Computing Hausdorff dimension of sets of continued fractions

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joint work with Mark Pollicott

University of Warwick

March 2021

A computation is a temptation that should be resisted as long as possible.

J.P. Boyd

Sets of continued fractions

Continued fraction of $x \in (0,1)$ is an expression

$$x = [0; a_1, \ldots, a_n, \ldots] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}, \quad a_n \in \mathbb{N}$$

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Goal

Give an effective and efficient method for computing Hausdorff dimension of subsets of an interval which are specified in terms of continued fraction expansions of their elements.

We apply our method to the sets:

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I will first present our results and then describe the method.

$\dim_H(E_2)$

E_2 is the Cantor set of numbers whose continued fraction expansions have digits 1 and 2.

$$\dim_H(E_2) = 0.5312805062\,7720514162\,4468647368\,4717854930\,5910901839\,8779888397\,8039275295\,3564383134\,5918109570\,1811852398$$

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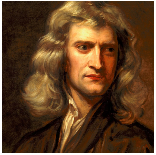

"I am ashamed to tell you to how many figures I carried these computations, having no other business"
— Isaac Newton
(on computing 15 digits for π in 1666)

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$$\dim(E_5) = 0.836829445 \pm 5 \cdot 10^{-9}$$

and in 2020 Mark Pollicott and I improved this to

$$\dim(E_5) = 0.8368294436\,8120882244\,159438727 \pm 10^{-29}.$$

Short forbidden subsequences (for Markov and Lagrange spectra)

C. Matheus and C. Moreira, Fractal geometry of the complement of Lagrange spectrum in Markov spectrum, arXiv:1803.01230

$$X := \Big\{[0; a_1, a_2, a_3, a_4, \dots],\ a_n \in \{1,2\} \quad 121 \text{ and } 212 \text{ forbidden}\Big\}$$

$$\dim_H(X) \overset{?}{<} 0.365$$

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M. Pollicott & P.V. (2020):

$$\dim_H(\Omega) = 0.5371534 \pm 3 \cdot 10^{-7}$$

Infinite set of partial denominators

V. Chousionis, D. Leykekhman, and M. Urbański. On the dimension spectrum of infinite subsystems of continued fractions. (2020)

r(N)	s_0	s_1
0 (2)	0.719360	0.719500
1 (2)	0.821160	0.821177
0 (3)	0.639560	0.640730
2 (3)	0.664900	0.665460
1 (3)	0.743520	0.743586

$$s_0 < \dim_H X_{r(N)} < s_1, \quad s_1 - s_0 \approx 10^{-4}$$

Fix $N \geq 2$, $0 < r \leq N$

$$X_{r(N)} = \left\{ [0; a_1, a_2, a_3, \cdots] \mid a_n \equiv r \mod N \right\}$$

Infinite set of partial denominators

Pollicott—V. (2020)

$r(N)$	$\dim_H(X_{r(N)})$
0 (2)	$0.7194980248\,3 \pm 10^{-11}$
1 (2)	$0.8211764906 \pm 4 \cdot 10^{-10}$
0 (3)	$0.6407253143\,8 \pm 10^{-11}$
2 (3)	$0.6654623380\,4 \pm 10^{-11}$
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I will first present the method in the simplest case:

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Step 1: Introduce a dynamical system

Idea

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$x \in X \iff$

there exists $y \in B$ and a sequence $\{j_n\} \in \{1, \ldots, k\}^{\mathbb{N}}$ such that

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In fact, since all T_j are uniformly contracting, i.e. $|T_j'| < 1 - \varepsilon$ for some $\varepsilon > 0$, the limit depends only on the sequence j_n, and not on the reference point y.

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(these are inverse branches of the Gauss map $x \mapsto \{\frac{1}{x}\}$). Then

$$\lim_{n\to\infty} T_{a_1} \circ T_{a_2} \circ \ldots \circ T_{a_n}(0) = \lim_{n\to\infty} \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots \cfrac{1}{a_n}}} \in E_N.$$

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\begin{aligned}
[\mathcal{L}_t w](x) &= \sum_{j=1}^{N} |T_j(x)'|^t \cdot w(T_j(x)) \\
&= \sum_{j=1}^{N} \frac{1}{(x+j)^{2t}} \cdot w\left(\frac{1}{x+j}\right) \qquad (t > 0)
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The operator is called the transfer operator for the iterated function scheme.

Spectral radius and dimension

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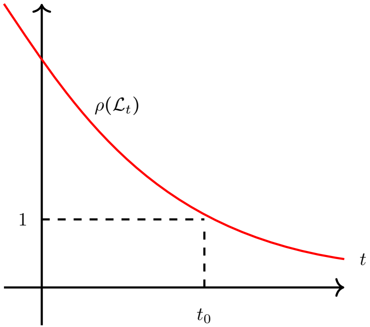

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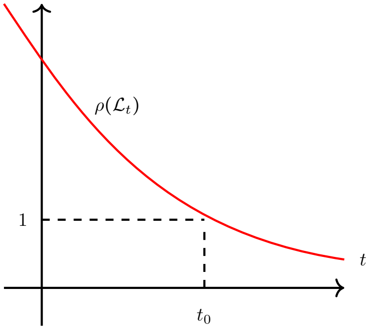

Lemma (after Bowen and Ruelle, from 1980s)

The map $t \mapsto \rho(\mathcal{L}_t)$ *is strictly monotone decreasing and the solution to* $\rho(\mathcal{L}_t) = 1$ *is* $t = \dim_H(E_N)$.

Approaches to the spectral radius $\rho(\mathcal{L}_t)$

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Useful fact (after Ruelle–Grothendieck):

In the case we consider, i.e. for the transformations $T_j\colon x \mapsto \frac{1}{x+a_j}$ with $a_j \in \mathbb{N}$ the operators $\mathcal{L}_t$ are nuclear and $\rho(t)$ is the isolated eigenvalue.

Step 3: Estimates on $\rho(\mathcal{L}_t)$

We can use a sort of "min-max" estimate:

Lemma

Let $t_0 < t_1$*.*

1. *If there exists a (positive) polynomial* $f : [0,1] \to \mathbb{R}^+$ *such that*

$$\inf_x \frac{\mathcal{L}_{t_0} f(x)}{f(x)} > 1 \implies \text{then } \rho(\mathcal{L}_{t_0}) > 1.$$

2. *If there exists a (positive) polynomial* $g : [0,1] \to \mathbb{R}^+$ *such that*

$$\sup_x \frac{\mathcal{L}_{t_1} g(x)}{g(x)} < 1 \implies \text{then } \rho(\mathcal{L}_{t_1}) < 1.$$

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$$\inf_x \frac{\mathcal{L}_{t_0} f(x)}{f(x)} > 1 \implies \text{ then } \rho(\mathcal{L}_{t_0}) > 1.$$

2. *If there exists a (positive) polynomial $g : [0,1] \to \mathbb{R}^+$ such that*

$$\sup_x \frac{\mathcal{L}_{t_1} g(x)}{g(x)} < 1 \implies \text{ then } \rho(\mathcal{L}_{t_1}) < 1.$$

This lemma gives us a way to estimate the dimension.

Corollary

If we can find f, g as above then $t_0 < \dim_H(E_N) < t_1$.

Summary — so far

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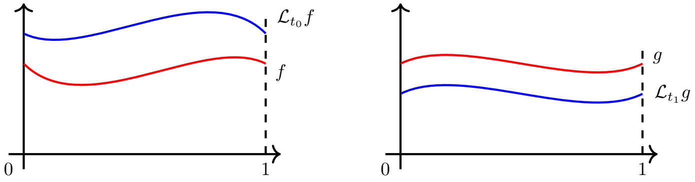

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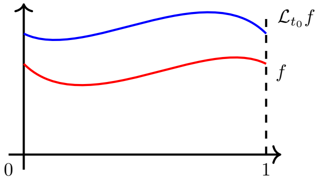

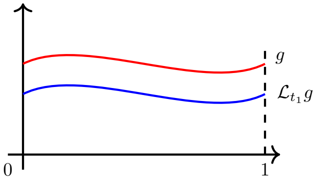

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It only remains to construct such functions f and g, which is the final step.

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- Let $w_t = (w_t^1, \cdots, w_t^m)$ be the (left) eigenvector for the largest eigenvalue.
- Finally, set $f_{m,t}(x) = \sum_{k=1}^{m} w_t^k p_k(x)$.

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To apply the "min-max" principle, we need to confirm that

1. $f_{m,t} > 0$; and
2. $\sup_x \frac{\mathcal{L}_t f_{m,t}(x)}{f_{m,t}(x)} < 1$ (or $\inf_x \frac{\mathcal{L}_t g_{m,t}(x)}{g_{m,t}(x)} > 1$)

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In the case of E_N, the numerator is sum a rational functions with coefficients $\left(\frac{1}{x+n}\right)^t$, $n = 1, \ldots, N$. It turns out that

$$(\mathcal{L}_t f_{m,t})' \cdot f_{m,t} - (f_{m,t})' \cdot \mathcal{L}_t f_{m,t} \to 0 \text{ as } m \to \infty$$

exponentially fast.

THE END OF PART I

NEXT: PART II
ABSTRACT SETTING AND TECHNICAL DETAILS

Intermission

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We can use the break to compute the dimension of some sets. Let $\mathcal{A}_N := \{d_1, d_2, \ldots, d_N\} \subset \mathbb{N}$, $d_j < 1000$ for all $1 \leq j \leq N$.

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$$X_{\mathcal{A}_N} := \Big\{[0; a_1, a_2, \dots] \mid \ a_n \in \mathcal{A}_N\Big\}, \ N \le 10$$

or

$$\begin{aligned} Y_{\mathcal{A}_N,\bar{r}} := \Big\{[0;a_1, a_2, \dots] \mid \ a_n \in \mathcal{A}_N, \ \text{with extra restrictions} \\ a_j a_{j+1} \dots a_{j+r_1} \ne d_{i_1} d_{i_2} \dots d_{i_{r_1}}, \ i_1 i_2 \dots i_{r_1} \in (\mathcal{A}_N)^{r_1} \\ a_j a_{j+1} \dots a_{j+r_2} \ne d_{i_1} d_{i_2} \dots d_{i_{r_2}}, \ i_1 i_2 \dots i_{r_2} \in (\mathcal{A}_N)^{r_2} \\ * \quad * \quad * \\ a_j a_{j+1} \dots a_{j+r_k} \ne d_{i_1} d_{i_2} \dots d_{i_{r_k}}, \ i_1 i_2 \dots i_{r_k} \in (\mathcal{A}_N)^{r_k}\Big\} \subsetneq X_{\mathcal{A}_N} \end{aligned}$$

with $N, k \le 5$ and $r_j \le 5$ for all $1 \le j \le k$.

Iterated function scheme

Let $\mathcal{A} \subset \mathbb{N}$ be a finite *alphabet*. For $a \in \mathcal{A}$ define

$$T_a\colon [0,1] \to [0,1], \qquad T_a : x \mapsto \frac{1}{x+a}$$

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To any word $\underline{w}_n = \{w_j\}_{j=1}^n$, $w_k \in \mathcal{A}$ associate

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The *limit set*

$$X_{\mathcal{A}} := \bigcup_{w} \lim_{n\to\infty} T_{\underline{w}_n}(0) \subset \mathbb{R}.$$

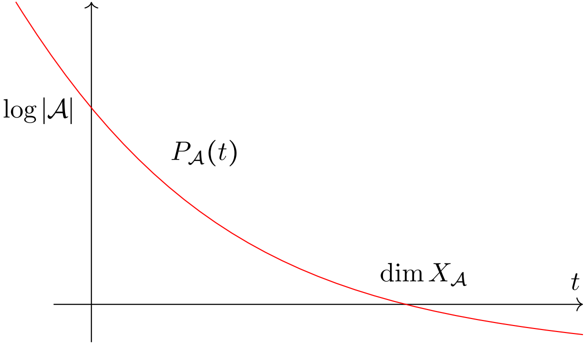
It is a Cantor set of numbers whose continued fractions have partial quotients $a_j \in \mathcal{A}$.

Pressure function

Given a system of contractions $\{T_a \mid a \in \mathcal{A}\}$ we define

$$P_{\mathcal{A}}(t) = \lim_{n\to+\infty} \frac{1}{n} \log \left(\sum_{\underline{w}_n} |(T_{w_n} \circ \cdots \circ T_{w_1})'(0)|^t \right),$$

It is a strictly decreasing (convex) analytic function, whose unique zero is the Hausdorff dimension of the limit set.



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$$H_\delta^{t_0}(X_{\mathcal{A}}) \lessapprox \sum_{\underline{w}_n} |(T_{w_n} \circ \cdots \circ T_{w_1})'(0)|^{t_0}$$

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6. The outer measure vanishes and thus $\dim_H(X_{\mathcal{A}}) \leq t_0$

Transfer operators

The transfer operator is a linear operator acting on a space of Hölder functions

$$\mathcal{L}_t : C^\alpha([0,1]) \to C^\alpha([0,1]) \qquad \mathcal{L}_t : f \mapsto \sum_{a \in \mathcal{A}} f(T_a)|T_a'|^t.$$

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$$\| e^{-nP(t)} \mathcal{L}_t^n f - \eta(f) \|_\infty \to 0 \text{ as } n \to +\infty.$$

More complicated sets

Alphabet $\mathcal{A} = \{1, 2, 3, 4\}$

$$X_{\mathcal{A}} = \Big\{[0; a_1, \ldots, a_n, \ldots] \mid a_j \in \mathcal{A},\ a_j a_{j+1} \notin \{14, 24, 41, 42\}\Big\}$$

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We define a Markov iterated function scheme, consisting of 4 maps and a transition matrix M

$$T_j(x) = \frac{1}{j+x}, \quad j \in \{1, 2, 3, 4\} \qquad M = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

The limit set of $\{T_j\}_{j \in \mathcal{A}}$ with respect to M is

$$\left\{\lim_{n \to +\infty} T_{j_1} \circ \cdots \circ T_{j_n}(0) \mid j_k \in \mathcal{A}, M_{j_k, j_{k+1}} = 1, 1 \le k \le n-1\right\} = X$$

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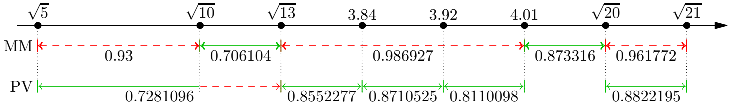

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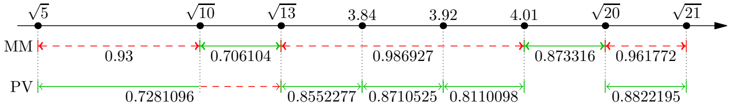

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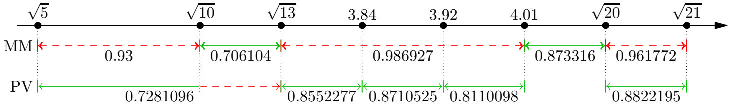

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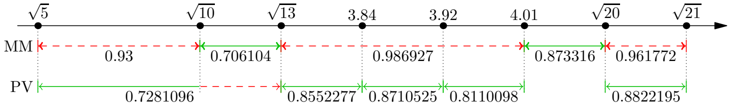

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- limit sets of finitely generated hyperbolic Schottky groups

- limit sets of Blaschke products

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Thank you for your time

Zaremba Conjecture, 1972

For any natural number $q \in \mathbb{N}$ there exists p (coprime to q) and $a_1, \cdots, a_n \in \{1, 2, 3, 4, 5\}$ such that

$$\frac{p}{q} = [a_1, \cdots, a_n].$$

Unfortunately, this conjecture is still open.

However, the conjecture is true *for most denominators*, there is a density one result.

Theorem (Bourgain-Kontorovich, Huang)

$$\lim_{Q \to +\infty} \frac{1}{Q} Card \left\{ 1 \leq q \leq Q \middle| \exists p \in \mathbb{N} : \frac{p}{q} = [a_1, \cdots, a_n], a_i \in \{1, 2, 3, 4, 5\} \right\} = 1$$

The proof is conditional on the fact $\dim_H(E_5) > \frac{5}{6}$.

Examples for numerical experiments

Alphabet $\mathcal{A} = \{1, 2, 3, 4\}$

$$X_{\mathcal{A}} = \left\{ [0; a_1, \ldots, a_n, \ldots] \mid a_j \in \mathcal{A} \right\}$$

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$$X_{\mathcal{A}} = \Big\{[0; a_1, \ldots, a_n, \ldots] \mid a_j \in \mathcal{A}\Big\}$$

$$Y_{\mathcal{A}} = \Big\{[0; a_1, \ldots, a_n, \ldots] \mid a_j \in \mathcal{A},\ a_j a_{j+1} \notin \{14, 24, 41, 42\}\Big\}$$