\frac{1}{\sqrt{2\pi}\delta}\Big(e^{-\frac{(x-t)^2}{2\delta^2}}\Big|_{t=-\infty}^{x} - e^{-\frac{(x-t)^2}{2\delta^2}}\Big|_{x}^{t=+\infty}\Big) \leq \\
&\leq \sup\frac{|\Omega^1_k|}{\delta}\|f\|_\infty.
\end{aligned}$$

Now we proceed to the weighted $\mathcal{L}_1$-norm. Using Lemma 6.1 we get

$$\begin{aligned}
&\|D_{\Omega^1}W_\delta f - W_\delta f\|_1 \leq 2^{-m-1}\int_{\mathbb{R}}\Big|\frac{\mathrm{d}}{\mathrm{d}x}W_\delta f(x)\Big|\mathrm{d}x = \\
&= 2^{-m-1}\int_{\mathbb{R}}\Big|\frac{\mathrm{d}}{\mathrm{d}x}\int_{\mathbb{R}} w_\delta(x-t)f(t)\mathrm{d}t\Big|\mathrm{d}x \leq 2^{-m}\int_{\mathbb{R}}\int_{\mathbb{R}}\Big|\frac{\mathrm{d}w_\delta(x-t)}{\mathrm{d}x}\Big|\cdot|f(t)|\mathrm{d}t\mathrm{d}x = \\
&= 2^{-m-1}\int_{\mathbb{R}}|f(t)|\int_{\mathbb{R}}\Big|\frac{\mathrm{d}w_\delta(x-t)}{\mathrm{d}x}\Big|\mathrm{d}x\mathrm{d}t = \frac{2^{-m}}{\sqrt{2\pi}\delta}\sum_{j\in\mathbb{Z}}\int_{\Omega_j}|f(t)|\mathrm{d}t \leq \frac{\sup|\Omega^2_j|}{\delta}\|f\|_{\Omega^2}.
\end{aligned}$$

■

Lemma 6.3. *Let Ω^1 and Ω^2 be two partitions of the class $\mathcal{G}(\delta, s_1, s_2)$. Then an upper bound of the norm of the Weierstrass transform is given by*

$$\|W_\delta f\|_2 \leq 2m\cdot\sup|\Omega^1_j|\cdot\frac{N_\delta}{\delta}\|f\|_1 \leq \frac{mN_\delta}{s_2^m\delta}\|f\|_1. \tag{64}$$

Proof. We estimate the norm of the operator W_δ on step functions first. Let $\phi\in\Phi_{\Omega^1}$ be a step function on Ω^1. Assume that $\phi = \sum\limits_{j\in\mathbb{Z}} c_j\chi_{\Omega^1_j}$ and $\|\phi\|_{\Omega^1} = 1$, that is

$$\max\Big(2^{-m}\sum_{j\in\mathbb{Z}}|c_j|, 2^{-m/2}\sup|c_j|\Big) = 1;$$

which implies

$$\sum_{j\in\mathbb{Z}}|c_j| \leq 2^m, \quad \sup|c_j| \leq 2^{m/2}.$$

Then

$$W_\delta\phi(x) = \sum_{j\in\mathbb{Z}}\int_{\Omega^2_j} c_j w_\delta(x-t)\mathrm{d}t.$$

So we calculate

$$\begin{aligned}\|W_\delta\phi\|_{\Omega^2,\mathcal{L}_1} &= \sum_{k\in\mathbb{Z}}\frac{2^{-m}}{|\Omega^2_k|}\int_{\Omega^2_k}\Big|\sum_{j\in\mathbb{Z}}c_j\int_{\Omega^1_j}w_\delta(x-t)\mathrm{d}t\Big|\mathrm{d}x \le\\ &\le \sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\frac{|c_j|}{2^m|\Omega^2_k|}\int_{\Omega^2_k}\int_{\Omega^1_j}w_\delta(x-t)\mathrm{d}t\mathrm{d}x =\\ &= \sum_{j\in\mathbb{Z}}\frac{|c_j|}{2^m}\int_{\Omega^1_j}\sum_{k\in\mathbb{Z}}\frac{1}{|\Omega^2_k|}\int_{\Omega^2_k}w_\delta(x-t)\mathrm{d}x\mathrm{d}t =\\ &= \sum_{j\in\mathbb{Z}}\frac{|c_j|}{2^m}\int_{\Omega^1_j}\Big(\sum_{|\Omega^2_k-\Omega^1_j|>m\delta}+\sum_{|\Omega^2_k-\Omega^1_j|<m\delta}\Big)\frac{1}{|\Omega^2_k|}\int_{\Omega^2_k}w_\delta(x-t)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

We know that

$$\frac{1}{|\Omega^2_k|}\int_{\Omega^2_k}w_\delta(x-t)\mathrm{d}x < \frac{1}{\delta}.$$

We also observe that for any $t\in\Omega^1_j$

$$\sum_{|\Omega^2_k-\Omega^1_j|>m\delta}\frac{1}{|\Omega^2_k|}\int_{\Omega^2_k}w_\delta(x-t)\mathrm{d}x \le \frac{1}{\inf|\Omega^2_k|}\Big(\int_{-\infty}^{-t-m\delta}w_\delta(x-t)+\int_{t+m\delta}^{+\infty}w_\delta(x-t)\mathrm{d}x\Big)\mathrm{d}x \le \frac{e^{-m}}{\inf|\Omega^2_k|}.$$

Therefore, taking into account that $2^{-m}\sum_{j\in\mathbb{Z}}|c_j|\le 1$,

$$\|W_\delta\phi\|_{\Omega^2,\mathcal{L}_1} \le \sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\frac{|c_j|}{2^m}\Big(\frac{e^{-m}}{\inf|\Omega^2_k|}+\frac{mN_\delta}{\delta}\Big)|\Omega^1_j| \le \sup|\Omega^1_j|\Big(\frac{e^{-m}}{\inf|\Omega^2_k|}+\frac{mN_\delta}{\delta}\Big).$$

Now we consider arbitrary function $f\in\mathcal{L}_1(\mathbb{R})\cap\mathcal{L}_\infty(\mathbb{R})$ with $\|f\|_1=1$. Then

$$\begin{aligned}\|W_\delta f\|_{\Omega^2,\mathcal{L}_1} &= \sum_{k\in\mathbb{Z}}\frac{2^{-m}}{|\Omega^2_k|}\int_{\Omega^2_k}\Big|\sum_{j\in\mathbb{Z}}\int_{\Omega^1_j}w_\delta(x-t)f(t)\mathrm{d}t\Big|\mathrm{d}x \le\\ &\le 2^{-m}\sum_{j\in\mathbb{Z}}\int_{\Omega^1_j}|f(t)|\sum_{k\in\mathbb{Z}}\frac{1}{|\Omega^2_k|}\int_{\Omega^2_k}w_\delta(x-t)\mathrm{d}x\mathrm{d}t =\\ &= 2^{-m}\sum_{j\in\mathbb{Z}}\int_{\Omega^1_j}|f(t)|\Big(\sum_{|\Omega^2_k-\Omega^1_j|>m\delta}+\sum_{|\Omega^2_k-\Omega^1_j|<m\delta}\Big)\frac{1}{|\Omega^2_k|}\int_{\Omega^2_k}w_\delta(x-t)\mathrm{d}x\mathrm{d}t \le\\ &\le 2^{-m}\sum_{j\in\mathbb{Z}}\int_{\Omega^1_j}|f(t)|\Big(\frac{e^{-m}}{\inf|\Omega^2_k|}+\frac{3mN_\delta}{\delta}\Big)\mathrm{d}t \le\\ &\le \sup|\Omega^1_j|\Big(\frac{e^{-m}}{\inf|\Omega^2_k|}+\frac{3mN_\delta}{\delta}\Big).\end{aligned}$$

In the last inequality we take into account that

$$\|f\|_{\Omega^1,\mathcal{L}_1} = 2^{-m}\sum_{j\in\mathbb{Z}}\frac{1}{|\Omega^1_j|}\int_{\Omega^1_j}|f(t)|\mathrm{d}t \le 1.$$

Now we recall that $\inf |\Omega^1_j| \geq s_1^{-m}/m$ and therefore, for $s_1 < e$

$$\frac{e^{-m}}{\inf |\Omega^2_k|} = \left(\frac{s_1}{e}\right)^m \ll 1,$$

while

$$\frac{N_\delta}{\delta} = 2^{m(1-\alpha \log_{s_1} 2+\alpha)} > 2^m.$$

Therefore we conclude

$$\|W_\delta f\|_{\Omega^2,\mathcal{L}_1} \leq 2 \sup |\Omega^1_j| \cdot \frac{3mN_\delta}{\delta}.$$

The upper bound of the supremum norm is easy.

$$\|W_\delta f\|_\infty = \sup_{x\in\mathbb{R}} \Big| \int_{\mathbb{R}} w_\delta(x-t) f(t) \mathrm{d}t \Big| \leq \sup_{x\in\mathbb{R}} |f(x)|.$$

■

Lemma 6.4. *Let* Ω *be a partition of the class* $\mathcal{G}(s_1, s_2, \delta, m)$ *where the parameters* s_1 *and* $\delta = 2^{-m\alpha}$ *satisfy the inequality* $\log_2 s_1 < 2\alpha$ *then*

$$\|W_\delta \chi_{[-1,1]} - \chi_{[-1,1]}\|_\Omega \leq 2^{-m/2}. \tag{65}$$

Proof. Obviously, $\sup |W_\delta \chi_{[-1,1]}(x) - \chi_{[-1,1]}(x)| \leq 1$. Now we have to find an upper bound for $\|W_\delta \chi_{[-1,1]} - \chi_{[-1,1]}\|_{\Omega,\mathcal{L}_1}$.

$$\|W_\delta \chi_{[-1,1]} - \chi_{[-1,1]}\|_{\Omega,\mathcal{L}_1} = \sum_{j\in\mathbb{Z}} \frac{2^{-m}}{|\Omega_j|} \int_{\Omega_j} \Big| \int_{\mathbb{R}} w_\delta(x-t)\chi_{[-1,1]}(t)\mathrm{d}t - \chi_{[-1,1]}(x) \Big| \mathrm{d}x =$$

We split the sum into two parts: over the intervals inside $[-1,1]$ and the rest

$$= \sum_{j=N_l}^{N_r} \frac{2^{-m}}{|\Omega_j|} \int_{\Omega_j} \Big(1 - \int_{-1}^{1} w_\delta(x-t)\mathrm{d}t\Big)\mathrm{d}x + \Big(\sum_{j>N_r} + \sum_{j<N_l}\Big) \frac{2^{-m}}{|\Omega_j|} \int_{\Omega_j} \int_{-1}^{1} w_\delta(x-t)\mathrm{d}t\mathrm{d}x. \tag{66}$$

We begin with the first term of (66), that is the sum of the intervals of partition inside the interval $[-1,1]$.

$$\sum_{j=N_l}^{N_r} \frac{2^{-m}}{|\Omega_j|} \int_{\Omega_j} \Big(1 - \int_{-1}^{1} w_\delta(x-t)\mathrm{d}t\Big)\mathrm{d}x =$$

$$= \Big(\sum_{j=N_l}^{N_l+mN_\delta} + \sum_{j=N_l+mN_\delta}^{N_r-mN_\delta} + \sum_{j=N_r-mN_\delta}^{N_r}\Big) \frac{2^{-m}}{|\Omega_j|} \int_{\Omega_j} \Big(1 - \int_{-1}^{1} w_\delta(x-t)\mathrm{d}t\Big)\mathrm{d}x. \tag{67}$$

We estimate each term separately. The first term of (67) has only mN_δ elements:

$$\sum_{j=N_l}^{N_l+mN_\delta} \frac{2^{-m}}{|\Omega_j|}\int_{\Omega_j}\Big(1-\int_{-1}^{1} w_\delta(x-t)\mathrm{d}t\Big)\mathrm{d}x \le \sum_{j=N_l}^{N_l+mN_\delta} \frac{2^{-m}}{|\Omega_j|}\int_{\Omega_j}\Big(1-\int_{-1}^{1} w_\delta(t+1)\mathrm{d}t\Big)\mathrm{d}x \le$$
$$\le m2^{-m}N_\delta\Big(1-\int_{0}^{2} w_\delta(t)\mathrm{d}t\Big).$$

We have the following upper bound for the second term of (67), since for $|t|<1-m\delta$ the integral $\int_{-1}^{1} w_\delta(x-t)\mathrm{d}x$ is close to 1:

$$\sum_{j=N_l+mN_\delta}^{N_r-mN_\delta} \frac{2^{-m}}{|\Omega_j|}\int_{\Omega_j}\Big(1-\int_{-1}^{1} w_\delta(x-t)\mathrm{d}t\Big)\mathrm{d}x \le$$
$$\le \sum_{j=N_l+mN_\delta}^{N_r-mN_\delta} \frac{2^{-m}}{|\Omega_j|}\int_{\Omega_j}\Big(1-\int_{-1}^{1} w_\delta(1-m\delta-t)\mathrm{d}t\Big)\mathrm{d}x \le$$
$$\le 2^{-m}(N_r-N_l-2mN_\delta)\Big(1-\int_{-m\delta}^{2-m\delta} w_\delta(t)\mathrm{d}t\Big).$$

The third term of (67) has only mN_δ elements, so we write

$$\sum_{j=N_r-mN_\delta}^{N_r} \frac{2^{-m}}{|\Omega_j|}\int_{\Omega_j}\Big(1-\int_{-1}^{1} w_\delta(x-t)\mathrm{d}t\Big)\mathrm{d}x \le$$
$$\le \sum_{j=N_r-mN_\delta}^{N_r} \frac{2^{-m}}{|\Omega_j|}\int_{\Omega_j}\Big(1-\int_{-1}^{1} w_\delta(1-t)\mathrm{d}t\Big)\mathrm{d}x \le mN_\delta 2^{-m}\Big(1-\int_{0}^{2} w_\delta(t)\mathrm{d}t\Big).$$

Putting all three inequalities together, we get the following upper bound for the first term of (66):

$$\sum_{j=N_l}^{N_r} \frac{2^{-m}}{|\Omega_j|}\int_{\Omega_j}\Big(1-\int_{-1}^{1} w_\delta(x-t)\mathrm{d}t\Big)\mathrm{d}x \le$$
$$\le \frac{2N_\delta}{2^m}\cdot\Big(\frac{1}{2}+\delta\Big)+\Big(1-\frac{N_\delta}{2^{m-1}}\Big)\Big(1-\int_{-m\delta}^{2-m\delta} w_\delta(t)\mathrm{d}t\Big) \le$$
$$\le \frac{2N_\delta}{2^m}\cdot\Big(\frac{1}{2}+\delta\Big)+\Big(1-\frac{N_\delta}{2^{m}}\Big)\Big(\int_{-\infty}^{-m\delta} w_\delta(t)\mathrm{d}t+\int_{2-m\delta}^{+\infty} w_\delta(t)\mathrm{d}t\Big) \le$$
$$\le \frac{2N_\delta}{2^m}\cdot\Big(\frac{1}{2}+\delta\Big)+\Big(1-\frac{N_\delta}{2^{m}}\Big)\Big(\frac{e^{-m}+e^{-1/\delta}}{\sqrt{\pi}}\Big) \le \frac{2N_\delta}{2^m}.$$

Recall that $N_\delta \le 2^{m(1-\alpha \log_{s_1} 2)}$ by definition of the partition of the class $\mathcal{G}$. Therefore we complete the estimation of the first term of (66) :

$$\sum_{j=N_l}^{N_r} \frac{2^{-m}}{|\Omega_j|}\int_{\Omega_j}\Big(1-\int_{-1}^{1} w_\delta(x-t)\mathrm{d}t\Big)\mathrm{d}x \le 2^{-m\alpha \log_{s_1} 2} \le 2^{-m/2}. \tag{68}$$

Now we proceed to the upper bound for the second term of (66).

$$\Big(\sum_{j>N_r}+\sum_{j<N_l}\Big)\frac{2^{-m}}{|\Omega_j|}\int_{\Omega_j}\int_{-1}^{1} w_\delta(x-t)\mathrm{d}t\mathrm{d}x \le$$

$$\le \frac{2^{-m}}{\inf|\Omega_j|}\Big(\int_{-1-m\delta}^{-1}+\int_{1}^{1+m\delta}\Big)\int_{-1}^{1} w_\delta(x-t)\mathrm{d}t\mathrm{d}x+\Big(\int_{-\infty}^{-1-m\delta}+\int_{1+m\delta}^{+\infty}\Big)\int_{-1}^{1} w_\delta(x-t)\mathrm{d}t\mathrm{d}x \le$$

$$\le \frac{m\delta}{2^m \inf|\Omega_j|}\int_0^2 w_\delta(t)\mathrm{d}t+\int_{-1}^{1}\int_{1+m\delta}^{+\infty} w_\delta(x+t)\mathrm{d}x\mathrm{d}t+\int_{-1}^{1}\int_{1+m\delta}^{+\infty} w_\delta(x-t)\mathrm{d}x\mathrm{d}t \le$$

$$\le \frac{2m\delta}{2^m \inf|\Omega_j|}\Big(\frac{1}{2}-\delta\Big)+\int_{-1}^{1}\int_{1+m\delta}^{+\infty} w_\delta(t-1)\mathrm{d}t\mathrm{d}x+\int_{-1}^{1}\int_{1+m\delta}^{+\infty} w_\delta(t-1)\mathrm{d}t\mathrm{d}x \le$$

$$\le \frac{2m\delta}{2^m \inf|\Omega_j|}+2e^{-m}.$$

We observe that

$$\frac{2m\delta}{2^m \inf|\Omega_j|}+2e^{-m} = \frac{2ms_1^m}{2^{m(1+\alpha)}}+e^{-m} \le 2^{-m/2-1},$$

under condition that $s_1 < 2^{1/2+\alpha}$. Therefore, we get the following upper bound for the second term of (66)

$$\Big(\sum_{j>N_r}+\sum_{j<N_l}\Big)\frac{2^{-m}}{|\Omega_j|}\int_{\Omega_j}\int_{-1}^{1} w_\delta(x-t)\mathrm{d}t\mathrm{d}x \le 2^{-m/2-1}. \tag{69}$$

Summing up (68) with (69), we get (65). ■

Proposition 6.1. *Let Ω^1 and Ω^2 be two partitions of the class $\mathcal{G}(\delta)$. Let $\varepsilon_1 = 2^{m(\gamma-1/2)}$. Let $\phi \in \mathrm{Cone}\,(\varepsilon_1,\Omega^1)$ be a step function. (See p. 3 for a general definition of cones). Then*

$$\|D_{\Omega^2}W_\delta\phi\|_2 > \frac{1}{4}\|\phi\|_1. \tag{70}$$

Proof. By Lemma 6.3 above, for any $\phi \in \mathrm{Cone}\,(\varepsilon_1,\Omega^1)$,

$$\|W_\delta\phi\|_2 \le \frac{mN_\delta}{s_2^m\delta}\|\phi\|_1.$$

By Lemma 6.2,

$$\|D_{\Omega^2}W_\delta\chi_{[-1,1]} - W_\delta\chi_{[-1,1]}\|_\infty \le 2. \tag{71}$$

We can find an upper bound for the weighted $\mathcal{L}_1$-norm using Lemma 6.1,

$$\|D_{\Omega^2} W_\delta \chi_{[-1,1]} - W_\delta \chi_{[-1,1]}\|_{\Omega^2,\mathcal{L}_1} \le 2^{-m-1} \int_{\mathbb{R}} \Big| \frac{\mathrm{d}}{\mathrm{d}x} W_\delta \chi_{[-1,1]}(x) \Big| \mathrm{d}x =$$
$$= 2^{-m-1} \int_{\mathbb{R}} \Big| \int_{-1}^{1} \frac{\mathrm{d}}{\mathrm{d}x} w_\delta(x-t)\mathrm{d}t \Big| \mathrm{d}x = 2^{-m-1} \int_{\mathbb{R}} |w_\delta(x+1) - w_\delta(x-1)| \mathrm{d}x \le 2^{-m}. \tag{72}$$

Therefore

$$\|D_{\Omega^2} W_\delta \chi_{[-1,1]} - W_\delta \chi_{[-1,1]}\|_2 \le 2^{1-m/2}. \tag{73}$$

Using Lemma 6.4,

$$\|W_\delta \chi_{[-1,1]}\|_{\Omega^1} \ge \|\chi_{[-1,1]}\|_2 - \|W_\delta \chi_{[-1,1]} - \chi_{[-1,1]}\|_2 \ge 1 - 2^{-m/2}.$$

Consider a step function $\eta = d\chi_{[-1,1]} + \psi \in \operatorname{Cone}(\varepsilon_1, \Omega^1)$, with $\|\psi\|_1 \le d$. By Lemma 6.2

$$\|W_\delta \psi - D_{\Omega^2} W_\delta \psi\|_2 \le \frac{1}{s_2^m \delta} \|\psi\|_1 \le d \frac{2^{m(\gamma_1 - 1/2)}}{s_2^m \delta}; \tag{74}$$

and by Lemma 6.3

$$\|W_\delta \psi\|_2 \le \frac{m N_\delta}{s_2^m \delta} \|\psi\|_1 \le d \frac{N_\delta 2^{m(\gamma_1 - 1/2)}}{s_2^m \delta}; \tag{75}$$

summing up the last two (74) and (75) together

$$\|D_{\Omega^2} W_\delta \psi\|_2 \le d 2^{m(\gamma_1 - 1/2)} \frac{N_\delta + 1}{s_2^m \delta}.$$

We have the following upper bound for the error of approximation for a function from the cone $\operatorname{Cone}(\varepsilon_1, \Omega^2)$, using the inequality (72), (73), and (74),

$$\|W_\delta \phi - D_{\Omega^2} W_\delta \phi\|_2 \le d\|W_\delta \chi_{[-1,1]} - D_{\Omega^2} W_\delta \chi_{[-1,1]}\|_2 + \|W_\delta \psi - D_{\Omega^2} W_\delta \psi\|_2 \le$$
$$\le d\Big(2^{1-m/2} + \frac{2^{m(\gamma_1 - 1/2)}}{s_2^m \delta}\Big). \tag{76}$$

We may also write using and Lemma 6.4 and (75)

$$\|W_\delta \phi\|_2 = \|d W_\delta \chi_{[-1,1]} + W_\delta \psi\|_2 \ge d\|W_\delta \chi_{[-1,1]}\|_2 - \|W_\delta \psi\|_2 \ge$$
$$\ge d\big(\|\chi_{[-1,1]}\|_2 - \|W_\delta \chi_{[-1,1]} - \chi_{[-1,1]}\|_2\big) - \|W_\delta \psi\|_2 \ge$$
$$\ge d\Big(\frac{1}{2} - 2^{-m/2} - \frac{N_\delta 2^{m(\gamma_1 - 1/2)}}{s_2^m \delta}\Big). \tag{77}$$

Hence we deduce from (76) and (77)

$$\|D_{\Omega^2}W_\delta\phi\|_2 \geq \|W_\delta\phi\|_2 - \|W_\delta\phi - D_{\Omega^2}W_\delta\phi\|_2 \geq \\ \geq d\Big(\frac{1}{2} - 2^{-m/2} - 2^{1-m/2} - 2^{m(\gamma_1-1/2)}\frac{(N_\delta+1)}{s_2^m\delta}\Big).$$

We can simplify and write, dividing by d,

$$\|D_{\Omega^2}W_\delta\phi\|_2 > \frac{1}{4}\|\phi\|_1.$$

■

6.2. **Constructing an invariant cone.** We shall construct an invariant cone around the cones for the discretized operator $\mathcal{T}$. First we extend the cones from Φ_{Ω^i} to Φ and obtain a pair of cones for $W_\delta\mathcal{T}$; which depend on the choice of the first partition and hence on the sequence ξ. Then we get rid of this dependence using estimeates from the previous Subsection.

Proposition 6.2. *Let Ω^1, Ω^2, Ω^3 be partitions of the class $\mathcal{G}(\delta)$. Let $\mathcal{T}\colon \Phi_{\Omega^1} \to \Phi_{\Omega^2}$ be a generalised toy dynamo. There exists a number $\frac{15}{16} < \alpha < 1$ such that for $\delta = 2^{-m\alpha}$ we may choose $\gamma_2 \stackrel{\text{def}}{=} \gamma_1 + \alpha(1-\log_{s_1}2) < 1/2$, and then for any $\eta \in \mathrm{Cone}\,(1,\Omega^1)$ we have*

$$D_{\Omega^3}W_\delta\mathcal{T}\colon \mathrm{Cone}\,\big(1,\Omega^1\big) \to \mathrm{Cone}\left(\frac{2^{m(\gamma_2+1/2)}}{s_2^m},\Omega^3\right) \tag{78}$$

$$\|D_{\Omega^3}W_\delta\mathcal{T}\eta\|_3 \geq 2^{m-3}\|\eta\|_1 \tag{79}$$

(See p. 3 for definition of the cone).

Proof. We define an operator $\mathcal{E}\colon \Phi_{\Omega^1} \to \Phi_{\Omega^2}$ as before in (27). According to Theorem 2 p. 16, we know that $\mathcal{T}\colon \mathrm{Cone}\,(1,\Omega^1) \to \mathrm{Cone}\,\big(2^{m(\gamma_1-1/2)},\Omega^2\big)$. Consider a step function $\eta = d\chi_{[-1,1]} + \psi \in \mathrm{Cone}\,(1,\Omega^1)$. Then $\mathcal{T}\eta = d(N_r^1 - N_l^1)\chi_{[-1,1]} + \psi_1$, where the norm is bounded $\|\psi_1\|_2 = \|\mathcal{E}\psi + (\mathcal{T}-\mathcal{E})\eta\|_2 \leq d2^{m(1/2+\gamma_1)}$. We may write

$$D_{\Omega^3}W_\delta\mathcal{T}\eta = D_{\Omega^3}W_\delta(d(N_r^1 - N_l^1)\chi_{[-1,1]} + \psi_1).$$

Using Lemmas 6.2 and 6.3

$$\|D_{\Omega^3}W_\delta\psi_1\|_3 \leq \|W_\delta\psi_1\|_3 + \|D_{\Omega^3}W_\delta\psi_1 - W_\delta\psi_1\|_3 \leq \frac{mN_\delta+1}{s_2^m\delta}\|\psi_1\|_2 \leq \\ \leq d2^{m(1/2+\gamma_1)}\frac{mN_\delta+1}{s_2^m\delta}. \tag{80}$$

So we conclude using Lemma 6.4 that

$$\|D_{\Omega^3}W_\delta\chi_{[-1,1]} - \chi_{[-1,1]}\|_3 \leq$$
$$\leq \|D_{\Omega^3}W_\delta\chi_{[-1,1]} - W_\delta\chi_{[-1,1]}\|_3 + \|W_\delta\chi_{[-1,1]} - \chi_{[-1,1]}\|_3 \leq 3\cdot 2^{-m/2} \quad (81)$$

Then we may write

$$D_{\Omega^3}W_\delta(d(N_r^1 - N_l^1)\chi_{[-1,1]}) = d(N_r^1 - N_l^1)\chi_{[-1,1]} + \psi_2,$$

where $\psi_2 \in \Phi_{\Omega^3}$ and (81)

$$\|\psi_2\|_3 \leq d(N_r^1 - N_l^1)\|D_{\Omega^3}W_\delta\chi_{[-1,1]} - \chi_{[-1,1]}\|_3 \leq d(N_r^1 - N_l^1)2^{1-m/2}.$$

Hence

$$D_{\Omega^3}W_\delta\mathcal{T}\eta = D_{\Omega^3}W_\delta(d(N_r^1 - N_l^1)\chi_{[-1,1]} + \psi_1) = D_{\Omega^3}W_\delta\psi_1 + d(N_r^1 - N_l^1)\chi_{[-1,1]} + \psi_2,$$

where using (80)

$$\frac{\|D_{\Omega^3}W_\delta\psi_1 + \psi_2\|_3}{d(N_r^1 - N_l^1)} \leq \frac{\|\psi_2\|_3 + \|D_{\Omega^3}W_\delta\psi_1\|_3}{d(N_r^1 - N_l^1)} \leq 2^{1-m/2} + \frac{2^{(\gamma_1+1/2)m}}{N_r^1 - N_l^1}\cdot\frac{mN_\delta + 1}{s_2^m\delta} \leq$$
$$\leq 2^{1-m/2} + \frac{2^{(\gamma_1-1/2)m+3}N_\delta}{s_2^m\delta}.$$

Substituting $\delta = 2^{-\alpha m}$ and $N_\delta = 2^{m(1-\alpha\log_{s_1}2)}$, we set $\gamma_2 := \gamma_1 + \alpha(1-\log_{s_1}2)$ and get

$$D_3W_\delta\mathcal{T}\eta = \tilde{d}\chi_{[-1,1]} + \psi_3, \text{ where } \|\psi_3\| \leq \tilde{d}\cdot\frac{2^{m(\gamma_2+1/2)}}{s_2^m}.$$

■

Definition 13. We extend the operator $\mathcal{E}$ defined between two spaces of step functions by (27) to bounded integrable functions. Given a partition Ω^1 of the class $\mathcal{G}$ we consider a map $g_0\colon \mathbb{R}\to\mathbb{R}$ by

$$g_0(x) = \begin{cases} 1 + \frac{2x-2b}{a-b}, & \text{if } a < x < b \text{ for some interval } (a,b) = \Omega_j^1 \subset [-1,1] \\ x, & \text{otherwise.} \end{cases} \quad (82)$$

and introduce *a linear operator* $\mathcal{E}\colon \mathcal{L}_1(\mathbb{R}) \to \mathcal{L}_\infty(\mathbb{R})$ defined by:

$$(\mathcal{E}f)(x) = \sum_{y\in g_0^{-1}(x)} f(y). \quad (83)$$

Lemma 6.5. *For any bounded integrable function* f

$$\int_{\mathbb{R}} |\ell_{\xi*}^m f(x)| \mathrm{d}x = \int_{\mathbb{R}} |\mathcal{E} f(x)| \mathrm{d}x + \frac{2^{m(3/2+\gamma_1)}}{s_2^m} \|f\|_1.$$

Where $0 < \gamma_1 \leq 1/8$ *is chosen such that*

$$m\delta \cdot \frac{s_1^{3m}}{2^m s_2^m} < 2^{m\gamma_1}.$$

Remark 6. The statement of Lemma 6.5 and the argument below hold true for the map $\widehat{\ell_\xi^m}$ as well.

Proof. Let $\mathfrak{a}^{(m)} \colon = \{-\infty = a_0^{(m)} < a_1^{(m)} < \ldots < a_{N+1}^{(m)} = +\infty\}$ be a set of points of discontinuity of the map $\ell_{\xi*}^m$, and let $\mathfrak{a}_j^{(m)} = (a_j^{(m)}, a_{j+1}^{(m)})$ be intervals of the partition. We can

Let us introduce a set of indices of long branches

$$I_l^{(m)} \stackrel{\text{def}}{=} \big\{1 \leq j \leq N \mid \mathfrak{a}_j^{(m)} \text{ is a domain of a long branch of the map } \ell_\xi^m\big\}.$$

split the integral into two

$$\int_{\mathbb{R}} |\ell_{\xi*}^m f(x)| \mathrm{d}x = \Big(\int_{-\infty}^{-1-m\delta} + \int_{1+m\delta}^{+\infty}\Big) |\ell_{\xi*}^m f(x)| \mathrm{d}x + \int_{-1-m\delta}^{1+m\delta} |\ell_{\xi*}^m f(x)| \mathrm{d}x.$$

To estimate the first term we recall that $\ell_{\xi*}^m(x) = (-1)^m x + \sum_{j=1}^m \xi(j)$ for $x < a_0^{(m)}$ and $x > a_N^{(m)}$. Since $\|\xi\|_\infty < \delta$, we see that $\big|\sum_{j=1}^m \xi(j)\big| < m\delta$ and write

$$\begin{aligned}
\Big(\int_{-\infty}^{-1-m\delta} + \int_{1+m\delta}^{+\infty}\Big) |\ell_{\xi*}^m f(x)| \mathrm{d}x &= \Big(\int_{-\infty}^{-1-m\delta} + \int_{1+m\delta}^{+\infty}\Big) \Big| \sum_{y \in \ell_\xi^{-m}(x)} \operatorname{sgn}(\ell_\xi^m)'(y) f(y) \Big| \mathrm{d}x = \\
&= \Big(\int_{-\infty}^{-1-m\delta} + \int_{1+m\delta}^{+\infty}\Big) \Big| (-1)^m f\big((-1)^m (x - \sum_{j=1}^m \xi(j))\big) \Big| \mathrm{d}x = \\
&= \Big(\int_{-\infty}^{-1-m\delta} + \int_{1+m\delta}^{+\infty}\Big) \Big| f\big((-1)^m (x - \sum_{j=1}^m \xi(j))\big) \Big| \mathrm{d}x \leq \\
&\leq \Big(\int_{-\infty}^{-1} + \int_{1}^{+\infty}\Big) |f(x)| \mathrm{d}x = \Big(\int_{-\infty}^{-1} + \int_{1}^{+\infty}\Big) |\mathcal{E} f(x)| \mathrm{d}x.
\end{aligned}$$

Consider the second term.

$$\int_{-1-m\delta}^{1+m\delta}\left|\ell_{\xi*}^m f(x)\right|\mathrm{d}x = \int_{-1-m\delta}^{1+m\delta}\Big|\sum_{y\in\ell_\xi^{-m}(x)}\operatorname{sgn}(\ell_\xi^m)'(y)f(y)\Big|\mathrm{d}x =$$

$$= \int_{-1-m\delta}^{1+m\delta}\Big|\sum_{j=1}^{N}\operatorname{sgn}(\ell_\xi^m)'(\ell_\xi^{-m}(x)\cap\mathfrak{a}_j^{(m)})f(\ell_\xi^{-m}(x)\cap\mathfrak{a}_j^{(m)})\Big|\mathrm{d}x =$$

$$= \int_{-1-m\delta}^{1+m\delta}\Big|\Big(\sum_{j\in I_l^{(m)}}+\sum_{j\notin I_l^{(m)}}\Big)\operatorname{sgn}(\ell_\xi^m)'(\ell_\xi^{-m}(x)\cap\mathfrak{a}_j^{(m)})f(\ell_\xi^{-m}(x)\cap\mathfrak{a}_j^{(m)})\Big|\mathrm{d}x \le$$

$$\le \int_{-1-m\delta}^{1+m\delta}\Big|\sum_{j\in I_l^{(m)}}f(\ell_\xi^{-m}(x)\cap\mathfrak{a}_j^{(m)})\Big|\mathrm{d}x +$$

$$+ \int_{-1-m\delta}^{1+m\delta}\Big|\sum_{j\notin I_l^{(m)}}\operatorname{sgn}(\ell_\xi^m)'(\ell_\xi^{-m}(x)\cap\mathfrak{a}_j^{(m)})f(\ell_\xi^{-m}(x)\cap\mathfrak{a}_j^{(m)})\Big|\mathrm{d}x \le$$

$$\le \sum_{j\in I_l^{(m)}}\int_{\mathfrak{a}_j^{(m)}}|f(y)|\mathrm{d}(\ell_\xi^m(y)) + \sum_{j\notin I_l^{(m)}}\int_{\mathfrak{a}_j^{(m)}}|f(y)|\mathrm{d}(\ell_\xi^m(y))\operatorname{sgn}(\ell_\xi^m)'(y) \le$$

$$\le \sum_{j\in I_l^{(m)}}\int_{\mathfrak{a}_j^{(m)}}|f(y)|\frac{|\ell_\xi^m(a_j^{(m)})|}{|\mathfrak{a}_j^{(m)}|}\mathrm{d}y + \sum_{j\notin I_l^{(m)}}\int_{\mathfrak{a}_j^{(m)}}|f(y)|\frac{|\ell_\xi^m(a_j^{(m)})|}{|\mathfrak{a}_j^{(m)}|}\mathrm{d}y \le$$

$$\le \sum_{j=N_l^1}^{N_r^1}\int_{\Omega_j^1}|f(y)|\frac{|g_0(\Omega_j^1)|}{|\Omega_j^1|}\mathrm{d}y + \sup|f(x)|\sum_{j\in I_l^{(m)}}|\ell_\xi^m(\mathfrak{a}_j^{(m)})| \le$$

$$\le \sum_{j=N_l^1}^{N_r^1}\int_{\Omega_j^1}|f(y)|\frac{|g_0(\Omega_j^1)|}{|\Omega_j^1|}\mathrm{d}y + \sup|f(x)|\cdot\sup|\tau_{ij}|\cdot\sup|\Omega_j^2|\cdot\#(\mathrm{D_{in}}).$$

Observe that

$$\sum_{j=N_l^1}^{N_r^1}\int_{\Omega_j^1}|f(y)|\mathrm{d}(g_0(y)) = \sum_{j=N_l^1}^{N_r^1}\int_{-1}^{1}|f(g_0^{-1}(x)\cap\Omega_j^1)|\mathrm{d}x = \int_{-1}^{1}|\mathcal{E}f(x)|\mathrm{d}x.$$

So we may proceed

$$\int_{-1-m\delta}^{1+m\delta}\left|\ell_{\xi*}^m f(x)\right|\mathrm{d}x \le 2\int_{-1}^{1}|\mathcal{E}f(y)|\mathrm{d}y + \sup|f(x)|\cdot\sup|\tau_{ij}|\cdot\sup|\Omega_j^2|\cdot\#(\mathrm{D_{in}}) \le$$

$$\le 2\int_{\mathbb{R}}|\mathcal{E}f(x)|\mathrm{d}x + 2^{m/2}\|f\|_1\cdot m^2\Big(\frac{s_1}{s_2}\Big)^m\cdot 3m^2\delta s_1^{2m}.$$

Recall that $m\delta \cdot \frac{s_1^{3m}}{2^m s_2^m} < 2^{m\gamma_1}$ so we may conclude

$$\int_{-1-m\delta}^{1+m\delta} \left|\ell_{\xi*}^m f(x)\right| \mathrm{d}x \le 2\int_{\mathbb{R}} |\mathcal{E}f(x)|\mathrm{d}x + \frac{2^{m(3/2+\gamma_1)}}{s_2^m}\|f\|_1.$$

■

Lemma 6.6. *Let* Ω^1, Ω^2, *and* Ω^3 *be partitions of the class* $\mathcal{G}$. *Let* $\mathcal{T}$ *be a linear operator on the main space such that* $\mathcal{T}\colon \Phi_{\Omega^1} \to \Phi_{\Omega^1}$ *is generalised toy dynamo. Assume that*

$$\int_{\mathbb{R}} |\mathcal{T}f(x)|\mathrm{d}x \le \int_{\mathbb{R}} |\mathcal{E}f(x)|\mathrm{d}x + \frac{2^{m(3/2+\gamma_1)}}{s_2^m}\|f\|_1;$$

where $0 < \gamma_1 \le 1/8$ *is chosen such that* $m\delta \cdot \frac{s_1^{3m}}{2^m s_2^m} < 2^{m\gamma_1}$. *Then for any essentially bounded and absolutely integrable function* f

$$\|W_\delta \mathcal{T} f\|_3 \le 5m\frac{N_\delta}{\delta}\|f\|_1. \tag{84}$$

Proof. We shall show that there exists a polynomial $\widetilde{Q}$ such that

$$\|W_\delta \mathcal{E} f\|_3 \le \frac{N_\delta}{\delta}\widetilde{Q}(m)\|f\|_1,$$

and the Lemma will follow. By direct calculation, substituting $N_\delta = 2^{m(1-\alpha\log_{s_1} 2)}$ and $\delta = 2^{-\alpha m}$ we see that

$$\frac{2^{m(3/2+\gamma_1)}}{s_2^m} \le \frac{N_\delta}{\delta},$$

under condition that $2^{1/2+\gamma_1+\alpha(\log_{s_1} 2-1)} \le s_2$, i.e. for $s_2 < 2$ sufficiently large, or, in other words, for $\varkappa = \log\frac{s_1}{s_2}$ small enough.

By definition of the norm we calculate,

$$2^m\|f\|_1 \ge \sum_{j\in\mathbb{Z}} \int_{\Omega_j^1} \frac{|f(y)|}{|\Omega_j^1|}\mathrm{d}y = \sum_{j=N_l^1}^{N_r^1} \int_{\Omega_j^1} \frac{|f(y)|}{|\Omega_j^1|}\mathrm{d}y +$$
$$+ \Bigg(\sum_{j=N_l^1-mN_\delta}^{N_l^1} + \sum_{j=N_r^1}^{N_r^1+mN_\delta}\Bigg)\int_{\Omega_j^1} \frac{|f(y)|}{|\Omega_j^1|}\mathrm{d}y + \Bigg(\sum_{j<N_l^1-mN_\delta} + \sum_{j>N_r^1+mN_\delta}\Bigg)\int_{\Omega_j^1} \frac{|f(y)|}{|\Omega_j^1|}\mathrm{d}y. \tag{85}$$

We estimate each of three terms separately. For the first term we have the following lower bound, using $|\Omega_j^1| \cdot \mathrm{d}g_0(y) = 2$ for any $y \in \Omega_j^1 \subset [-1, 1]$.

$$\sum_{j=N_l^1}^{N_r^1} \int_{\Omega_j^1} \frac{|f(y)|}{|\Omega_j^1|} \mathrm{d}y = \sum_{j=N_l^1}^{N_r^1} \int_{\Omega_j^1} |f(y)| \frac{\mathrm{d}g_0(y)}{2} \mathrm{d}y = \frac{1}{2} \sum_{j=N_l^1}^{N_r^1} \int_{\Omega_j^1} |f(y)| \mathrm{d}g_0(y) =$$

$$= \frac{1}{2} \sum_{j=N_l^1}^{N_r^1} \int_{-1}^{1} |f(g_0^{-1}(x) \cap \Omega_j^1)| \mathrm{d}x \geq \frac{1}{2} \int_{-1}^{1} |(\mathcal{E}f)(x)\mathrm{d}x.$$

Thus for any function f

$$\int_{-1}^{1} |\mathcal{E}f(x)| \mathrm{d}x \leq 2^{m+1} \|f\|_1. \tag{86}$$

Consider the second term of (85) now:

$$\Big(\sum_{j=N_l^1 - mN_\delta}^{N_l^1} + \sum_{j=N_r^1}^{N_r^1 + mN_\delta} \Big) \int_{\Omega_j^1} \frac{|f(y)|}{|\Omega_j^1|} \mathrm{d}y \geq \frac{1}{\sup |\Omega_j^1|} \Big(\int_{-1-m\delta}^{-1} + \int_{1}^{1+m\delta} \Big) |f(y)| \mathrm{d}y \geq$$

$$\geq \frac{1}{\sup |\Omega_j^1|} \Big(\int_{-1-m\delta}^{-1} + \int_{1}^{1+m\delta} \Big) |(\mathcal{E}f)(y)| \mathrm{d}y$$

Thus

$$\Big(\int_{-1-m\delta}^{-1} + \int_{1}^{1+m\delta} \Big) |(\mathcal{E}f)(y)| \mathrm{d}y \leq 2^m \cdot \sup |\Omega_j^1| \cdot \|f\|_1. \tag{87}$$

We have for the remaining term of (85)

$$\Big(\sum_{j < N_l^1 - mN_\delta} + \sum_{j > N_r^1 + mN_\delta} \Big) \int_{\Omega_j^1} \frac{|f(y)|}{|\Omega_j^1|} \mathrm{d}y = 2^m \Big(\int_{1+m\delta}^{+\infty} + \int_{-\infty}^{-1-m\delta} \Big) |\mathcal{E}f(y)| \mathrm{d}y. \tag{88}$$

Summing up the three inequalities (86), (87) and (88) together, we get

$$\int_{\mathbb{R}} |\mathcal{E}f(y)| \mathrm{d}y \leq 2^{m+2} \|f\|_1. \tag{89}$$

Taking the last inequality (89) into account, we estimate the norm

$$\begin{aligned}
\|W_\delta \mathcal{E} f\|_3 &= 2^{-m} \sum_{j\in\mathbb{Z}} \frac{1}{|\Omega_j^3|} \int_{\Omega_j^3} \Big| \int_{\mathbb{R}} w_\delta(x-t)(\mathcal{E} f)(t)\mathrm{d}t \Big| \mathrm{d}x \le \\
&\le 2^{-m} \sum_{j\in\mathbb{Z}} \frac{1}{|\Omega_j^3|} \int_{\Omega_j^3} \int_{\mathbb{R}} w_\delta(x-t)|\mathcal{E} f(t)|\mathrm{d}t\mathrm{d}x = \\
&= 2^{-m} \sum_{j\in\mathbb{Z}} \frac{1}{|\Omega_j^3|} \int_{\Omega_j^3} \sum_{k\in\mathbb{Z}} \int_{\Omega_k^1} w_\delta(x-t)|\mathcal{E} f(t)|\mathrm{d}t\mathrm{d}x = \\
&= 2^{-m} \sum_{k\in\mathbb{Z}} \int_{\Omega_k^1} |\mathcal{E} f(t)| \Big(\sum_{|\Omega_j^3-\Omega_k^1|>m\delta} + \sum_{|\Omega_j^3-\Omega_k^1|<m\delta} \Big) \frac{1}{|\Omega_j^3|} \int_{\Omega_j^3} w_\delta(x-t)\mathrm{d}x\mathrm{d}t \le \\
&\le 2^{-m} \Big(\frac{e^{-m}}{\inf|\Omega_j^3|} + \frac{mN_\delta}{\delta} \Big) \int_{\mathbb{R}} |\mathcal{E} f(t)|\mathrm{d}t \le \\
&\le \frac{4mN_\delta}{\delta} \|f\|_1.
\end{aligned}$$

Taking into account

$$\int_{\mathbb{R}} |\mathcal{T} f(t)|\mathrm{d}t \le \int_{\mathbb{R}} |\mathcal{E} f(t)|\mathrm{d}t + \frac{2^{m(3/2+\gamma_1)}}{s_2^m} \|f\|_1,$$

we calculate in a similar way

$$\begin{aligned}
\|W_\delta \mathcal{T} f\|_3 &\le 2^{-m} \Big(\frac{e^{-m}}{\inf|\Omega_j^3|} + \frac{mN_\delta}{\delta} \Big) \int_{\mathbb{R}} |\mathcal{T} f(t)|\mathrm{d}t \le \\
&\le 2^{-m} \Big(\frac{e^{-m}}{\inf|\Omega_j^3|} + \frac{mN_\delta}{\delta} \Big) \Big(\int_{\mathbb{R}} |\mathcal{E} f(t)|\mathrm{d}t + \frac{2^{m(3/2+\gamma_1)}}{s_2^m} \|f\|_1 \Big) \le \\
&\le \frac{N_\delta}{\delta} \cdot \Big(4m + \frac{2^{m(1/2+\gamma_1)}}{s_2^m} \Big) \|f\|_1 \\
&< 5m \frac{N_\delta}{\delta} \|f\|,
\end{aligned}$$

for $0 < \gamma_1 < 1/8$ and m large enough. ■

Recall general definition of cones associated to a partition Ω (p. 3):

$$\mathrm{Cone}\,(r,\Omega) = \Big\{ \eta = d\chi_{[-1,1]} + \varphi \mid \varphi = \sum_{j\in\mathbb{Z}} c_j \chi_{\Omega_j};\ \sum_{j=N_l}^{N_r} c_j = 0;\ \|\varphi\|_\Omega \le dr \Big\}. \qquad (10);$$

$$\widehat{\mathrm{Cone}}\,(r,\varepsilon,\Omega) \stackrel{\mathrm{def}}{=} \Big\{ f = \eta + g,\ \eta \in \mathrm{Cone}\,(r,\Omega)\,,\ \|g\|_\Omega \le \varepsilon \|\eta\|_\Omega \Big\} \qquad (11).$$

Theorem 4. *Let W_δ be the Weierstrass transform defined by (62). Let Ω^1, Ω^2, and Ω^3 be three partitions of the class $\mathcal{G}$. Let a linear operator $\mathcal{T}\colon \mathcal{L}_1(\mathbb{R}) \to \mathcal{L}_\infty(\mathbb{R})$ be such that $\mathcal{T}(\Phi_{\Omega^1}) \subset \Phi_{\Omega^2}$ is a generalised toy dynamo. Then for any m sufficiently large and*

$\varkappa = \log \frac{s_1}{s_2}$ sufficiently small there exists $\frac{3}{4} < \alpha < 1$, $r_2(m) \ll 1$, $\varepsilon_2(m) \ll \varepsilon_1(m) \ll 1$ such that $W_\delta \mathcal{T}(\widehat{\mathrm{Cone}}\,(1, \varepsilon_1, \Omega^1)) \subset \widehat{\mathrm{Cone}}\,(r_2, \varepsilon_2, \Omega^3)$ with $\delta = 2^{-m\alpha}$. Moreover, the norm of any function $f \in \widehat{\mathrm{Cone}}\,(1, \varepsilon_1, \Omega^1)$ grows exponentially fast $\|W_\delta \mathcal{T} f\|_3 \ge 2^{m-5}\|f\|_1$.

Proof. By Theorem 2 on p. 16 we know that $\mathcal{T}(\mathrm{Cone}\,(1, \Omega^1)) \subset \mathrm{Cone}\,\big(2^{m(\gamma_1 - 1/2)}, \Omega^2\big)$. Consider a function $\eta = d\chi_{[-1,1]} + \psi \in \mathrm{Cone}\,(1, \Omega^1)$, such that $\int_{-1}^{1} \mathcal{E}\psi = 0$. By Proposition 5.2, for any step function $\varphi \in \Phi_{\Omega^1}$ we have $\|(\mathcal{T} - \mathcal{E})\varphi\|_2 \le 2^{m(1/2+\gamma_1)}\|\varphi\|_1$. Using Lemma 5.3, we calculate

$$\|\mathcal{T}\eta\|_2 \ge d\|\mathcal{T}\chi_{[-1,1]}\|_2 - \|\mathcal{T}\psi\|_2 \ge d\|\mathcal{E}\chi_{[-1,1]} + (\mathcal{T} - \mathcal{E})\chi_{[-1,1]}\|_2 - \|(\mathcal{T} - \mathcal{E})\psi + \mathcal{E}\psi\|_2 \ge$$
$$\ge d(N_r^1 - N_l^1) - 2d(2^{m(1/2+\gamma_1)} + 1) > \frac{d}{2}(N_r^1 - N_l^1) \ge d2^{m-3} \quad (90)$$

Consider a function $f = \eta + g \in \widehat{\mathrm{Cone}}\,(1, \varepsilon_1, \Omega^1)$, where $\eta \in \mathrm{Cone}\,(1, \Omega^1)$ as above is a piecewise constant part; and $\|g\|_1 < d\varepsilon_1$. We may write $W_\delta \mathcal{T} f = W_\delta \mathcal{T}\eta + W_\delta \mathcal{T} g$.

We shall show that for $\delta = 2^{-m\alpha}$ large enough compared to the size of particles of the partition, $W_\delta \mathcal{T} f$ may be approximated by a step function from Φ_{Ω^3}. We write each term as a sum of a step function with remainder, and estimate the Ω^3 norm of every term. Let

$$W_\delta \mathcal{T}\eta = \phi_1 + g_1, \text{ where } \phi_1 = D_{\Omega^3} W_\delta \mathcal{T}\eta, \text{ and } g_1 = W_\delta \mathcal{T}\eta - D_{\Omega^3} W_\delta \mathcal{T}\eta; \quad (91)$$
$$W_\delta \mathcal{T} g = \phi_2 + g_2, \text{ where } \phi_2 = D_{\Omega^3} W_\delta \mathcal{T} g, \text{ and } g_2 = W_\delta \mathcal{T} g - D_{\Omega^3} W_\delta \mathcal{T} g. \quad (92)$$

Using Lemma 6.2 and Proposition 5.2 we estimate the Ω^3 norm of the first remainder term $\|g_1\|$.

$$\|g_1\|_3 = \|W_\delta \mathcal{T}\eta - D_{\Omega^3} W_\delta \mathcal{T}\eta\|_3 \le \frac{\|\mathcal{T}\eta\|_2}{s_2^m \delta} \le \frac{2d(N_r^1 - N_l^1)}{s_2^m \delta} \le \frac{d2^m}{s_2^m \delta}, \quad (93)$$

since

$$\|\mathcal{T}\eta\|_2 = \|(\mathcal{T} - \mathcal{E})\eta\|_2 + \|\mathcal{E}\eta\|_2 \le d2^{m(1/2+\gamma_1)} + d(N_r^1 - N_l^1) \le 2d(N_r^1 - N_l^1).$$

We also know that $\|\mathcal{T} g\|_2 \le s_1^m \|g\|_1$, therefore we have the following upper bound for the second remainder term $\|g_2\|_3$:

$$\|g_2\|_3 = \|W_\delta \mathcal{T} g - D_{\Omega^3} W_\delta \mathcal{T} g\|_3 = \frac{\|\mathcal{T} g\|_2}{s_2^m \delta} \le \frac{d s_1^m \varepsilon_1}{s_2^m \delta}. \quad (94)$$

Since $\mathcal{T}\eta \in \mathrm{Cone}\,\big(2^{(\gamma_1 - 1/2)}, \Omega^2\big)$ we may apply Proposition 6.1 to estimate $\|\phi_1\|_3$, using (90)

$$\|\phi_1\|_3 = \|D_{\Omega^3} W_\delta \mathcal{T}\eta\|_3 \ge \frac{1}{4}\|\mathcal{T}\eta\|_2 \ge d2^{m-5}.$$

Finally, for $\|\phi_2\|_3$ we get, using Lemma 6.6

$$\|\phi_2\|_3 = \|D_{\Omega^3} W_\delta \mathcal{T} g\|_3 \le \|W_\delta \mathcal{T} g\|_3 + \|W_\delta \mathcal{T} g - D_{\Omega^3} W_\delta \mathcal{T} g\|_3 \le$$
$$\le 5m \frac{N_\delta}{\delta} \|g\|_1 + \|g_2\| \le d \frac{\varepsilon_1}{\delta} \Big(5mN_\delta + \frac{s_1^m}{s_2^m} \Big). \quad (95)$$

We would like to find a number $0 < r_2(m) \ll 1$ such that for some d_0

$$\phi_1 + \phi_2 = d_0 \chi_{[-1,1]} + \psi \text{ with } \|\psi\|_3 \le d_0 r_2; \quad (96)$$

and two numbers $0 < \varepsilon_2(m) \ll \varepsilon_1(m) < 1$ such that the following inequality holds true

$$\|g_1 + g_2\|_3 \le d_0 \varepsilon_2. \quad (97)$$

We apply Proposition 6.2 p. 46 to the function $\eta \in \mathrm{Cone}\,(1, \Omega^1)$, and get

$$\phi_1 = D_{\Omega^3} W_\delta \mathcal{T} \eta = \tilde{d} \chi_{[-1,1]} + \psi_1 \text{ where } \|\psi_1\|_3 \le \tilde{d} \frac{2^{m(\gamma_2 + 1/2)}}{s_2^m} \text{ and } 2^{m-5} d < \tilde{d} < 2^m d. \quad (98)$$

with $\gamma_2 : = \gamma_1 + \alpha(1 - \log_{s_1} 2)$. Using the inequalities (95) and (98) above we write

$$\|\psi\|_3 = \|\phi_2 + \psi_1\|_3 \le d \frac{\varepsilon_1}{\delta} \Big(N_\delta + \frac{s_1^m}{s_2^m} \Big) + d 2^{m(\gamma_2 + 3/2)} \frac{1}{s_2^m}. \quad (99)$$

Therefore the condition (96) on r_2 holds true if

$$\frac{\varepsilon_1}{\delta} \Big(N_\delta + \frac{s_1^m}{s_2^m} \Big) < r_2 2^{m-3}; \quad (100)$$
$$\frac{2^{m(\gamma_2 + 3/2)}}{s_2^m} < r_2 2^{m-3}. \quad (101)$$

We can find a lower bound on d_0 from (96), using upper bound for $\|\psi\|_3$ from (99)

$$\|d_0 \chi_{[-1,1]}\|_3 = \|\phi_1 + \phi_2 - \psi\|_3 = \|\tilde{d} \chi_{[-1,1]} + \psi_1 + \phi_2 - \psi\|_3 \ge$$
$$\ge \|\tilde{d} \chi_{[-1,1]}\|_3 - \|\psi_1 + \phi_2\|_3 - \|\psi\|_3 \ge d 2^{m-4} - 2\|\psi\|_3 \ge$$
$$\ge d 2^{m-4} - d r_2 2^{m-1} \ge d 2^{m-2}, \quad (102)$$

for all $r_2 < 1/2$.

We can find an upper bound for $\|g_1 + g_2\|$ summing up (93) with (94). Then the second inequality (97) on ε_2 will follow from

$$\frac{2^m}{\delta s_2^m} + \frac{\varepsilon_1 s_1^m}{\delta s_2^m} \le 2^{m-2} \varepsilon_2. \quad (103)$$

We claim that the three inequalities (100), (101), (103), and conditions of Theorem 2 on p. 16 hold true with $\alpha = \frac{15}{16}$, $\gamma_1 = \frac{1}{8}$, $r_2 = \delta^{\frac{1}{64}}$, and $\varepsilon_1 = r_2^2$, $\varepsilon_2 = r_2^4$, if $\varkappa = \log \frac{s_1}{s_2} \le \frac{1}{25}$ is small enough. In particular, we get

$$W_\delta \mathcal{T}(\mathrm{Cone}\left(1, r_2^2, \Omega^1\right)) \subset \mathrm{Cone}\left(r_2, r_2^4, \Omega^3\right),$$

for $r_2 = \delta^{\frac{1}{64}}$. The condition on the norm $\|W_\delta \mathcal{T} f\|_3 \ge 2^{m-5}\|f\|_1$ follows from (93), (94), (99) and (102). ■

Corollary 1. *Under the hypotheses and in the notations of Theorem 4 on p. 52, we have for $r_2 = \delta^{\frac{1}{64}}$:*

$$W_{\frac{\delta}{m}} \mathcal{T} \colon \mathrm{Cone}\left(1, r_2^2, \Omega^1\right) \to \mathrm{Cone}\left(r_2, r_2^4, \Omega^3\right); \tag{104}$$

$$\forall f \in \mathrm{Cone}\left(1, r_2^2, \Omega^1\right) : \|W_{\frac{\delta}{m}} \mathcal{T} f\|_3 \ge 2^{m-5}\|f\|_1. \tag{105}$$

Proof. The theorem follows from Propositions 6.1 and 6.2 and Lemma 6.6. If we replace δ in the Gaussian kernel by $\frac{\delta}{m}$, we shall multiply the upper bounds in the inequalities by polynomials. Since the estimates are based on comparison powers of 2, the results still hold true. ■

Theorem 5. *Let W_δ be the Weierstrass transform defined by* (62). *Consider a sequence $\xi \in \ell_\infty(\mathbb{R})$ with $\|\xi\| \le \delta$ and a partition Ω of the class $\mathcal{G}$. Then there exist four numbers $r_2(m) \ll r_1(m)$ and $\varepsilon_2(m) \ll \varepsilon_1(m) \ll 1$ such that*

$$W_\delta \ell_{\xi*}^m W_\delta \colon \widehat{\mathrm{Cone}}\left(r_1, \varepsilon_1, \Omega\right) \to \widehat{\mathrm{Cone}}\left(r_2, \varepsilon_2, \Omega\right) \subset \widehat{\mathrm{Cone}}\left(r_1, \varepsilon_1, \Omega\right). \tag{106}$$

$$\forall f \in \mathrm{Cone}\left(r_1, \varepsilon_1, \Omega\right) : \|W_\delta \ell_{\xi*}^m W_\delta f\|_\Omega \ge 2^{m-2}\|f\|_\Omega. \tag{107}$$

Proof. Let Ω^1 be the canonical partition of the perturbation ℓ_ξ^m. First of all, we shall find a number r_1 such that for any $\eta \in \mathrm{Cone}\left(r_1, \Omega\right)$ we have $D_{\Omega^1} W_\delta \eta \in \mathrm{Cone}\left(1, \Omega^1\right)$.

Since $\eta \in \mathrm{Cone}\left(r_1, \Omega\right)$, we may write $\eta = d\chi_{[-1,1]} + \psi$, where $\psi = \sum\limits_{j\in\mathbb{Z}} c_j \chi_{\Omega_j}$, $\sum\limits_{j=N_l}^{N_r} c_j = 0$; and $\|\psi\|_{\Omega^1} \le dr_1$. Then

$$D_{\Omega^1} W_\delta \eta = d D_{\Omega^1} W_\delta \chi_{[-1,1]} + D_{\Omega^1} W_\delta \psi.$$

Using Lemmas 6.2 and 6.3 we get

$$\|D_{\Omega^1} W_\delta \psi\|_1 \le \|W_\delta \psi\|_1 + \|D_{\Omega^1} W_\delta \psi - W_\delta \psi\|_1 \le dr_1 \frac{mN_\delta + 1}{s_2^m \delta} \le dr_1 \frac{2mN_\delta}{\delta s_2^m}$$

and for the supremum norm we have $\|D_{\Omega^1} W_\delta \psi\|_\infty \le \|\psi\|_\infty$. Summing up,

$$\|D_{\Omega^1} W_\delta \psi\|_1 \le dr_1 \frac{2mN_\delta}{\delta s_2^m}. \tag{108}$$

Using Lemma 6.4, we calculate

$$\|D_{\Omega^1}W_\delta\chi_{[-1,1]} - \chi_{[-1,1]}\|_1 \leq$$
$$\leq \|D_{\Omega^1}W_\delta\chi_{[-1,1]} - W_\delta\chi_{[-1,1]}\|_1 + \|W_\delta\chi_{[-1,1]} - \chi_{[-1,1]}\|_1 \leq 2^{1-m/2}; \quad (109)$$

which implies $dD_{\Omega^1}W_\delta\chi_{[-1,1]} = d\chi_{[-1,1]} + \psi_1$, where $\psi_1 \in \Phi_{\Omega^1}$, $\|\psi_1\|_1 \leq d2^{1-m/2}$. Hence $D_{\Omega^1}W_\delta\eta = d\chi_{[-1,1]} + D_{\Omega^1}W_\delta\psi + \psi_1$, where

$$\|D_{\Omega^1}W_\delta\psi + \psi_1\|_1 \leq dr_1\frac{2mN_\delta}{\delta s_2^m} + d2^{1-m/2}.$$

By Lemma 5.2 p. 13, in order to guarantee $D_{\Omega^1}W_\delta\eta \in \mathrm{Cone}\,(1, \Omega^1)$, it is sufficient to choose $r_1 \ll 1$ such that

$$\frac{2mN_\delta}{\delta s_2^m} < \frac{1}{r_1};$$

Let us set

$$r_1 \stackrel{\text{def}}{=} \frac{\delta s_2^m}{4mN_\delta}. \quad (110)$$

We can also notice using Lemma 6.2, that

$$\|(D_{\Omega^1}W_\delta - W_\delta)\eta\|_1 \leq \frac{1}{s_2^m\delta}dr_1 = \frac{d}{4mN_\delta}. \quad (111)$$

Taking into account that $D_{\Omega^1}W_\delta\eta \in \mathrm{Cone}\,(1, \Omega^1)$ and (111) we conclude

$$D_{\Omega^1}W_\delta\eta + (D_{\Omega^1}W_\delta - W_\delta)\eta \in \widehat{\mathrm{Cone}}\left(1, \frac{1}{4mN_\delta}, \Omega^1\right). \quad (112)$$

Let $\mathcal{T}\colon \Phi_{\Omega^1} \to \Phi_{\Omega^2}$ be a generalised toy dynamo, approximating the operator $\ell_{\xi*}^m$, constructed as described in Theorem 3 on p. 31. By straightforward calculation we see that the cone $\widehat{\mathrm{Cone}}\left(1, \frac{1}{4mN_\delta}, \Omega^1\right)$ satisfies the assumptions of Theorem 4 on p. 52 for any $\frac{15}{16} < \alpha < 1$:

$$\frac{1}{4mN_\delta} \leq 2^{m(\alpha\log_{s_1}2 - 1)} < 2^{m(\alpha-1)} < 2^{-\frac{m\alpha}{32}} = \delta^{\frac{1}{32}}.$$

Therefore, by Theorem 4,

$$W_\delta\mathcal{T}(D_{\Omega^1}W_\delta\eta + (D_{\Omega^1}W_\delta - W_\delta)\eta) \in \widehat{\mathrm{Cone}}\left(\delta^{\frac{1}{64}}, \delta^{\frac{1}{16}}, \Omega\right).$$

We may write for any partition Ω^3 of the class $\mathcal{G}$ and for any $f \in \widehat{\mathrm{Cone}}\,(r_1, \varepsilon_1, \Omega^1)$

$$W_\delta\ell_{\xi*}^m W_\delta f = W_\delta\ell_{\xi*}^m W_\delta(\eta + g) = W_\delta\mathcal{T}D_{\Omega^1}W_\delta\eta + W_\delta\mathcal{T}(W_\delta - D_{\Omega^1}W_\delta)\eta +$$
$$+ D_{\Omega^3}W_\delta\ell_{\xi*}^m W_\delta g + W_\delta(\ell_{\xi*}^m - \mathcal{T})W_\delta\eta + (\mathrm{Id} - D_{\Omega^3})W_\delta\ell_{\xi*}^m W_\delta g. \quad (113)$$

We are interested in the coefficient in front of the term $\chi_{[-1,1]}$, which corresponds to the "cone axis". Let $\mathcal{E}\colon \Phi_{\Omega^1} \to \Phi_{\Omega^2}$ be a linear operator defined by (27), p. 14. Then

$$W_\delta \mathcal{T} D_{\Omega^1} W_\delta \eta = W_\delta \mathcal{T}(d\chi_{[-1,1]} + \psi_1) = W_\delta(\mathcal{T} - \mathcal{E})(d\chi_{[-1,1]} + \psi_1) + W_\delta \mathcal{E}(d\chi_{[-1,1]} + \psi_1) = \\ = W_\delta(\mathcal{T} - \mathcal{E})(d\chi_{[-1,1]} + \psi_1) + W_\delta \mathcal{E}\psi_1 + d(N_r^1 - N_l^1)(W_\delta \chi_{[-1,1]} - \chi_{[-1,1]}) + \\ + d(N_r^1 - N_l^1)\chi_{[-1,1]} = d(N_r^1 - N_l^1)\chi_{[-1,1]} + \psi_2; \quad (114)$$

where

$$\psi_2 = W_\delta(\mathcal{T} - \mathcal{E})(d\chi_{[-1,1]} + \psi_1) + W_\delta \mathcal{E}\psi_1 + d(N_r^1 - N_l^1)(W_\delta \chi_{[-1,1]} - \chi_{[-1,1]}),$$

and its norm may be bounded using Lemmas 5.2 p. 13, 5.3 p. 14, 6.3 p. 40, 6.4 p. 42, and Proposition 5.2 p. 14:

$$\|\psi_2\|_3 \le \|W_\delta(\mathcal{T} - \mathcal{E})(d\chi_{[-1,1]} + \psi_1)\|_3 + \|W_\delta \mathcal{E}\psi_1\|_3 + \|d(N_r^1 - N_l^1)(W_\delta \chi_{[-1,1]} - \chi_{[-1,1]})\|_3 \le \\ \le d 2^{m(1/2+\gamma_1)} \frac{mN_\delta}{\delta s_2^m} + 2\frac{mN_\delta}{\delta s_2^m} + d2^{m-1}2^{1-m/2} \le d\delta^{\frac{1}{16}} 2^{m-3}; \quad (115)$$

for a suitable choice of $s_2 < 2 < s_1$ and $\gamma_1 = \frac{1}{8}$.

By Theorem 3 p. 31 we get, using Lemma 6.3

$$\|W_\delta(\ell_{\xi*}^m - \mathcal{T})W_\delta \eta\|_3 \le \frac{mN_\delta}{s_2^m \delta} \cdot \|(\ell_{\xi*}^m - \mathcal{T})W_\delta \eta\|_2 \le \\ \le \frac{m^2 N_\delta}{s_2^m \delta} \cdot \Big(\frac{s_1^3}{2^{1/2+\alpha} s_2}\Big)^m \|\eta\|_1 \le dm^2 N_\delta \Big(\frac{s_1^3}{2^{1/2} s_2^2}\Big)^m. \quad (116)$$

Using Lemmas 6.2 and 6.3 we obtain, taking into account that $\|g\|_\Omega \le d\varepsilon_1$,

$$\|(\mathrm{Id} - D_{\Omega^3})W_\delta \ell_{\xi*}^m W_\delta g\|_3 \le \frac{\|\ell_{\xi*}^m W_\delta g\|_3}{s_2^m \delta} \le \frac{d\varepsilon_1}{s_2^m \delta} \cdot m^2 \Big(\frac{2s_1}{s_2}\Big)^m \cdot \frac{mN_\delta}{s_2^m \delta} \le 2d\varepsilon_1 \frac{m^3 N_\delta}{\delta^2} \Big(\frac{2s_1}{s_2^3}\Big)^m. \quad (117)$$

Combining (116) and (117), we have the following upper bound for the sum of the last two terms in (113)

$$\|W_\delta(\ell_{\xi*}^m - \mathcal{T})W_\delta \eta\| + \|(\mathrm{Id} - D_{\Omega^3})W_\delta \ell_{\xi*}^m W_\delta g\| \le \\ \le dm^2 N_\delta \Big(\frac{s_1^3}{2^{1/2} s_2^2}\Big)^m + d\varepsilon_1 \frac{m^3 N_\delta}{\delta^2} \Big(\frac{2s_1}{s_2^3}\Big)^m. \quad (118)$$

Applying Lemma 6.3 and Theorem 3 p. 31 again, we get

$$\|W_\delta(\ell_{\xi*}^m - \mathcal{T})W_\delta g\|_3 \le \frac{mN_\delta}{s_2^m\delta}\|(\ell_{\xi*}^m - \mathcal{T})W_\delta g\|_3 \le \frac{m^2N_\delta}{s_2^m\delta}\cdot\frac{s_1^{3m}}{2^{m(1/2+\alpha)}s_2^m}\|g\|_3 \le$$
$$\le d\varepsilon_1 N_\delta m^2\Big(\frac{s_1^3}{2^{1/2}s_2^2}\Big)^m.$$

By Lemma 6.6, taking into account Lemma 6.3,

$$\|W_\delta\mathcal{T}W_\delta g\| \le 5m\frac{N_\delta}{\delta}\|W_\delta g\| \le 5d\varepsilon_1 m^2\frac{N_\delta^2}{s_2^m\delta^2}.$$

Hence summing up the last three inequalities we obtain:

$$\|D_{\Omega^3}W_\delta\ell_{\xi*}^m W_\delta g\|_3 \le \|(\mathrm{Id}-D_{\Omega^3})W_\delta\ell_{\xi*}^m W_\delta g\|_3 + \|W_\delta(\ell_{\xi*}^m - \mathcal{T})W_\delta g\|_3 + \|W_\delta\mathcal{T}W_\delta g\|_3 \le$$
$$\le d\varepsilon_1\frac{m^3N_\delta}{\delta^2}\Big(\frac{2s_1}{s_2^3}\Big)^m + d\varepsilon_1 N_\delta m^2\Big(\frac{s_1^3}{2^{1/2}s_2^2}\Big)^m + 5d\varepsilon_1 m^2\frac{N_\delta^2}{s_2^m\delta^2} \le$$
$$\le d\varepsilon_1 m^3\frac{N_\delta}{\delta^2}\cdot\frac{s_1^m}{s_2^{2m}}\Big(\frac{2^m}{s_2^m} + \delta^2\frac{s_1^{2m}}{2^{m/2}} + N_\delta\frac{s_2^m}{s_1^m}\Big).$$

We see that for $\varkappa = \log\frac{s_1}{s_2}$ sufficiently small and α is as chosen above,

$$\delta^2\Big(\frac{s_1^2}{2^{1/2}}\Big)^m \ll 1 \quad\text{and}\quad N_\delta\Big(\frac{s_2}{s_1}\Big)^m \gg 1.$$

Therefore, we may write

$$\|D_{\Omega^3}W_\delta\ell_{\xi*}^m W_\delta g\|_3 \le d\varepsilon_1 m^3\frac{N_\delta^2}{\delta^2 s_2^m}. \tag{119}$$

Therefore we deduce from (114), (119), and (118) that in order to get the inclusion $W_\delta\ell_{\xi*}^m W_\delta f \in \mathrm{Cone}\,(r_2,\varepsilon_2,\Omega)$ we need to make sure that for some $1 \gg \varepsilon_1 > \varepsilon_2$ the following inequalities holds true:

$$2dr_2(N_r^1 - N_l^1) \gg d\varepsilon_1 m^3\frac{N_\delta^2}{\delta^2 s_2^m}; \tag{120}$$

$$2d\varepsilon_2(N_r^1 - N_l^1) \gg dm^2N_\delta\Big(\frac{s_1^3}{2^{1/2}s_2^2}\Big)^m + d\varepsilon_1\frac{m^3N_\delta}{\delta^2}\Big(\frac{2s_1}{s_2^3}\Big)^m. \tag{121}$$

We know that $N_r^1 - N_l^1 \ge 2^{m-1}$, therefore we may choose $\varepsilon_1 = \delta^{\frac{1}{32}}$ and get in the first inequality

$$r_2 \ge \delta^{\frac{1}{32}}\frac{m^2N_\delta^2}{4\delta^2 2^m s_2^m} = \delta^{\frac{1}{32}}\frac{m^2 2^{m2(1-\alpha\log_{s_1}2)}}{4s_2^m\cdot 2^{-2\alpha m}\cdot 2^m} = \delta^{\frac{1}{32}}\frac{m^2}{4}\cdot\frac{2^{m(1+2\alpha(1-\log_{s_1}2))}}{s_2^m}.$$

It holds true, if we set $r_2 = \delta^{\frac{1}{64}}$, as in Theorem 4 on p. 52. Comparing it with the value of $r_1 = \frac{\delta s_2^m}{4mN_\delta}$, we see that $r_2 < r_1$ provided $\log_2 s_2 + \alpha\log_{s_1}2 > \alpha\frac{63}{64} + 1$.

It remains to check for the second inequality that

$$\varepsilon_2 \geq m^2 N_\delta \Big(\frac{s_1^3}{2^{3/2} s_2^2}\Big)^m + \delta^{\frac{1}{32}} \frac{m^2 N_\delta}{4\delta^2} \Big(\frac{s_1}{s_2^3}\Big)^m. \tag{122}$$

We see immediately that we may choose s_1 and s_1 such that $\frac{1}{25} > \log \frac{s_1}{s_2} > \frac{1}{2r_2}$ and then

$$m^2 N_\delta \Big(\frac{s_1^3}{2^{3/2} s_2^2}\Big)^m = m^2 \Big(\frac{s_1^3}{s_2^2 \cdot 2^{1/2+\alpha \log_{s_1} 2}}\Big)^m \leq \delta^{\frac{1}{32}} \frac{m^2 N_\delta}{4\delta^2} \Big(\frac{s_1}{s_2^3}\Big)^m \leq m^2 \delta^{\frac{1}{32}} \Big(\frac{s_1 2^{1+\alpha}}{s_2^3}\Big)^m \ll \delta^{\frac{1}{24}}.$$

Hence we conclude that for $r_1 = \frac{\delta s_2^m}{4mN_\delta}$, $r_2 = \delta^{\frac{1}{64}}$, $\varepsilon_1 = \delta^{\frac{1}{32}}$ and $\varepsilon_2 = \delta^{\frac{1}{24}}$ we have

$$W_\delta \ell_{\xi*}^m W_\delta \colon \widehat{\mathrm{Cone}}\big(r_1, \varepsilon_1, \Omega^1\big) \to \widehat{\mathrm{Cone}}\,(r_2, \varepsilon_2, \Omega) \subset \widehat{\mathrm{Cone}}\,(r_1, \varepsilon_1, \Omega)\,. \tag{106}$$

The second inequality on the norm

$$\|W_\delta \ell_{\xi*}^m W_\delta \,|_{\widehat{\mathrm{Cone}}(r_1,\varepsilon_1,\Omega)}\, \|_\Omega \geq 2^{m-2}$$

follows from (120), (121) and (114) immediately. ■

Corollary 1. *Under the hypotheses and in the notations of Theorem 5 p. 55, let us choose four constants* $r_1 = \frac{\delta s_2^m}{4mN_\delta}$, $r_2 = \delta^{\frac{1}{64}}$, $\varepsilon_1 = \delta^{\frac{1}{32}}$ *and* $\varepsilon_2 = \delta^{\frac{1}{24}}$. *Then we have*

$$W_{\frac{\delta}{m}} \ell_{\xi*}^m W_{\frac{\delta}{m}} : \widehat{\mathrm{Cone}}\,(r_1, \varepsilon_1, \Omega) \to \widehat{\mathrm{Cone}}\,(r_2, \varepsilon_2, \Omega) \subset \widehat{\mathrm{Cone}}\,(r_1, \varepsilon_1, \Omega)$$
$$\forall f \in \mathrm{Cone}\,(r_1, \varepsilon_1, \Omega) : \|W_{\frac{\delta}{m}} \ell_{\xi*}^m W_{\frac{\delta}{m}} f\|_\Omega \geq 2^{m-2} \|f\|_\Omega.$$

The constructive proof of the existence of an invariant cone is complete. Fast Dynamo Theorem 1 now follows as described in the Section 3.

References

[1] Arnold, V. I. and Khesin, B. A. Topological methods in hydrodynamics. Applied Mathematical Sciences, v. 125, Springer-Verlag, 1998.

[2] Arnold, V. I., Zeldovich, Ya. B., Ruzmaikin, A. A., and Sokolov, D. D., A magnetic field in a stationary flow with stretching in Riemannian space, Soviet Physics JETP 81 (1981), no. 6, 2052–2058.

[3] Arnold, V. I., Zeldovich, Ya. B., Ruzmaikin, A. A., and Sokolov, D. D. Magnetic field in a moving conducting liquid, Uspekhi Mat. Nauk (in Russian) 36 (1981), no. 5, 220–221; Stationary magnetic field in a periodic flow, Doklady Akad. Nauk SSSR 266 (1982), no. 6, 1357–1358.

[4] Bayly, B. J. Scalar dynamo models. Geophys. Astrophys. Fluid Dyn. 73 (1993), 61–74.

[5] Bayly, B. J. and Childress, S. Construction of fast dynamos using unsteady flows and maps in three dimensions. Geophys. Astrophys. Fluid Dyn. 44 (1988), 211–240.

[6] Bayly, B. J. and Gilbert, A. D. Magnetic field intermittency and fast dynamo action in random helical flows. J. Fluid Mech. 241 (1992), 199–214.

[7] Childress, S. Construction of steady-state hydromagnetic dynamos. I. Spatially periodic fields, Report MF-53, Courant Inst. of Math. Sci. (1967); New solutions of the kinematic dynamo problem, J. Math. Phys. 11 (1970), 3063–3076.

[8] Childress, S. Fast dynamo theory, Topological aspects of the dynamics of fluids and plasmas (Eds: H.K. Moffatt, G.M. Zaslavsky, M. Tabor, and P. Comte), Kluwer Academic Publishers, Dordrecht (1992), 111–147.

[9] Childress, S. On the geometry of fast dynamo action in unsteady flows near the onset of chaos, Geophys. Astrophys. Fluid Dyn. 73 (1993), 75–90.

[10] Childress, S. and Gilbert, A. D. Stretch, Twist, Fold: The Fast Dynamo. Lecture Notes in Physics: Monographs, v. 37, Springer-Verlag, 1995.

[11] Cowling, T. G. The magnetic field of sunspots, Mon. Not. Roy. Astr. Soc. 140 (1934), 39–48

[12] Du, Y. and Ott, E. Growth rates for fast kinematic dynamo instabilities of chaotic fluid flows. (English summary) With an appendix by B. J. Bayly and A. Rado. J. Fluid Mech. 257 (1993), 265–288.

[13] Finn, J. M. and Ott, E. Chaotic flows and fast magnetic dynamos. Phys. Fluids 31 (1988), no. 10, 2992–3011.

[14] Gilbert, A. D. Towards a realistic fast dynamo: Models based on cat maps and pseudo-Anosov maps. Proc. Roy. Soc. Lond. 443 (1993), 585–606.

[15] Jiang, Y. P., Morita, T., and Sullivan, D. Expanding direction of the period doubling operator. Comm. Math. Phys. 144 (1992), no. 3, 509–520.

[16] Jones, S. E. and Gilbert, A. D. Dynamo action in the *ABC* flows using symmetries. Geophys. Astrophys. Fluid Dyn. 108 (2014), no. 1, 83–116.

[17] Keller, G. and Liverani, C. Rare events, escape rates and quasistationarity: some exact formulae. J. Stat. Phys. 135 (2009), no. 3, 519–534.

[18] Keller, G. and Liverani, C. Stability of the spectrum for transfer operators. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 28 (1999), no. 1, 141–152.

[19] Klapper, I. and Young, L. S. Rigorous bounds on the fast dynamo growth rate involving topological entropy. Comm. Math. Phys. 173 (1995), no. 3, 623–646.

[20] Kogan, V. R., Kolokolov, I. V., and Lebedev, V. V. Kinematic magnetic dynamo in a random flow with strong average shear. J. Phys. A 43 (2010), no. 18, 182001.

[21] Kozlovski, O. S. An integral formula for topological entropy of C^∞ maps. Ergodic Theory Dynam. Systems 18 (1998), no. 2, 405–424.

[22] Krause, F. and Rädler, K.-H. Mean-field magnetohydrodynamics and dynamo theory, Pergamon Press, Oxford - Elmsford, NY, 1980, 271 pp.

[23] Oseledets, V. I. Fast dynamo problem for a smooth map on a two-torus, Geophys. Astrophys. Fluid Dyn. 73 (1993), no. 1-4, 133–145.

[24] Gilbert A. D. and M. R. E. Proctor. Lectures on Solar and Planetary Dynamos. Publications of the Newton Institute. (1994), 389pp.

[25] Roberts, G. O. and Soward, A. M. Dynamo theory, Annual Review of Fluid Mechanics 24 (1992), 459–512.

[26] D. Ruelle, Thermodynamic formalism, Addison-Wesley, Reading, MA, 1978.

[27] Soward, A. M. Fast dynamo action in a steady flow, J. Fluid Mech. 180 (1987), 267–295.

[28] Soward, A. M. An asymptotic solution of a fast dynamo in a two-dimensional pulsed flow, Geophys. Astrophys. Fluid Dyn. 73 (1993), no. 1-4, 179–215.

[29] Soward, A. M. On the role of stagnation points and periodic particle paths in a two-dimensional pulsed flow fast dynamo model. Physica D 76, (1994), 181–201.

[30] Shukurov, A., Stepanov, R., and Sokoloff, D. Dynamo action in Möbius flow. Phys. Rev. E (3) 78 (2008), no. 2, 4 pp.

[31] Vainshtein, S. I. and Zeldovich, Ya. B. Origin of magnetic fields in astrophysics. Soviet Phys. Usp. 15, (1972), 159–172.

[32] Vishik, M. M. Magnetic field generation by the motion of a highly conducting fluid, Geophys. Astrophys. Fluid Dyn. 48 (1989), no. 1-3, 151–167.

[33] Widder, D. V. The heat equation. Pure and Applied Mathematics, Vol. 67. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1975, xiv+267pp.

[34] Yomdin, Y. Volume growth and entropy. Israel J. Math. 57 (1987), no. 3, 285–300.

[35] Zeldovich, Ya. B.. Collected works, (in Russian) Nauka, Moscow, 1986.