Estimating fast dynamo growth rate using zeta functions

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Mathematics is the part of physics, where experiments are cheap
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The kinematic fast dynamo problem

Ignoring the Lorenz force, the system of magnetohydrodynamics may be reduced to a Navier-Stokes type equation.

The kinematic dynamo equations

$$\begin{cases} \dfrac{\partial B}{\partial t} = (B \cdot \nabla)v - (v \cdot \nabla)B + \varepsilon \Delta B; \\ \nabla \cdot v = \nabla \cdot B = 0. \end{cases}$$

- v is the (known) velocity field of a fluid filling a certain compact domain M;
- B is the (unknown) magnetic field;
- ε is a dimensionless parameter reflecting the magnetic diffusion through the boundary of M.

Problem (Main fast dynamo problem)

Does there exist a divergence-free velocity field v in a bounded domain M tangent to the boundary, such that the energy of the magnetic field $B(t)$ grows exponentially in time for some initial field B_0 in the presence of small diffusion ($\varepsilon > 0$)?

From flows to diffeomorphisms

1. Dynamo problem is a Cauchy problem (for a Navier-Stokes type equation).
2. A case of special interest are stationary velocity fields in three-dimensional domains, diffeomorphic to $\mathbb{R}^3$.

The problem has a discrete version.

Lemma

The exponent of the Laplacian is the Weierstrass transform.

$$(\exp(\varepsilon\Delta)B)(z) = (W_\varepsilon B)(z) \stackrel{\mathrm{def}}{=} \int_{\mathbb{R}^d} \frac{1}{(\sqrt{2\pi}\varepsilon)^d} \exp\Big(-\frac{|z-t|^2}{2\varepsilon^2}\Big) B(t)\mathrm{d}t$$

The Lemma gives a natural discretization of the dynamo equation, where the action of piecewise diffeomorphisms is used instead of the transport by a flow

$$F_\varepsilon\colon B \mapsto (W_\varepsilon g_*)B, \qquad g \text{ is a piecewise diffeomorphism.}$$

Problem (Discrete version)

Does there exist a volume preserving diffeomorphism $g\colon \overline{M} \to \overline{M}$, such that the energy of the magnetic field B grows exponentially with number of iterations of the map F_ε for some initial field B_0 in the presence of small diffusion ($\varepsilon > 0$)?

Dynamo Theorems

Theorem (The case $\varepsilon = 0$ is easy)

On an arbitrary n-dimensional manifold any divergence-free vector field with a stagnation point with a unique positive eigenvalue is a non-dissipative kinematic fast dynamo.

Theorem (Dissipative dynamos on surfaces)

Let $g\colon M \to M$ be an area-preserving diffeomorphism of the two-dimensional compact Riemannian manifold M. Then g is a dissipative fast dynamo if and only if the induced linear operator g_{*1} on the first homology group has an eigenvector with eigenvalue $|\lambda| > 1$. The dynamo increment is independent of ε:

$$\lim_{n\to\infty} \frac{1}{n} \ln \|B_n\| = \ln |\lambda|$$

for almost any initial vector field B_0. (Here $B_{n+1} = \exp(\varepsilon\Delta) g_* B_n$.)

Theorem (Antidynamo theorem)

A transitionally, helically, or axially symmetric magnetic field in $\mathbb{R}^3$ cannot be maintained by a dissipative dynamo action.

Integral operator viewpoint

In order to solve the problem, we should

1. provide a manifold M;
2. specify an area-preserving diffeomorphism $g\colon \overline{M} \to \overline{M}$;
3. give an initial magnetic field B_0 on M.

In particular,

1. *Anosov diffeomorphisms will do, but don't exist in* $\mathbb{R}^3$
2. *growth rate estimations is a question of independent interest*

Is there any general theory to help us?

We have an integral operator

$$F_\varepsilon\colon B \mapsto \int_M \frac{1}{(\sqrt{2\pi}\varepsilon)^d} \exp\Big(\frac{|g^{-1}x - y|^d}{2\varepsilon^2}\Big) \mathrm{d}g^{-1}(g(x)) B(y) \mathrm{d}y,$$

on bounded analytic fields on M, which is nuclear, if the kernel is $\mathcal{L}_2$ and has a weak singularity on diagonal. The kernel

$$G_\varepsilon(x, y) = \frac{1}{(\sqrt{2\pi}\varepsilon)^d} \exp\Big(\frac{|g^{-1}x - y|^d}{2\varepsilon^2}\Big) \mathrm{d}g^{-1}(g(x))$$

Fredholm determinant

$$\det(1 - zF_\varepsilon) = \exp\Big(\sum_{k=1}^{\infty} -\frac{z^n}{n}\mathrm{Tr}F_\varepsilon^n\Big)$$

The trace of the integral operator

$$\mathrm{Tr}F_\varepsilon^n = \int_M \mathrm{Tr}\prod_{j=1}^{n} \mathrm{d}g^{-1}(g(x_j))\mathrm{Tr}\prod_{j=1}^{n-1} w_\varepsilon(g^{-1}x_j - x_{j+1})w_\varepsilon(g^{-1}x_n - x_1)\mathrm{d}x_1 \ldots \mathrm{d}x_n$$

The limit operator, corresponding to $\varepsilon = 0$ is acting on the space of bounded analytic vector fields on M and satisfies hypotheses of Ruelle – Grothendieck theory; we deduce

1. Fredholm determinant $\det(1 - zF_0)$ is an entire function;
2. there exist a power series expansion $\det(1 - zF_0) = 1 + \sum_{j=1}^{\infty} a_j z^n$;
3. the coefficients a_j can be calculated from periodic orbits of g;
4. zeros of the truncated expansion converge to the largest eigenvalue superexponentially fast.

Examples

We can use the method to analyse some models.

1. stretch-fold map with shear. Let's identify $\mathbb{R}^2$ with $\mathbb{C}$ and consider

$$g(x,y) = \begin{cases} (2x, \frac{1}{2}y), & \text{if } 0 < x < \frac{1}{2}, \\ (2-2x, 1-\frac{1}{2}y), & \text{if } \frac{1}{2} < x < 1; \end{cases} \qquad B \mapsto \exp(i\alpha y) g_* B;$$

   (this model is very popular in physics literature); and get the eigenvalue $2\cos(\alpha/2)\exp(i\alpha/2)$.

2. CAT map on the two-dimensional torus with shear

$$B \mapsto \begin{pmatrix} \exp(2\pi i k_1 x) \\ \exp(2\pi i k_2 y) \end{pmatrix} \cdot g_* B$$

   and get $\frac{3+\sqrt{5}}{2}$, the eigenvalue of the determinant of the CAT's map.

Alas, the convergence is not uniform and there is no justification for interchanging the limits $\varepsilon \to 0$ and $n \to \infty$ in the determinant approximation.

The *principal* reason is that multipliers of periodic orbits don't keep data about induced action on vector fields. *Technically*, smooth bounded vector fields in $\mathbb{R}^3$ don't form a Banach space, and we can't use zeta functions.

The provisional flow

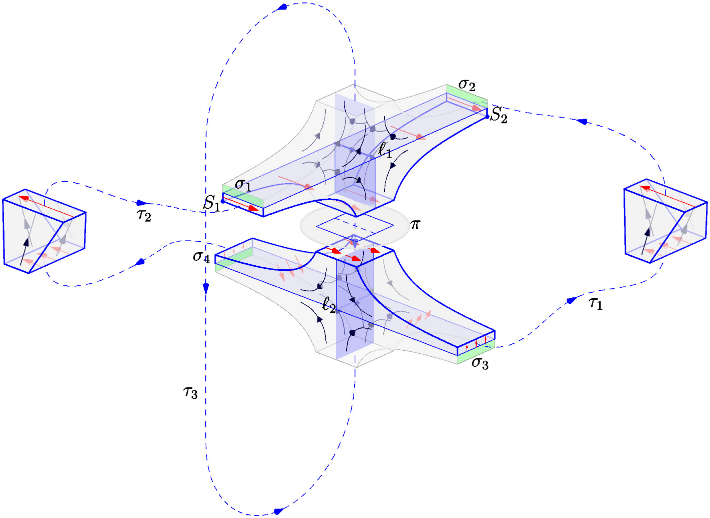

Figure: Dynamo manifold with the fluid flow (blue) and magnetic induction field (red). The labels S_1 and S_2 mark periodic saddle points. $\tau_{1,2,3,4}$ stand for manifolds equivalent to cylinders.

The integral operator

We consider a steady vector field and ignore the diffusion, in order to get an operator acting on Banach space.

$$\frac{\partial B}{\partial t} = (B \cdot \nabla)v - (v \cdot \nabla)B$$
$$\nabla \cdot v = \nabla \cdot B = 0.$$

Let ν^t be the fluid flow defined by v, and let $V(x,t)$ be the trajectory of x, that we consider as a map of the flow into itself.

$$B_i(z,t) = \sum_j \frac{\partial \nu_i^t}{\partial x_j}\Big|_y B_j(y,0) = \int_M \sum_j \delta(y - \nu^{-t}x) \frac{\partial \nu_i^t}{\partial x_j}\Big|_y B_j(y,0)\mathrm{d}y$$

This is an integral operator with the kernel

$$G_{ij}(x,y,t) = \delta(y - \nu^{-t}x) \frac{\partial \nu_i^t}{\partial x_j}\Big|_y$$

- Using chaos theory of Cvitanovic, one can calculate the trace of G from periodic orbits of the flow ν^t;
- In the case of real analytic hyperbolic flow, Fredholm determinant is an entire function;
- The leading eigenvalue can be calculated from the power series expansion.

By definition,

$$\mathrm{Tr} G(t) = \int_M G(x, x, t) \mathrm{d}x = \int_M \delta(x - \nu^{-t} x) \frac{\partial \nu_i^t}{\partial x_j}$$

Let $\ell(\gamma)$ be the length of the periodic orbit γ; and let P_γ be the differential of the Poincare map at the intersection with the periodic orbit. Then

$$\mathrm{Tr} G(t) = \sum_\gamma \ell(\gamma) \sum_{s=1}^{\infty} \frac{\mathrm{Tr} P_\gamma^s}{|\det(1 - P_\gamma^{-s})|} \delta(t - s\ell(\gamma))$$

Using limit cycles data, we can

calculate the leading eigenvalue of the Fredholm determinant using zeta function

References

- Arnold, V. I. and Khesin, B. A. Topological methods in hydrodynamics. Applied Mathematical Sciences, v. 125, Springer-Verlag, 1998.
- Aurell, E. and Gilbert, A. D. Fast dynamos and determinants of singular operators. Geophys. Astrophys. Fluid Dynam. 73, (1993), 5–32.
- Balmforth, N. J., Cvitanovic, P., Ierley, G. R., Spiegel, E. A., and Vattay, G. Advection of vector fields by chaotic flows. Stochastic Processes in Astrophysics, Annals New York Acad. Sci., vol. 706, (1993), 148–160.
- Vainshtein, S. I. and Zeldovich, Ya. B. Origin of magnetic fields in astrophysics. Soviet Phys. Usp. 15, (1972), 159–172.

Thank you!