Computing Hausdorff dimension of Bernoulli convolutions

(on a joint work with M. Pollicott and V. Kleptsyn)

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Bernoulli convolution: a special probability measure

Consider the sum of a geometric series with randomly chosen signs:

$$\xi = \sum_{j=0}^{\infty} \pm\lambda^j = \sum_{j=0}^{\infty} \xi_j \lambda^j,$$

where $\lambda \in [0,1)$ and the signs ($\xi_j = \pm 1$) are chosen independently.

Definition

The probability measure μ_λ given by the distribution of values of ξ is called a *Bernoulli convolution.*

In this talk we will be concerned with properties of μ_λ.

For instance, ξ always takes values between $-\frac{1}{1-\lambda}$ and $\frac{1}{1-\lambda}$, because

$$\sum_{j=0}^{\infty} \lambda^j = \frac{1}{1-\lambda}.$$

Dynamical viewpoint

It is convenient to shift and to rescale the variable ξ to $\left[0, \frac{1}{1-\lambda}\right)$

$$\xi \mapsto \tilde{\xi} = \frac{\xi + c_\lambda}{2} = \sum_{j=0}^{\infty} \tilde{\xi}_j \lambda^j,$$

where $\tilde{\xi}_j$ are i.i.d. assuming values 0 and 1 with equal probability. Observe that

$$\tilde{\xi} = \sum_{j=0}^{\infty} \tilde{\xi}_j \lambda^j = \tilde{\xi}_0 + \lambda \sum_{j=0}^{\infty} \tilde{\xi}_{j+1} \lambda^j.$$

The probability measure $\tilde{\mu}_\lambda$ corresponding to the distribution of $\tilde{\xi}$ is the stationary measure for the iterated function system

$$f_0(x) = \lambda x, \quad f_1(x) = 1 + \lambda x :$$

$$\tilde{\mu}_\lambda = \frac{1}{2}(f_0)_* \tilde{\mu}_\lambda + \frac{1}{2}(f_1)_* \tilde{\mu}_\lambda.$$

A stationary measure for a system of contracting maps is unique.

Dependence $\tilde{\mu}_\lambda$ on λ

Let $\tilde{\mu}_\lambda$ be the stationary measure for the iterated function system

$$f_0(x) = \lambda x, \quad f_1(x) = 1 + \lambda x.$$

Then

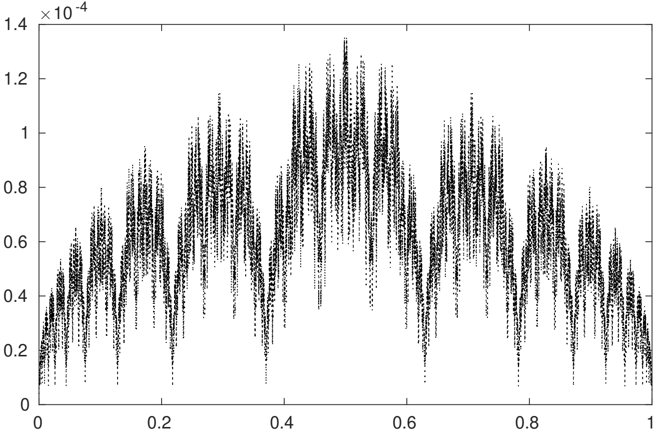
- For $\lambda \in \left(0, \frac{1}{2}\right)$, the measure $\tilde{\mu}_\lambda$ is supported on a Cantor set. For $\lambda = \frac{1}{3}$, this is a (rescaled) standard "mid-$\frac{1}{3}$" Cantor set.
- For $\lambda = \frac{1}{2}$, the measure $\tilde{\mu}_\lambda$ is the Lebesgue measure on $[0, 2]$.
- For $\lambda \in \left(\frac{1}{2}, 1\right)$, the measure $\tilde{\mu}_\lambda$ is fully supported on $\left(0, \frac{1}{1-\lambda}\right)$.

Question

What can we say about $\tilde{\mu}_\lambda$ for $\lambda > \frac{1}{2}$? For example, is it absolutely continuous or singular (with respect to Lebesgue measure)?

It turns out, properties of $\tilde{\mu}_\lambda$ depend on algebraic properties of λ.

First look



For $\lambda = 1/1.7$ the measure μ_λ is fully supported and is singular with respect to Lebesgue measure. The plot shows the graph of the map $D_k \to \mu_\lambda(D_k)$ for the uniform partition $\{D_k\}_{k=1}^{2^{16}}$ into 2^{16} equal intervals.

Pisot numbers

Definition

A Pisot number is a real algebraic integer $a > 1$ such that all its algebraic conjugates a_j (that is, other roots of its minimal polynomial) are less than 1 in absolute value:

$$P(z) = (x - a)(x - a_2)\dots(x - a_n) \in \mathbb{Z}[x], \quad \text{and } |a_j| < 1,\ j = 2, \dots, n.$$

Example

The golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$ is a root of $P(x) = x^2 - x - 1$ with the second root $-\frac{1}{\varphi}$.

Lemma

If a is a Pisot number, then the fractional parts $\{a^k\} \to 0$ as $k \to \infty$.

Proof.

Consider $a^k + a_2^k + \dots + a_n^k$. □

Erdös: singularity for inverse Pisot

Theorem (Erdös, 1939)

Let $\lambda \in (\frac{1}{2}, 1)$ be the inverse of a Pisot number. Then μ_λ is singular.

Proof.

Rewrite the measure as a countable convolution of scaled Bernoulli variables

$$\mu_\lambda = \mathop{\ast}_{j=1}^{\infty} \left(\frac{1}{2}\delta_{-\lambda^j} + \frac{1}{2}\delta_{\lambda^j} \right)$$

and apply Fourier transform:

$$f(t) := \widehat{\mu_\lambda}(t) = \prod_{j=0}^{\infty} \widehat{\left(\tfrac{1}{2}\delta_{-\lambda^j} + \tfrac{1}{2}\delta_{\lambda^j}\right)}(t) = \prod_{j=0}^{\infty} \cos(2\pi t \lambda^j).$$

Main idea: along the sequence $t_k = \lambda^{-k}$ the values $f(t_k)$ do not tend to zero. Indeed, for $j \le k$

$$\cos(2\pi t_k \lambda^j) = \cos(2\pi \lambda^{k-j}),$$

and λ^{k-j} are very close to integers, because λ is a Pisot number. This implies singularity of the measure. □

Absolute continuity for almost all λ

Theorem (Solomyak, 1995; Erdös conjecture, 1940)

For almost every $\lambda \in (\frac{1}{2}, 1)$ the corresponding measure μ_λ is absolutely continuous and its density is an L_2 function.

Idea of the proof.

- The random variable ξ can be considered as a family of functions of the variable λ, which is "parametrised" by the choice of signs;
- "Transversality": two such functions are "usually" not very close to each other.
- In order for the measure μ_λ to be singular, many these random values should be close to each other.

$\square$

Theorem (Shmerkin, 2013)

The set of $\lambda \in (\frac{1}{2}, 1)$ such that the corresponding measure μ_λ is singular, has Hausdorff dimension zero.

Dimension of a measure

Definition

The Hausdorff dimension of a measure μ is the infimum of dimensions of sets of the full measure:

$$\dim_H \mu := \inf\{\dim_H(B) \mid \mu(B) = 1\}.$$

Lemma (Folklore)

The measure μ_λ is exact-dimensional. In other words, for μ_λ-almost all x we have

$$\lim_{r\to 0} \frac{\log(\mu_\lambda([x-r, x+r]))}{\log r} = \dim_H \mu_\lambda.$$

Question

What can we say about Hausdorff dimension of Bernoulli convolutions?

$\lambda \in (\frac{1}{2}, 1)$ with $\dim_H \mu_\lambda < 1$

Any absolutely continuous measure has Hausdorff dimension 1, thus

$$\dim_H(\{\lambda \mid \dim_H \mu_\lambda < 1\}) = 0$$

Theorem (Garsia, 1963)

If λ is a Pisot number, then $\dim_H \mu_\lambda < 1$.

n	λ	$\dim_H \mu_\lambda$
2	0.61803399...	0.99571312..
3	0.54368901...	0.98040931..
4	0.51879006...	0.98692647..
5	0.50866039...	0.99258530..
6	0.50413825...	0.99603259..
7	0.50201705...	0.99793744..
8	0.50099418...	0.99894491..
9	0.50049311...	0.99946536..
10	0.50024546...	0.99973060..

Estimates for the dimension of the Bernoulli measure for the Pisot root of $x^n - x^{n-1} - \ldots - 1 = 0$, by Grabner at al. (2002)

Uniform lower bounds

Theorem (Varjú, 2018)

For all transcendental $\lambda \in (\frac{1}{2}, 1)$ *we have* $\dim_H(\mu_\lambda) = 1$ *(based on Garsia entropy approach and a result of Hochman).*

Theorem (Hare, Sidorov, 2018)

For all $\lambda \in (\frac{1}{2}, 1)$ *one has* $\dim_H \mu_\lambda \geq 0.82$ *(based on Garsia entropy approach and a result of Hochman). Furthermore*
$\dim_H \mu_\lambda \geq \dim_H \mu_{\lambda^2}$.

Theorem (V. Kleptsyn, M. Pollicott, P.V., 2021)

For all $\lambda \in (\frac{1}{2}, 1)$ *one has* $\dim_H \mu_\lambda \geq 0.96399$*. Moreover,*
$\dim_H \mu_\lambda \geq G(\lambda)$ *for an explicit piecewise-constant function* G *(based on random processes methods).*

... a couple of weeks later ...

Theorem (D.-J. Feng, Z. Feng, 2021)

For all $\lambda \in (\frac{1}{2}, 1)$ *one has* $\dim_H \mu_\lambda \geq 0.9804085$*, the dimension corresponding to the root of* $\lambda^3 - \lambda^2 - \lambda - 1 = 0$ *(based on partition entropy approach).*

The correlation dimension

The existing methods for computing Hausdorff dimension of $\dim_H \mu_\lambda$ use the properties of the minimal polynomial of λ and therefore are not suitable for uniform estimates.

Correlation dimension:

$$\dim_{cor} \mu := \sup \left\{ \alpha \;\middle|\; \iint |x-y|^{-\alpha} \, d\mu(x) \, d\mu(y) < +\infty \right\}$$

For any probability measure μ,

$$\dim_{cor} \mu \leq \dim_H \mu.$$

The correlation dimension is known to be easier to estimate numerically and gives surprisingly good lower bounds on $\dim_H \mu$.

Approach

Let $\alpha < \dim_{cor} \mu_\lambda$. Then the function

$$\psi(r) = \iint |x - y + r|^{-\alpha} d\mu_\lambda(x)\, d\mu_\lambda(y)$$

is continuous and decreasing as $r \to \infty$.

Stationarity of the measure μ_λ with respect to the iterated function scheme $f_0(x) = \lambda x$, $f_1(x) = 1 + \lambda x$, implies

$$\psi(r) = \lambda^{-\alpha} \left(\frac{1}{4}\psi\left(\frac{r-1}{\lambda}\right) + \frac{1}{2}\psi\left(\frac{r}{\lambda}\right) + \frac{1}{4}\psi\left(\frac{r+1}{\lambda}\right)\right)$$

In other words, ψ is the fixed point for the operator

$$\mathcal{D}_{\alpha,\lambda} \colon \varphi \mapsto [D_{\alpha,\lambda}\varphi](r) := \lambda^{-\alpha} \left(\frac{1}{4}\varphi\left(\frac{r-1}{\lambda}\right) + \frac{1}{2}\varphi\left(\frac{r}{\lambda}\right) + \frac{1}{4}\varphi\left(\frac{r+1}{\lambda}\right)\right)$$

How to obtain a lower estimate for the correlation dimension

Theorem (V. Kleptsyn, M. Pollicott, P.V.)

Let ψ be a positive function, bounded away from 0 and ∞ on an interval $J \supset \left[-\frac{1}{1-\lambda}, \frac{1}{1-\lambda}\right] \supset \operatorname{supp} \mu_\lambda$, such that everywhere on this interval

$$[\mathcal{D}_{\alpha,\lambda}\psi](x) < \psi(x).$$

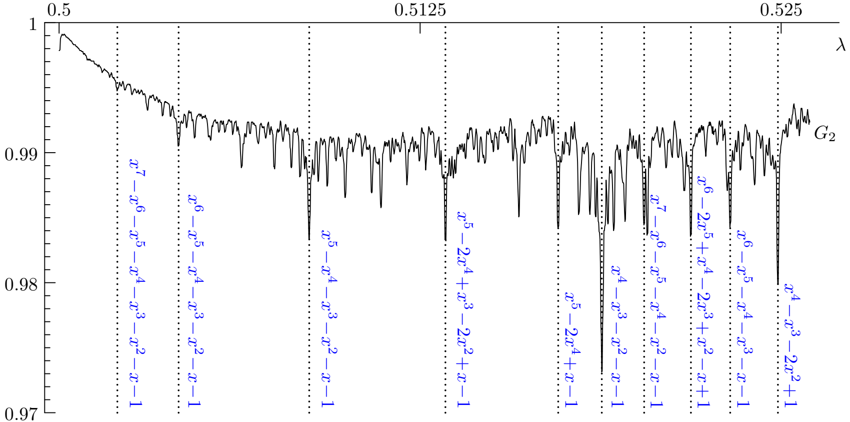
Then $\alpha \leq \dim_{cor} \mu_\lambda$.
Moreover, for any $\alpha < \dim_{cor} \mu_\lambda$ there exists a piecewise-constant function ψ such that the inequality holds.

To find ψ numerically, consider the iterations of the pointwise minimum

$$\varphi \mapsto \min(\mathcal{D}_{\alpha,\lambda}\varphi(x), \varphi(x))$$

applied to the indicator function of J.

Main result: a lower bound



The dimension of Bernoulli convolutions μ_λ is bounded from below by a piecewise-constant function G_2 corresponding to approximately 10000 intervals $\dim_H \mu_\lambda \geq G_2(\lambda)$.

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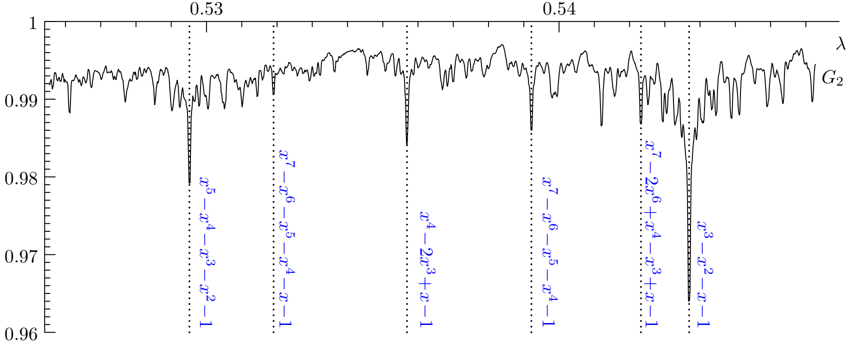

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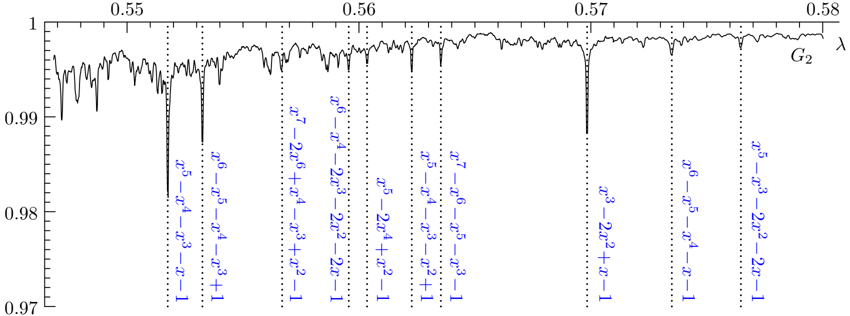

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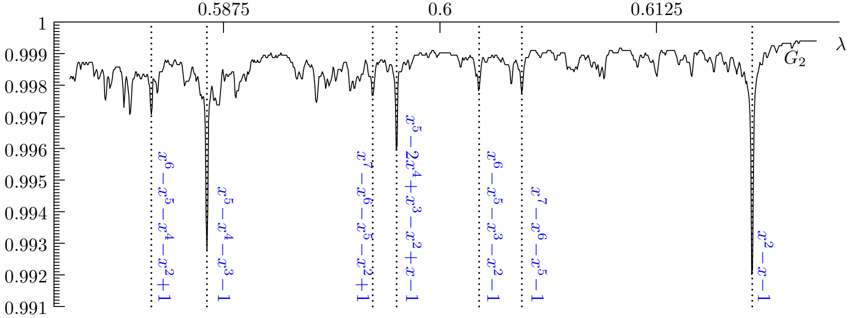

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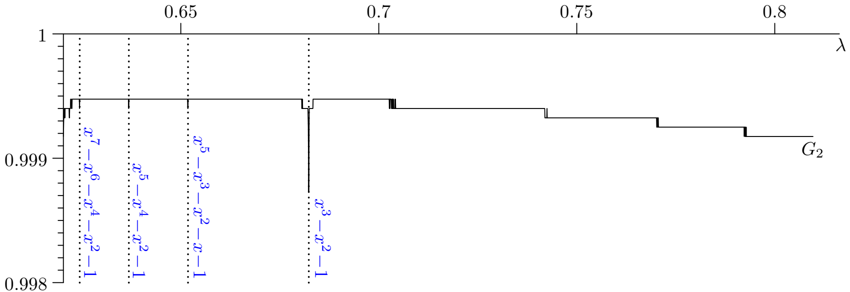

The dimension of Bernoulli convolutions μ_λ is bounded from below by a piecewise-constant function G_2 corresponding to approximately 10000 intervals $\dim_H \mu_\lambda \geq G_2(\lambda)$. Due to result by Hare and Sidorov $\dim_H \mu_\lambda \geq \dim_H \mu_{\lambda^2}$ and we only need to consider $\lambda < 1/\sqrt{2}$.

Generalisations

The approach we have developed applies to any iterated function scheme of similarities

$$f_j: \mathbb{R} \to \mathbb{R} \qquad f_j(x) = \lambda x + c_j, \quad 1 \le j \le k,$$

and gives a lower bound on the Hausdorff dimension of the stationary measure.

Thank you for your attention!