On the chaotic properties of quadratic maps over non-archimedean fields.

V.Dremov*, G.Shabat† and P.Vytnova**

*Moscow State University, Moscow, Russia
†Moscow State University for Humanities, Moscow, Russia
**Moscow Independent University, Moscow, Russia

Abstract. [1] We study dynamic properties of the quadratic maps over arbitrary non-archimedean fields. We find conditions under which these maps demonstrate the chaotic behavior. For the quadratic maps defined over a global field the chaos occurs only over a finite number of valuations.

INTRODUCTION

0.0. Consider a general discrete dynamical system on a *countable* set (=*phase space*). Formally it is a *deterministic* model of motion (we know *everything* about the orbit of any point) and there seems to be no context for the chaotic considerations.

However, if we are going to study and *describe* the orbits, we need some additional structures on the phase space.

First of all, we need some *language* to specify the points of the phase space. It can be formalized as a *recursive* structure, i.e. the distinguished class of numbering (= bijections with natural numbers) up to recursive renumberings.

For the most dynamical systems the *amount of information* needed to specify a point (it can be formalized in terms of *Kolmogorov complexity*) generically grows along the orbit. In most cases not all this information is valuable for describing the system qualitatively; e.g., if an orbit "goes to infinity" (in some sense) we might be not interested in the details of the positions of the points that are terribly far away.

Thus we impose some *topologies* on the phase space in order to be able to describe the orbits approximately. We emphasize the specific feature of the *nonclassical* discrete dynamics: it is not assumed that the phase space carries some distinguished topology; we rather consider the *set* of natural topologies. The product of the completions of the phase space with respect to all these topologies is provided by a suitable *product topology*; the diagonal embedding of the phase space into this product should induce its true *discrete* topology.

---

[1] The work is partially supported by RFBR grant 04-01-00640.

The adelic dynamics provides a perfect framework for this approach, the phase space being global number fields; the topologies are defined by their non-archimedean valuations.

In the present paper we consider the simplest non-linear model of this kind — the iterations of quadratic maps. Conceptually our main result is the theorem 5, according to which the system demonstrates the chaotic behavior only over the finite number of valuations — precisely over those ones over which the quadratic map is in some sense *averagely expanding* in the fixed points.

The results of the paper generalize the earlier results of two of the authors Shabat [2] and Dremov [1]. The similar results over p-adic fields with $p \neq 2$ were obtained considerably earlier in Thiran at al. [3].

0.1. The paper is organized as follows. Sections 1 and 2 are devoted to certain elementary properties of the quadratic maps over non-archimedean fields. Sections 3 and 4 are technical: under some assumptions the preimages of 0 and of a "large disc" around it are described. In the section 5 the filled Julia sets for all the quadratic maps over all the non-archimedean local fields are described. In the section 6 under the assumptions of the section 3 the isomorphism between the quadratic dynamics on the filled Julia set and some sequence dynamics (Bernoulli shift on the left-infinite sequences) is established. In the section 7 the main results are formulated; the 2-adic case is considered separately. In the section 8 some adelic interpretation of our results is suggested.

0.2. Some of the notations we use are not quite standard.

For a map $T\colon X \mapsto X$ and for $n \in \mathbf{N}$ we denote by $T^{n\circ}$ its nth iterate and by $T^{-n\circ}$ its n inverse iterate (possibly multivalued). By $T^{\mathbf{N}\circ}(x)$ we denote the T- orbit of $x \in X$; finally, for $Y \subseteq X$ denote by $T^{-\mathbf{N}\circ}Y := \bigcup_{n\in\mathbf{N}} T^{-n\circ}Y$ and $T^{-\infty}Y := \bigcap_{n\in\mathbf{N}} T^{-n\circ}Y$.

When X is a metric space denote by $\mathcal{FJ}(T)$ the *filled Julia set*, i.e. the set of elements of X with bounded T- orbits.

For an alphabet (=finite set of characters) A denote by $A^{-\mathbf{N}} = \{\ldots a_2 a_1 a_0\}$ (where $a_0, a_1, a_2 \ldots \in A$) the set of sequences of elements of A, infinite *to the left.* For a finite sequence ε we denote its length by $|\varepsilon|$.

For a field $\mathbf{k}$ denote its set of squares by $\mathbf{k}^{2\cdot} := \{x^2 \mid x \in \mathbf{k}\}$.

For a field $\mathbf{k}$ with the norm $\|\cdot\|$ for $a \in \mathbf{k}$ and $r \in \mathbf{R}_{>0}$ denote the open and closed discs by

$$D(a,r) := \{x \in \mathbf{k} \mid \|x-a\| < r\}$$
$$D[a,r] := \{x \in \mathbf{k} \mid \|x-a\| \leq r\}$$

CANONICAL FORMS OF QUADRATIC MAPS

1.0. We fix a field $\mathbf{k}$ with $\operatorname{char}\mathbf{k} \neq 2$ and consider the general quadratic map

$$q\colon \mathbf{A}^1(\mathbf{k}) \mapsto \mathbf{A}^1(\mathbf{k})$$

defined by

$$q(x) = Ax^2 + Bx + C$$

with $A, B, C \in \mathbf{k}$ and $A \neq 0$.

1.1. The dynamical properties of the above q depend only on the similarity class of q; it means that we consider the action of the group of affine transformation of argument

$$x \mapsto L(x) := mx + n \text{ with } m \in \mathbf{k}^{\bullet}, n \in \mathbf{k}$$

on the set of quadratic transformations. This action is defined by

$$L \bullet q = L \circ q \circ L^{-1\circ};$$

q and thus defined $L \bullet q$ are called *similar*. The problem is to find the simplest (and traditional) representatives of similarity classes of the quadratic map.

1.2. It is easy to check that in all the cases the transformation

$$L(x) := Ax + \frac{B}{2}$$

sends

$$q(x) = Ax^2 + Bx + C$$

to

$$[L \bullet q](y) = y^2 + c$$

with

$$c = AC - \frac{B^2}{4} + \frac{B}{2}$$

thus the standard form of the quadratic map

$$x \mapsto x^2 + c$$

is universal, and we are going to stick to it in this paper.

The invariant meaning of c is as follows. Denote by $\operatorname{Fix}(q)$ the (generally 2-element) set of fixed points of q, i.e., the set of solutions of the quadratic equation

$$Ax^2 + Bx + C = x.$$

It belongs to $\mathbf{k}$ or to its quadratic extension depending on whether or not the discriminant of the above equation

$$(B-1)^2 - 4AC$$

is a square in **k**. But one checks that

$$c := AC - \frac{B^2}{4} + \frac{B}{2} = \frac{1}{4} \prod_{x \in \mathrm{Fix}(q)} q'(x)$$

is always in **k**. We'll see that in the case when **k** is equipped with a (usually non-archimedean) metric the dynamical properties of q depend drastically on the norm of c; in particular, q generates the chaotic behavior iff $\|c\| > 1$, i.e., when q is *averagely expanding in the fixed points*. We are not aware of any reasonable generalization of this observation.

1.3. The map q is not always similar to another standard form (the *logistic* map)

$$[L \bullet q](y) = \lambda y(1-y).$$

(hence the results of this paper are a bit stronger than those in Shabat [2] even in the case $\mathbf{k} = \mathbf{Q}_p$). The obvious necessary condition is the existence of fixed points of q defined over **k**. It is easy to show that this condition is sufficient as well.

BEHAVIOR OF NORMS ALONG THE ORBITS

2.0. We fix a field **k** with the non-archimedean norm $\|\cdot\|$ and for any element $c \in \mathbf{k}$ consider the quadratic map

$$T_c : \mathbf{A}^1(\mathbf{k}) \mapsto \mathbf{A}^1(\mathbf{k})$$

defined by

$$T_c(x) := x^2 + c$$

2.1. Every $x \in \mathbf{k}$ defines a sequence $\|T_c^{n\circ}(x)\|$. In most cases the behavior of the norm is quite simple.

Theorem 1 *According to the values of $\|c\|$ and $\|x\|$ the following statements hold:*

	$\|c\| < 1$	$\|c\| = 1$	$\|c\| > 1$
$\|x\| < 1$	$\lim_{n\to\infty} \|T_c^{n\circ}(x)\| = \|c\|,$	*No general statement*	$\lim_{n\to\infty} \|T_c^{n\circ}(x)\| = \infty$
$\|x\| = 1$	$\|T_c^{n\circ}(x)\| \equiv 1$	*No general statement*	$\lim_{n\to\infty} \|T_c^{n\circ}(x)\| = \infty$
$\|x\| > 1$	$\lim_{n\to\infty} \|T_c^{n\circ}(x)\| = \infty$	$\lim_{n\to\infty} \|T_c^{n\circ}(x)\| = \infty$	$\|T_c^{n\circ}\|$ *is either constant or* $\to \infty$

Proof. All the statements about existing limits and about the norms $\|T_c^{n\circ}\|$ being constant are obvious. In the case $\|c\| = \|x\| = 1$ the $\lim_{n\to\infty} \|T_c^{n\circ}(x)\|$ can exist. E.g., in any field where $\|2\| = 1$, $x = -1$ is a fixed point of $x \mapsto x^2 - 2$. But it is possible as well that $\|c\| = \|x\| = 1$, but $\lim_{n\to\infty} \|T_c^{n\circ}(x)\|$ does not exist. Over any field the map

$$x \mapsto x^2 - 1$$

provides a cycle that gives a sequence of norms $0,1,0,1,\dots$

In the case $\|c\| > 1, \|x\| > 1$ the trajectories generally tend to ∞. E.g., for $\mathbf{k} = \mathbf{Q}_3$ and $x = c = \frac{1}{3}$ we have the orbit

$$\frac{1}{3} \to \frac{4}{9} \to \frac{43}{81} \to \dots$$

with the sequence of norms $3, 9, 81, \dots$ But in some special cases (which are the most interesting from the viewpoint of the present paper) the norms along the orbits are constant. E.g., over $\mathbf{k} = \mathbf{Q}_5$ the map

$$x \to x^2 - \frac{1}{25}$$

has two fixed points $\frac{1}{2} \pm \frac{\sqrt{21}}{16} \in \mathbf{Q}_5$ of the norm 5. **QED**

THE PREORBIT OF 0.

3.0. We fix the triple $\mathbf{k} \supset \mathcal{O} \supset \mathcal{M}$ consisting of a local field, its valuation ring and its maximal ideal; let $p = \mathrm{char}(\mathcal{O}/\mathcal{M})$. We fix the non-archimedean norm $\|\cdot\|$ on $\mathbf{k}$, normalized by the condition $\|p\| = \frac{1}{p}$ and the element $c \in \mathbf{k} \setminus \mathcal{O}$ (i.e. $\|c\| > 1$; this is the only case we'll need). Our goal is to describe the set $T_c^{-\mathbf{N}\circ}(0)$.

3.1. Informally,

$$T_c^{-1\circ}(0) = \{x \mid x^2 + c = 0\} = \pm\sqrt{-c},$$
$$T_c^{-2\circ}(0) = \{x \mid x^2 + c \in T_c^{-1\circ}(0)\} = \{x \mid x^2 = -c \pm \sqrt{-c}\} = \pm\sqrt{-c \pm \sqrt{-c}}$$

and so on. We should is to give the precise sense to the expressions with nested roots

$$\pm\sqrt{\dots \pm \sqrt{-c \pm \sqrt{-c \pm \sqrt{-c}}}}$$

(continued recursively to the *left*).

Note that if the roots do not belong to the corresponding fields our notations would be just the convenient names of the elements of their quadratic extensions; however, we are most interested in the case where these roots belong to $\mathbf{k}$ and we are going rather to provide for our nested roots certain *analytic* sense.

3.2 Proposition. *The following statements are equivalent:*

3.2.0 $-c \in \mathbf{k}^{2\cdot}$;
3.2.1 $T_c^{-1\circ}(0)$ *is non-empty*
3.2.2 *For any positive natural n the set* $T_c^{-n\circ}(0)$ *is non-empty and, moreover,*

$$\#\{T_c^{-n\circ}(0)\} = 2^n.$$

Proof. Implications 3.2.0 $\Longleftrightarrow$ 3.2.1 $\Longleftarrow$ 3.2.2 are trivial; concentrate on 3.2.0 $\Longrightarrow$ 3.2.2. The assumption 3.2.0 implies $c=-a^2$ for some $a\in\mathbf{k}$ with $\|a\|>1$. In fact, we have *arbitrarily* attributed the signs to $\pm\sqrt{-c}$. Further,

$$\pm\sqrt{-c\pm\sqrt{-c}}=\pm\sqrt{a^2\pm a}=\pm a(1\pm\frac{1}{a})^{\frac{1}{2}}:=$$
$$=\pm a\Big[1+\frac{\frac{1}{2}}{1!}\Big(\pm\frac{1}{a}\Big)+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}\Big(\pm\frac{1}{a}\Big)^2+\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}\Big(\pm\frac{1}{a}\Big)^3+\dots\Big],$$

and this series converges p-adically (we use $p\neq 2$); see lemma 1 below.

The longer expressions with nested roots are also defined by the convergent series; see the next subsection. A similar description in terms of *dichotomic variables* can be found in Thiran at al. [3]. **QED**

3.3. Notations of the elements of $T_c^{-\mathbf{N}\circ}(0)$. We assume $c=-a^2$ for all $a\in\mathbf{k}$ and introduce recursively the numbers $b_\varepsilon\in\mathbf{k}$ labeled by the strings ε of +'s and -'s

$$b:=0,$$
$$b_\pm:=\pm a,$$
$$\dots\dots\dots$$
$$b_{\pm\varepsilon}:=\{\text{ solution of } x^2-a^2=b_\varepsilon\}.$$

In order to choose the signs for $b_{\pm\varepsilon}$ we introduce recursively the following Laurent series $B_\varepsilon\in\mathbf{Q}((\frac{1}{A}))$:

$$B_\pm:=\pm A,$$

$$B_{\pm\varepsilon}:=\pm\sqrt{A^2+B_\varepsilon}:=\pm A\Big(1+\frac{B_\varepsilon}{A^2}\Big)^{\frac{1}{2}}=\pm A\Big[1+\frac{\frac{1}{2}}{1!}\frac{B_\varepsilon}{A^2}+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}\Big(\frac{B_\varepsilon}{A^2}\Big)^2+\dots\Big],$$

and it makes sense since one proves inductively that

$$B_\varepsilon\in\pm A+\mathbf{Z}[\frac{1}{2}]\Big[\Big[\frac{1}{A}\Big]\Big]$$

We check that after substituting the free variable A by $a\in\mathbf{k}$ all the B_ε ' s converge in $\|\cdot\|$-norm and hence define $b_\varepsilon\in\mathbf{k}$.

LARGE DISC AND THE INVERSE DYNAMICS ON IT

4.0. We keep the same notations, including $c=-a^2$. Besides, for any $S\subset\mathbf{k}$ we denote by $\sqrt{S}$ the set $\{x\in\mathbf{k}\mid x^2\in S\}$.

Lemma 1 (Effective openness of the set of squares.) *Let* $x_0\in\mathbf{k}^{2\cdot}$. *Then* $B(x_0,\|x_0\|)\subset\mathbf{k}^{2\cdot}$.

Proof. Let $y \in \mathbf{k}$ be such that $y^2 = x_0$. By Taylor formula for any x with $\|x\| < \|x_0\|$

$$(y^2+x)^{1/2} = y\Big(1+\frac{x}{y^2}\Big)^{1/2} = y\sum_{n=0}^{\infty}\frac{1(-1)(-3)\dots(3-2n)}{2^n n!}\cdot\Big(\frac{x}{y^2}\Big)^n$$

In order to prove the convergence of this series estimate the norm of its general term. Using

$$-\log_p\|n!\|_p = \Big[\frac{n}{p}\Big]+\Big[\frac{n}{p^2}\Big]+\ldots\sim\frac{n}{p}\cdot\frac{1}{1-1/p}=\frac{n}{p-1}$$

We see that $\sqrt[n]{\|n!\|_p}\sim p^{-\frac{1}{(p-1)}}$, $\sqrt[n]{\|(2n-1)!!\|_p}=\sqrt[n]{\Big\|\frac{(2n)!}{2^n n!}\Big\|_p}\sim p^{-\frac{1}{(p-1)}}$. Then nth root of general term satisfies

$$\sqrt[n]{\Big\|y\frac{(-1)(-3)\dots(3-2n)}{2^n n!}\cdot\Big(\frac{x}{y^2}\Big)^n\Big\|}=\sqrt[n]{\Big\|y\frac{(2n-3)!!}{2^n n!}\Big\|}\cdot\Big\|\frac{x}{y^2}\Big\|\sim\sqrt[n]{\Big\|\frac{(2n-1)!!}{n!}\Big\|_p}\Big\|\frac{x}{x_0}\Big\|<1$$

QED

4.1. By definition, for all $\varepsilon\in\bigsqcup_{n=0}^{\infty}\{\pm\}^{\{-n\dots0\}}$

$$D_\varepsilon := D\Big[b_\varepsilon;\frac{1}{\|a\|^{|\varepsilon|-1}}\Big].$$

In particular, the one marked by the empty word is

$$D = D[0,\|a\|].$$

Theorem 2 *For any* $n\in\mathbf{N}$

$$T_{-a^2}^{-n\circ}(D)=\bigsqcup_{|\varepsilon|=n}D_\varepsilon.$$

Lemma 2 *Let* $a\in\mathbf{k}$ *and* $r\in\mathbf{R}_{>0}$ *satisfy* $\|a\|>1$ *and* $D[a^2,r^2]\subset\mathbf{k}^{2\cdot}$. *Then*

$$\sqrt{D[a^2,r^2]}=D\Big[a,\frac{r^2}{\|a\|}\Big]\sqcup D\Big[-a,\frac{r^2}{\|a\|}\Big]$$

Proof. First of all note that $\|a\|>r$, since $D[a^2,r^2]\subset\mathbf{k}^{2\cdot}$.

We are going to show that $\sqrt{D(a^2,r^2)}\supseteq D[a,\frac{r^2}{\|a\|}]\sqcup D[-a,\frac{r^2}{\|a\|}]$. Let $x\in D[a,\frac{r^2}{\|a\|}]\sqcup D[-a,\frac{r^2}{\|a\|}]$, then $\|x\|=\|a\|$, as $\|x-a\|<\|a\|$ or $\|x+a\|<\|a\|$. For one of the choices of the sign $\|x\mp a\|=\max(\|x\|,\|a\|)=\|a\|$. Then $\|x\pm a\|<\|a\|$, and

$$\|x^2-a^2\|=\|x\mp a\|\cdot\|x\pm a\|\le\frac{r^2}{\|a\|}\cdot\|a\|\le r^2.$$

Hence $x^2\in D[a^2,r^2]$.

Now show that $\sqrt{D[a^2,r^2]} \subseteq D[a,\frac{r^2}{\|a\|}] \sqcup D[-a,\frac{r^2}{\|a\|}]$. Let $x \in \sqrt{D[a^2,r^2]}$, then (as in the previous case), $\|x\| = \|a\|$. Therefore $\|x \mp a\| = \|a\|$. Hence $\frac{\|a^2\|}{\|a^2-x^2\|} = \frac{\|a\|}{\|x \pm a\|}$. Therefore $\|a \pm x\| = \frac{\|a^2-x^2\|}{\|a\|} \leq \frac{r^2}{\|a\|}$. So $x \in D[a,\frac{r^2}{\|a\|}] \sqcup D[-a,\frac{r^2}{\|a\|}]$. **QED**

Now we prove the theorem 2 by the induction in n. It follows from the effective openness of $\mathbf{k}^{2\cdot}$ that for the disc $D[a^2,\|a\|]$ belongs to $\mathbf{k}^{2\cdot}$. Therefore by lemma 2 $T_{-a^2}^{-1\circ}D[0,\|a\|] = \sqrt{D[a^2,\|a\|]} = D[a,1] \sqcup D[-a,1] = \bigsqcup_{|\varepsilon|=1} D_\varepsilon$. Since $\pm a \in D[0,\|a\|]$, we have

$$T_{-a^2}^{-1\circ}D[0,\|a\|] \subset D[0,\|a\|].$$

So for any n

$$T_{-a^2}^{-n\circ}D[0,\|a\|] = \bigsqcup_{|\varepsilon|=n} D_\varepsilon \subset D[0,\|a\|],$$

and the lemma 2 is applicable to every disk it is used for. The theorem 2 is proved.

Corollary 1

$$T_{-a^2}^{-\infty}(D) = \bigcap_{n=0}^{\infty} \bigsqcup_{|\varepsilon|=n} D_\varepsilon$$

THE FILLED JULIA SETS

Keep the notations of the previous section (with the exception of c that now is arbitrary).

Theorem 3 *If $\|c\| \leq 1$, then $\mathcal{FJ}(T_c) = O = D[0,1]$. If $\|c\| > 1$, then*

(a) if $-c \notin \mathbf{k}^{2\cdot}$, then $\mathcal{FJ}(T_c) = \emptyset$;
(b) if $-c \in \mathbf{k}^{2\cdot}$, i.e. $c = -a^2$ for some $a \in \mathbf{k}$, then

$$\mathcal{FJ}(T_{-a^2}) = T^{-\infty}D[0,\|a\|].$$

Proof. The statement in the case $\|c\| \leq 1$ follows from the properties of the norm sequence for $T^{n\circ}(x)$, see section 2.

In the case $\|c\| > 1$ we see that if $\|x\| > \sqrt{\|c\|}$, then $\|T^{n\circ}(x)\| = \|x\|^{2^n} \to \infty$ and if $\|x\| < \sqrt{\|c\|}$, then $\|T^{n\circ}(x)\| = \|c\|^{2^{n-1}} \to \infty$. Hence the $\mathcal{FJ}$ lies on the circle defined by $\|x\| = \sqrt{\|c\|}$.

Consider the case *(a)*. The assumption $-c \notin \mathbf{k}^{2\cdot}$ for any x satisfying $\|x\| = \sqrt{\|c\|}$ implies $\|x^2+c\| \geq \|c\|$. Indeed, if $\|x^2+c\| < \|c\|$, then $-c \in D(x^2,\|x^2\|) \subset \mathbf{k}^{2\cdot}$ by the effective openness of squares. Hence $\|T^{n\circ}(x)\| \geq \|c\|^{2^{n-1}} \to \infty$.

In the case *(b)* we just use our construction of indexed discs:

$$\mathcal{FJ} \subset D = D[0,\|a\|].$$

Then $\mathcal{FJ} \subseteq T^{-n\circ}(D) = \bigsqcup_{|\varepsilon|=n} D_\varepsilon$, so $\mathcal{FJ} \subseteq \bigcap_{n=0}^{\infty} T^{-n\circ}(D) = T^{-\infty}D[0, \|a\|\,]$

The opposite inclusion $\mathcal{FJ} \supseteq T^{-\infty}D[0, \|a\|\,]$ is obvious. **QED**

ISOMORPHISM WITH THE SEQUENCE DYNAMICS

Keep the notations of the section 4. Consider the space $\{\pm\}^{-\mathbf{N}} := \{\ldots \varepsilon_2, \varepsilon_1, \varepsilon_0 \mid \varepsilon_n \in \{+,-\}\}$ of sequences of pluses and minuses infinite *to the left* endowed with Tikhonov topology. Denote by

$$\sigma\colon \{\pm\}^{-\mathbf{N}} \mapsto \{\pm\}^{-\mathbf{N}} \colon \ldots \varepsilon_2\varepsilon_1\varepsilon_0 \mapsto \ldots \varepsilon_3\varepsilon_2\varepsilon_1$$

the *Bernoulli shift.*

Theorem 4 *For any a satisfying $\|a\| > 1$ there is an isomorphism of dynamical systems (i.e. compacts with continuous endomorphisms)*

$$(\mathcal{FJ}(T_{-a^2}), T_{-a^2}) \simeq (\{\pm\}^{-\mathbf{N}}, \sigma).$$

Proof. For any $x \in \mathcal{FJ}(T_{-a^2})$ there exists a unique sequence of embedded discs.

$$D_{\varepsilon_0\varepsilon_1\varepsilon_2} \subset D_{\varepsilon_0\varepsilon_1} \subset D_{\varepsilon_0} \subset D$$

such that $\{x\} = \ldots \cap D_{\varepsilon_0\varepsilon_1} \cap D_{\varepsilon_0} \cap D$ and $\{T(x)\} = \ldots \cap D_{\varepsilon_1} \cap D \cap T(D)$. This construction defines

$$I\colon \mathcal{FJ}(T_{-a^2}) \mapsto \{\pm\}^{\mathbf{N}} \colon x \mapsto \ldots \varepsilon_2\varepsilon_1\varepsilon_0,$$

and it is easy to check that I is a homeomorphism satisfying $I \circ T_{-a^2} = \sigma \circ I$. **QED**

CHAOTIC PROPERTIES OF QUADRATIC MAPS

Restore the notations $\mathbf{k} \supset \mathcal{O} \supset \mathcal{M}$ (a local field, its valuation ring and its maximal ideal); $p := \mathrm{char}(\mathcal{O}/\mathcal{M})$. Extend the polynomial maps we consider from $\mathbf{A}^1(\mathbf{k})$ to the projective line $\mathbf{P}^1(\mathbf{k})$, sending infinity to infinity.

Here are the main results of the paper.

Theorem 5 *If $p \neq 2$, then the map*

$$T_c\colon \mathbf{P}^1(\mathbf{k}) \to \mathbf{P}^1(\mathbf{k}) \colon x \mapsto x^2 + c$$

has positive topological entropy iff $\|c\| > 1$ and $-c \in \mathbf{k}^{2\cdot}$.

Proof. Follows from the theorem 4 and the results of Nitecki [4] and Adlet et al. [5]. See details in Shabat [2]. **QED**

Theorem 6 *If $p = 2$, then the map*

$$\mathbf{P}^1(\mathbf{k}) \to \mathbf{P}^1(\mathbf{k}) \colon x \mapsto x^2 + c$$

has positive topological entropy iff $\|4c\| > 1$ and $(1-4c) \in \mathbf{k}^{2\cdot}$.

Proof. We formulate and outline the proofs of the analogues of our main statements for $p = 2$.

Consider the case $\|c\| \le \|1/4\|$. Denote the roots of $T_c(x) - x$ by x_1 and x_2. We have $\mathbf{K} := \mathbf{k}[x_1] = \mathbf{k}[x_2]$, with $(\mathbf{K} : \mathbf{k}) \in \{1,2\}$. Our norm can be extended to the field $\mathbf{K}$. Then $\|2x_1\| \le 1$, $\|2x_2\| \le 1$ and moreover $\|x_1 - x_2\| = \|\sqrt{1-4c}\| \le 1$. So $D[x_1,1] = D[x_2,1]$.

Now prove the formula $\mathcal{FJ}(T_c) = \mathbf{k} \cap D_{\mathbf{K}}[x_1,1]$. For $t := x - x_1$ we obtain $\|T(x) - x_1\| = = \|(x_1+t)^2 + c - x_1\| = \|t(2x_1+t)\|$. Hence for $\|t\| \le 1$ we have $\|T_c^{n\circ}(x) - x_1\| \le 1$ and for $\|t\| > 1$ we have $\|T_c^{n\circ}(x) - x_1\| = \|t\|^{2^n}$.

For any two points $x, y \in \mathcal{FJ}(T_c)$ we have

$$\|T_c(x) - T_c(y)\| = \|(x-y)(x+y)\| \le \|x-y\|\|2x_1 + (x-x_1) + (y-x_1)\| \le \|x-y\|.$$

Hence if $\|c\| \le 1/4$, then the topological entropy of T_c equals zero.

Consider the case $\|c\| > 1/4$. Now we have two distinct disks $D[x_1,1]$ and $D[x_2,1]$, with $\|x_1\| = \|x_2\| = \sqrt{\|c\|}$ and $\|x_1 - x_2\| = \sqrt{\|4c\|}$. We introduce $b_\pm := x_{1,2}$, and construct the b_ε's and D_ε as in the subsections **3.3, 4.1** (excluding the empty word). We argue similarly to the case $p \ne 2$, but have to introduce some modifications.

As in the case $p \ne 2$, $\|T_c(x) - x_1\| = \|(x_1+t)^2 + c - x_1\| = \|t(2x_1+t)\|$.

For $x_1 \notin \mathbf{k}$ we have $\|T_c^{n\circ}(x) - x_1\| = \|t\|^{2^n}$ for $\|t\| > \|2x_1\|$ and $\|T_c(x) - x_1\| = \|2x_1\| \cdot \|x - x_1\| > > \|x - x_1\|$ for $0 < \|t\| \le \|2x_1\|$. Hence the filled Julia set is empty and the entropy is equals zero.

But for $x_1 \in \mathbf{k}$ we have $x_2 = 1 - x_1 \in \mathbf{k}$ and moreover all the discs D_ε lie within $\mathbf{k}$ since lemma 1 holds for the disks $D(x_0, \|4x_0\|)$.

Lemma 2 is replaced by the statement $\sqrt{D[a^2, r^2]} = D[a, \frac{r^2}{\|2a\|}] \sqcup D[-a, \frac{r^2}{\|2a\|}]$ for all the discs $D[a^2, r^2]$ with $r^2 < \|4a^2\|$ (in particular, for all the shifted disks in the proof of the theorem 2). Hence for D_ε we obtain the formula $D_\varepsilon = D[b_\varepsilon, \|2a\|^{1-|\varepsilon|}]$.

So we prove that on $\mathcal{FJ}(T_c)$ our dynamical system is equivalent to the Bernoulli shift as in the theorem 4. Its topological entropy is positive. **QED**

ADELIC INTERPRETATION

Let $\mathcal{K}$ be a global number field, $(\mathcal{K} : \mathbf{Q}) < \infty$. Consider $c \in \mathcal{K}$ and

$$T_c \colon \mathcal{K} \longrightarrow \mathcal{K} \colon x \mapsto x^2 + c.$$

For any c there is only a finite number of v's such that $T_c \colon \mathcal{K}_v \mapsto \mathcal{K}_v$ demonstrates chaotic behavior. For any non-archimedean valuation

$$v \colon \mathcal{K} \longrightarrow \mathbf{Z} \sqcup \{\infty\}$$

we extend T_c to

$$T_c\colon \mathcal{K}_v \longrightarrow \mathcal{K}_v.$$

According to the theorems 5 and 6 we can introduce the quantitative measure of global chaos:

$$\mathrm{chao}(c) := \#\{v \in \mathrm{val}(\mathcal{K}) \mid T_c\colon \mathcal{K}_v \longrightarrow \mathcal{K}_v \text{ is chaotic } \} = \#\{v \in \mathrm{val}(\mathcal{K}) \mid \|c\|_v > 1, c \in \mathcal{K}_v^{2\cdot}\}.$$

Perhaps, it deserves further study.

REFERENCES

1. Dremov V.A. On a certain *p*-adic set (in russian) Uspekhi Mat. Nauk 58(2003) no.6(354)
2. Shabat G.B. *p*-adic entropies of logistic maps. Tr. Mat. Inst. Steklova 245(2004), Izbr.Vopr.*p*-adich. Mat. Fiz. i Anal., 257 — 263 translation in Proc. Steklov Inst. Math. 2004, no.2(245), 243 — 249
3. Thiran E., Verstegen D., Weyers J., *p*-adic dynamics. J. Statist. Phys. 54(1989), no.3-4, 893 — 913
4. Nitecki, Zbiqniew H., Topological entropy and the preimage structure of maps. Real Anal. Exchange 29 (2003/04) no.1 9 — 41.
5. Roy G.Adler, A.G.Konheim and M.H.McAndrew, Topological entropy, Trans.Am.Math.Soc.114(1965)