Hausdorff dimension of limit sets via zeta functions

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Mathematics is the part of physics, where experiments are cheap

V. Arnold

Toolbox in pictures

Figure: The Hammer Law of Abraham Maslow. If the only tool you have is a hammer, everything looks like a nail.

Known (to me) “nails”

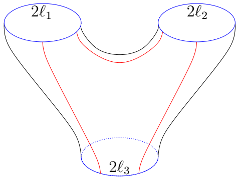
Quontitative problems of hyperbolic dynamics

- Estimating the rate of mixing for Anosov flows
- Estimating the rate of mixing for geodesic flows
- Estimating entropy of hyperbolic flows
- Estimating Hausdorff dimension of Julia sets
- Estimating Hausdorff dimension of limit sets (sometimes coincide with the entropy of geodesic flow)
- Estimating singularity dimension of self-affine sets
- Estimating Hausdorff dimension of graphs of functions on metric spaces
- **• • • ?**

Dynamical zeta function is our hammer

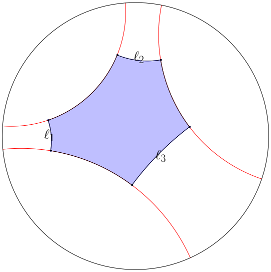
- A proper Banach space of Hölder continuous functions on the manifold;
- A nuclear transfer operator acting on the Banach space;
- The determinant of the transfer operator, which is an analytic function;
- Ruelle-Pollicott dynamical zeta function;
- The zeta function turns to be an analytic function, which is closely related to the determinant it contains contains intrinsic information about the dynamical system;
- The zeta function can be computed very efficiently and give numerical information of high accuracy, using an eight-years-old laprop (dob March 2007).

A pair of pants



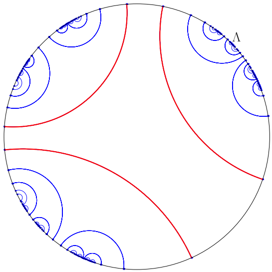
- Topologically pair of pants P is a 3-punctured sphere;
- It is a surface of constant negative curvature -1 and cannot be embedded into $\mathbb{R}^3$ by Efimov's theorem;
- As a metric space, it is uniquelly defined by the lengths of the three boundary geodesics: $P = P(\ell_1, \ell_2, \ell_3)$;
- It possess countably many of closed geodesics $\{\gamma_n\}$ of the lengths $0 < \lambda(\gamma_1) < \lambda(\gamma_2) < \ldots < \lambda(\gamma_n) \ldots \to \infty$

A Fuchsian group



- Cutting the pair of pants along the red geodesics, we obtain a pair of hexagons;
- The hexangons can be immersed into the hyperbolic upper plane as right-angled hexagons;
- The Fuchsian group Γ, generated by reflections with respect to the "cuts", gives the pair of pants as the factor space $P(\ell_1, \ell_2, \ell_3) = \mathbb{H}/\Gamma$.
- Any closed geodesics is uniquely defined by the sequence of crossings with the cuts, and thus can be associated to an element of the Fuchsian group.

The limit set



- The Fuchsian group is acting on the boundary of the hyperbolic plane, and the action is hyperbolic if the surface has no cusps, or $\ell_j \neq 0$;
- We may consider *the limit set* Λ, the smallest invariant set with respect the group action;
- The boundary is topologically equivalent to the unit circle;
- We are interested in the Hausdorff dimension of the limit set as a function of the lengths $\dim_{H,\Lambda}(\overline{\ell})$, where $\overline{\ell} = (\ell_1, \ell_2, \ell_3) \in \mathbb{R}^3$.

Selberg zeta function

Definition

Selberg zeta function for a infinite area hyperbolic surface is defined by

$$Z_{\bar{\ell}} \colon \mathbb{C} \to \mathbb{C} \qquad Z(s) = \prod_{n=0}^{\infty} \prod_{\gamma} \big(1 - \exp(-s + n)\lambda(\gamma)\big)$$

Theorem (Ruelle)

There exists $0 < d < 1$ *such that the Selberg zeta function converges to an analytic function on* $\Re(s) > d$ *and has a simple zero at* $s = d$*. It also has an analytic extension to* $\mathbb{C}$*.*

The largest real zero

Consider a simplex

$$\sigma_\ell \stackrel{\mathrm{def}}{=} \{\overline{\ell} = (\ell_1, \ell_2, \ell_3) \in \mathbb{R}^3 \mid \ell_1 + \ell_2 + \ell_3 = \ell\}$$

and define a function

$$\mathrm{d}\colon \sigma_\ell \to (0,1) \qquad \mathrm{d}(\ell_1, \ell_2, \ell_3) = \sup_{\mathbb{R}}\{x \mid Z_{\overline{\ell}}(x) = 0\}$$

Lemma

The following are equaivalent

- *d is largest real zero of $Z_{\overline{\ell}}(s)$;*
- *$d = dim_{H,\Lambda}(\overline{\ell})$, Hausdorff dimension of the limit set;*
- *$d = \lim_{T\to\infty} \frac{1}{T} \log \#\{\text{closed geodesics } \gamma \mid \lambda(\gamma) \leq T\}$.*

Computing zeta function

- We choose the Banach space $\mathcal{B}$ of analytic functions on the union of disjoint neighbourhoods of the disks bounded by geodesics and their reflected images in $\mathbb{C}$.

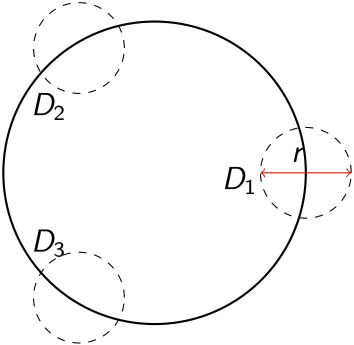


Figure: Three circles of reflection on the unit circle

Transfer operator

We define a transfer operator $\mathcal{L}_s$ on the space $\mathcal{B}$ by

$$(\mathcal{L}_s f)\mid_{D_1}(z) = R'_{\ell_1}(z_2)f(z_2) + R'_{\ell_1}(z_3)f(z_3),$$

where z_2, z_3 are preimages of $z \in D_1$ with respect to reflection with respect to the geodesics $\ell_1 = \partial D_1$.

Lemma (Grothendieck - Ruelle)

The operator $\mathcal{L}_s$ is nuclear, i.e. $\mathrm{Tr}\mathcal{L}_s$ is finite.

The determinant of the transfer operator is given by

$$\zeta(z,s) \stackrel{\mathrm{def}}{=} \exp\Big(-\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{Tr}\mathcal{L}_s^n\Big).$$

Zeta function magic

Lemma (Grothendieck - Ruelle)

The trace of the transfer operator can be explicitly computed in terms of the bounded closed geodesics.

$$\mathrm{Tr}\mathcal{L}_s^n = \sum_{|\gamma|=n} \frac{\exp(-s\lambda(\gamma))}{1-\exp(-\lambda(\gamma))}$$

Theorem (Ruelle)

There exists a constant C such that the determinant is an analytic function in both variables in a strip $0 < s < C$, and

$$\zeta(1,s) = Z(s) = \exp\Big(-\sum_{n=1}^{\infty} \frac{1}{n} \sum_{|\gamma|=n} \frac{\exp(-s\lambda(\gamma))}{1-\exp(-\lambda(\gamma))}\Big)$$

Computing zeros numerically

Theorem

The determinant can be expanded in a Taylor series $\zeta(z,s) = 1 + \sum_{n=1}^{\infty} a_n(s) z^s$ *with the coefficients* a_n *are bounded by* $|a_n(s)| \leq C\theta^{n^2}$ *where:*

1. $0 < \theta = \theta(\ell) < 1$ *is independent of* s *and there exists* $K > 0$ *such that for all* $\ell > 0$ *we have* $\theta = O(e^{-K\ell})$*; and*
2. $C = C(\ell, s) > 0$ *does depend on* s *and* ℓ*, but it is bounded and in a small neighbourhood of the iterval* $(0, dim_{H,\Lambda}(\overline{\ell})) \subset \mathbb{R}$*.*

Plotting Selberg zeta function

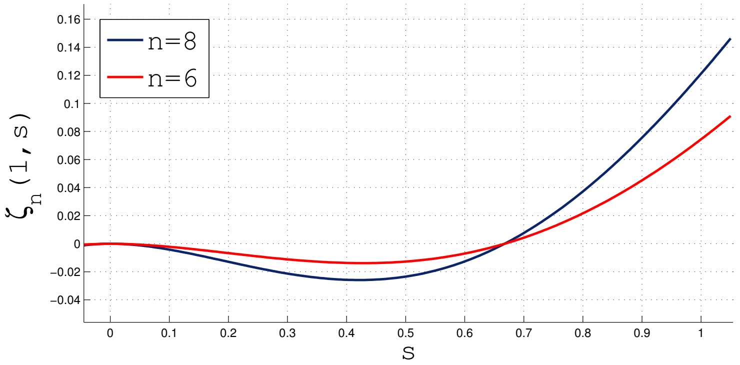


Figure: A typical graph of the zeta function, computed using the closed geodesics, crossing the cuts $n = 6$ and $n = 8$ times.

Plotting Hausdorff dimension

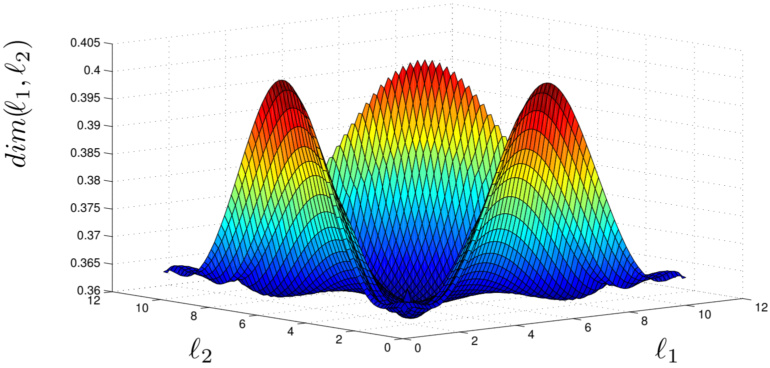


Figure: A typical plot of the Hausdorff dimension on the simplex $\ell_1 + \ell_2 + \ell_3 = 11$, computed using the closed geodesics, crossing the cuts $n = 10$ times.

Hausdorff dimension contour plot

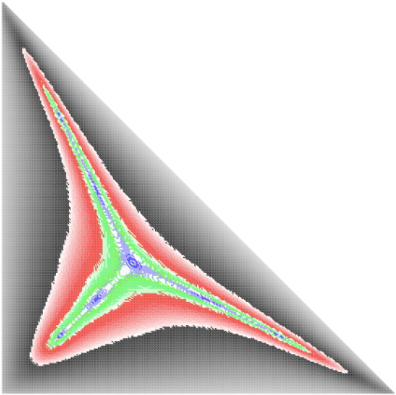


Figure: A projection of the graph of the Hausdorff dimension on the simplex $\ell_1 + \ell_2 + \ell_3 = 11$, computed using the closed geodesics, crossing the cuts $n = 10$ times.

Properties of $dim_{H,\Lambda}(\bar{\ell})$

The following properties hold for $dim_{H,\Lambda}(\bar{\ell})$ on the simplex σ_ℓ, for $\ell > 0$:

1. The function is (real) analytic;
2. There is a critical point at $(\ell/3, \ell/3, \ell/3)$;
3. There are critical points at $(\ell/6, \ell/6, 2\ell/3)$ and 2 other permutations;
4. There are critical points at $(\ell/2, \ell/4, \ell/4)$ and 2 other permutations;

Problem

The nature of the critical points remains a complete mystery.

Behaviour of $dim_{H,\Lambda}(\bar{\ell})$ near the boundary $\partial\sigma_\ell$

Fix $\bar{\ell}$ then at the edge of σ_ℓ :

1. The dimension function extends continuously to the boundary $\partial\sigma_\ell$;
2. For $(\ell_1, \ell_2, \ell_3) \in \partial\sigma_\ell$ we have $dim_{H,\Lambda}(\bar{\ell}) \geq 1/2$ (although numerical methods are not applicable);
3. The dimension at the middle point $dim_{H,\Lambda}(2\varepsilon, \ell/2 - \varepsilon, \ell/2 - \varepsilon) - dim_{H,\Lambda}(0, \ell/2, \ell/2) \asymp \varepsilon$.

References

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Thank you!