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On Dynamical Systems with 2-adic Time[1]

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A general concept of dynamical system with nonarchimedean time is suggested. It is illustrated by a certain limit on the dynamics on the sets of 2^n-periodic points of real quadratic maps.

1. INTRODUCTION AND MAIN RESULT

1.1. Periodic orbits and chaos. The onset of chaos (in families of discrete dynamical systems) is related to the appearance of periodic orbits of various orders. We are going to consider some nonarchimedean structures related to discrete periodic dynamics. The previous papers of the authors [4, 8, 3], as well as the one by Thiran, Verstegen and Weyers [10], were devoted to dynamics in nonarchimedean *spaces*; the present paper is devoted to nonarchimedean *time*. We will show that at least in one typical example the same dynamical system admits two compatible descriptions: a classical description with discrete time and a description with 2-adic time. Remarkably, the 2-adic version turns out to be considerably simpler.

1.2. General setting. We treat as "time" an arbitrary semigroup or group acting on the phase space

$$\mathbb{T} \colon X.$$

Classically, $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}$ for continuous and discrete dynamics, respectively. In the case of periodic processes, the groups $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $\mathbb{T} = \mathbb{Z}/N\mathbb{Z}$ are more suitable. The product of the latter groups over all $N \in \mathbb{N}$ acts in the obvious way on all sets of periodic orbits, but the spirit of nonarchimedean dynamics rather suggests considering the group of "universal periodic time"

$$\widehat{\mathbb{Z}} := \varprojlim \mathbb{Z}/N\mathbb{Z} \cong \prod_{\text{Prime } p} \mathbb{Z}_p,$$

acting on the periodic points by means of its finite cyclic factors. However, in the present paper we consider only $\mathbb{Z}_2$ acting on the 2^n-periodic points.

For a map $T \colon X \mapsto X$ and for $n \in \mathbb{N}$ we denote by $T^{n\circ}$ its nth iterate and by $T^{-n\circ}$ its nth inverse iterate (possibly multivalued). By $T^{\mathbb{N}\circ}(x)$ we denote the T-orbit of $x \in X$; finally, for $Y \subseteq X$ denote $T^{-\mathbb{N}\circ}Y := \bigcup_{n\in\mathbb{N}} T^{-n\circ}Y$ and $T^{-\infty}Y := \bigcap_{n\in\mathbb{N}} T^{-n\circ}Y$.

1.3. The main example. Consider the family of quadratic maps

$$f_c \colon x \mapsto x^2 + c.$$

The bifurcation values of c, where the 2^n-periodic orbits loose their stability and the 2^{n+1}-stable orbits appear, are well known. They are $0.25 = c_0 > c_1 > \ldots > c_\infty = -1.416\ldots$ and can be defined by the following property: for any $c \in (c_{n+1}, c_n)$ there is exactly one stable 2^n-cycle of f_c.

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The stable periodic points constitute the set

$$\mathrm{StabPer}_n(c),$$

on which the generator of the group $\mathbb{Z}/2^n\mathbb{Z}$ acts by $x \mapsto x^2 + c$. For every c there exists a distinguished element

$$x_0(c) := \lim_{N\to\infty} f_c^{2^N\circ}(0) \in \mathrm{StabPer}_n(c).$$

The orbits of $x_0(c)$'s are pasted together to define a *function of a real and a 2-adic variable*

$$X\colon\ (c_\infty, c_0] \times \mathbb{Z}_2 \to \mathbb{R},$$

where for $c \in (c_{n+1}, c_n]$ we set $X(c,t) := f_c^{[t]_n\circ}(x_0(c))$, denoting by $[t]_n$ the image of the *moment* under the projection $\mathbb{Z}_2 \to \mathbb{Z}/2^n\mathbb{Z}$. This function satisfies

$$X(c, t+1) = X(c,t)^2 + c.$$

1.4. Statement of the main result. The 2^n-element sets of connected components

$$\mathcal{X}_n := \pi_0\bigl(\{c, X(\mathbb{Z}_2, c) \,|\, c \in (c_{n+1}, c_n)\}\bigr)$$

are related by adjacency maps

$$\mathcal{X}_{n+1} \to \mathcal{X}_n$$

that send every element ξ of $\mathcal{X}_{n+1}$ to the only element of $\mathcal{X}_n$ representing a component whose closure has a nonempty intersection with the closure of ξ.

The sets $\mathcal{X}_n$ are acted upon by $\mathbb{Z}_2$ in a compatible manner, so that the projective limit

$$\mathcal{X} := \varprojlim \mathcal{X}_n$$

is also acted upon by $\mathbb{Z}_2$.

Denote by $\mathcal{X}_\infty$ the closure of the orbit $f_{c_\infty}^{\mathbb{N}\circ}(0)$. It is well known [11] that $\mathcal{X}_\infty$ is the attractor of f_{c_∞}. It is also true that *the map f_{c_∞} is invertible on $\mathcal{X}_\infty$—this follows easily from Theorem 1 below.*

Thus $\mathcal{X}_\infty$ is a $\mathbb{Z}$-set. The goal of the present paper is to prove the following result.

Main theorem. *There exists a $(\mathbb{Z} \to \mathbb{Z}_2)$-equivariant homeomorphism $\mathcal{X}_\infty \to \mathcal{X}$ sending 0 to the "distinguished element." Under this homeomorphism the action of $\mathbb{Z}_2$ is a continuous extension of the $\mathbb{Z}$-action.*

Though the statement seems quite natural, the analysis of the limiting behavior of the 2^n-periodic orbits of f_c is rather hard—and the difficulties are purely *archimedean*. We were unable to find easy proofs of our statements and had to use the deep results from the very well-developed one-dimensional dynamics, a survey of which is presented below.

2. PRELIMINARIES

2.1. Unimodal maps. Let $I = [-\alpha, \alpha] \subset \mathbb{R}$ be an interval and $f\colon I \to I$ a smooth even map.

Definition 1. A map f is called *unimodal* if it is monotone on each of the parts of $I \setminus \{0\}$, if it has a unique nondegenerate extremum at 0 and if it has no other critical points.

Definition 2. The Schwartzian derivative of f is defined as

$$Sf := \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

Theorem (Collet, Eckmann [2]). *Let f_c be a family of unimodal maps of the interval I to itself with negative Schwartzian derivatives, where c belongs to some interval, such that $c \mapsto f_c$ is a smooth nontrivial map. Then there exists a strictly monotone sequence of values $\{c_n\}_{n=1}^{\infty}$ providing the period-doubling bifurcation.*

Definition 3. We call a family of unimodal maps of the interval I into itself with negative Schwartzian derivatives a *Collet–Eckmann family.*

Every function of the classical family $f_c(x) = x^2 + c$, $c \in [-3/2, 1/4]$, is obviously unimodal on the interval $I_c = [-\beta, \beta]$, where β is a positive root of the equation $f_c(x) = x$. The properties of the existence of periodic orbits, as well as the type of their stability, are invariant under conjugation by diffeomorphisms. Instead of the classical family, we consider a truncated family $f_c(x) = x^2 + c$, $c \in [-3/2, -3/4]$. This family is conjugate to the family $\varphi_\gamma(x) = \frac{3}{2}x^2 + \gamma$ by the map $S(x) = \frac{3}{2}x$; i.e., $\varphi_\gamma = S \circ f_c \circ S^{-1}$, $\gamma \in [-1, -1/2]$. Hence the interval $[-1, 1]$ is invariant under the map φ_γ for $\gamma \in [1/2, 1]$; then φ_γ is a Collet–Eckmann family, as well as f_c for $c \in [-3/2, -3/4]$. Thus the families f_c and φ_γ provide the bifurcation sequences of period-doubling values $\{c_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$, respectively. These sequences converge with a universal rate (see, e.g., [2]): if c_∞ and γ_∞ are their limits, then

$$c_n - c_\infty \sim \delta^{-n}, \qquad \gamma_n - \gamma_\infty \sim \delta^{-n},$$

where $\delta = 4.669\dots$ is the famous Feigenbaum constant [5].

Our main functional space is the space $\mathcal{U}$ of smooth unimodal maps of the interval $[-1, 1]$ into itself. Let us equip it with the topology of uniform convergence.

2.2. Doubling transformation. Define a doubling transformation $T\colon \mathcal{U}_0 \to \mathcal{U}_0$, where

$$\mathcal{U}_0\colon = \bigl\{f \in \mathcal{U} \mid f^{2\circ}(0) \neq 0 \text{ and } x = 0 \text{ is a maximum point}\bigr\}.$$

Denote

$$\alpha = \alpha(f) = -\frac{f(0)}{f(f(0))}.$$

Suppose that $\alpha > 0$, $f(f(\alpha^{-1})) < \alpha^{-1} < f(\alpha^{-1})$ and $f(0) > 0$. Then

$$h(x) = -\alpha f(f(\alpha^{-1}x))$$

is also a unimodal map of the interval $[-1, 1]$ into itself, and $h(0) = f(0)$. Define the doubling map $T\colon \mathcal{U}_0 \to \mathcal{U}_0$ by

$$(Tf)(x) = -\alpha f(f(\alpha^{-1}x)), \qquad \alpha = \alpha(f) = -\frac{f(0)}{f(f(0))}.$$

Consider the functional equation

$$-\alpha g^{2\circ}\Bigl(\frac{x}{\alpha}\Bigr) \equiv g(x), \qquad \alpha = \alpha(g) = -\frac{g(0)}{g(g(0))}, \tag{1}$$

in the space of functions $\mathbb{R} \to \mathbb{R}$. This equation is called the *doubling equation.* According to the general theory (see the survey [11]), it has a unique solution, which satisfies the following conditions:

(i) $g(x) = -\alpha g(g(\alpha^{-1}x))$, $\alpha = 2.503\dots$;

(ii) $g(0) = 1$, and 0 is a maximum point;

(iii) $g(x) = g(-x)$.

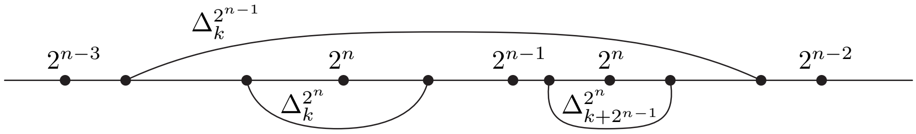

Fig. 1. Intervals and periodic points of period 2^n, 2^{n-1}, 2^{n-2} and 2^{n-3}

This function was thoroughly studied in the papers of Campanino and Epstein [1] and Lanford [6]. It was shown that all coefficients of the Taylor expansion of g, except the first one, are irrational. The first several terms are

$$g(x) = 1 - 1.527\ldots x^2 + 0.104\ldots x^4 + \ldots.$$

In the remaining part of the paper $\alpha = \alpha(g) = 2.503\ldots$.

Consider the sequence of maps $\{g_k\}_{k=1}^{\infty}$,

$$g_k(x) := \lim_{n\to\infty} (-\alpha)^n f_{c_{n+k}}^{2^n\circ}\left(\frac{x}{(-\alpha)^n}\right).$$

In the papers of Collet and Eckmann [2] and Lanford [6] one can find the following results. The sequence $\{g_k\}$ is well defined (all the limits exist). The functions g_k are unimodal and related by the doubling map:

$$g_{k-1} = Tg_k. \tag{2}$$

The sequence converges, and because of (2) its limit is the fixed point of the doubling operator:

$$\lim_{k\to\infty} g_k = g.$$

2.3. The partition tower of the interval. To prove Theorem 1, we use the construction suggested by Misiurewicz [7]. He considered the class Φ of maps φ of the interval $[-1,1]$ into itself with the following properties:

(1) $\varphi \in C^1([-1,1])$ and $\varphi \in C^2([-1,1]\setminus\{0\})$;

(2) $\varphi(-1) = -1$ and $\varphi'(-1) > -1$;

(3) $\varphi'(x) \neq 0$ if $x \neq 0$ and $S\varphi(x) < 0$ if $x \neq 0$;

(4) for every $n > 0$ the map φ has exactly one periodic orbit of order 2^n, and φ has no other periodic orbits.

For the maps of the class Φ one can construct a system of closed intervals $\{\Delta_i^{(n)}\}$, $n \geq 1$, $0 \leq i < 2^n$, with the following properties:

(1*) $\Delta_i^{(n)} \cap \Delta_j^{(n)} = \varnothing$ for $i \neq j$;

(2*) $f(\Delta_i^{(n)}) = \Delta_{i+1}^{(n)}$ for $0 \leq i < 2^n - 1$, $f(\Delta_{2^i-1}^{(n)}) \subset \Delta_0^{(n)}$, and the endpoints of the intervals $f(\Delta_{2^i-1}^{(n)})$ and $\Delta_0^{(n)}$ are different;

(3*) for every n the inclusion $\Delta_i^{(n)} \supset \Delta_i^{(n+1)} \cup \Delta_{i+2^n}^{(n+1)}$ holds and $\Delta_i^{(n)}$ contains no other intervals of level $n+1$.

Convention. We say that an interval $\Delta_k^{(N)}$ has *number* k and *level* N. We say that an interval $\Delta_k^{(N)}$ is less then $\Delta_m^{(L)}$ if $x < y$ for all points $x \in \Delta_k^{(N)}$ and $y \in \Delta_m^{(L)}$ (see Fig. 1).

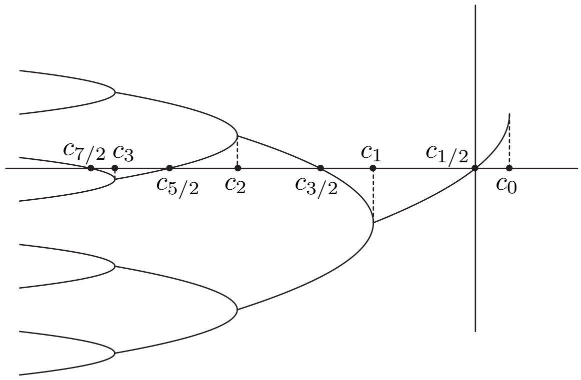

Fig. 2. Bifurcation diagram for the family f_c

3. REFORMULATION OF THE MAIN THEOREM

Definition 4. The *bifurcation diagram* (see Fig. 2) is the set of points of the (c,x)-plane defined as follows:

$$\mathrm{BD} := \bigsqcup_{n=0}^{\infty}\Big\{(c,x)\mid c\in[c_{n+1},c_n],\ f_c^{2^n\circ}(x)=x,\ \big|\big(f_c^{2^n\circ}\big)'(x)\big|\le 1\Big\}.$$

For each $c\in\mathbb{R}$ and each $n\in\mathbb{N}$ the equation $f_c^{2^n\circ}(x)=x$ defines an algebraic curve α_n. The bifurcation diagram consists of pieces of the curves α_n (with various n) defined by the conditions $\big|\big(f_c^{2^n\circ}(x)\big)\big|<1$, as well as of the points of neutral cycles at which the stability is lost: $\big\{x\colon f_{c_n}^{2^n\circ}(x)=x\mid \big(f^{2^n\circ}\big)'(x)=1\big\}$.

Now we define some special numeration of the components of BD over each interval (c_{n+1},c_n). As in the introduction, we denote these components by $\mathcal{X}_n$ with $\#\mathcal{X}_n=2^n$. We number the elements of $\mathcal{X}_n$ by the elements of the group $\mathbb{Z}/2^n\mathbb{Z}$, thus defining a bijection $\mathrm{num}\colon \mathcal{X}_n\to\mathbb{Z}/2^n\mathbb{Z}$. The only component that intersects the line $x=0$ acquires number 0; the other components are numbered uniquely by the condition

$$\mathrm{num}(f_c(\xi))=\mathrm{num}(\xi)+1.$$

Considering the groups $\mathbb{Z}/2^n\mathbb{Z}$ as factors of the additive group $\mathbb{Z}_2$ of 2-adic numbers, we paste these numerations to the global map

$$X(c,t)\colon (c_\infty,c_1]\times\mathbb{Z}_2\to\mathbb{R};$$

it is a real-valued function of a real and a 2-adic argument. By definition, it satisfies the equation

$$X(c,t+1)=X(c,t)^2+c.$$

The main technical result of the paper can be formulated as follows:

Theorem 1. *For any "2-adic moment" $t\in\mathbb{Z}_2$ there exists $\lim_{c\to c_\infty}X(c,t)=:X(c_\infty,t)$.*

4. IMPORTANT PARTICULAR CASE

First we prove a theorem that is a particular case of Theorem 1 (the case of *zero moment*).

Theorem 2.

$$\lim_{c\to c_\infty}X(c,0)=0. \tag{3}$$

Proof. The product of the derivatives of f_c, taken over the stable 2^n-periodic orbit, varies from 1 to -1 while c decreases from c_{n+1} to c_n, and $c_{n+1/2}$ is the value at which this product vanishes. The number $0 \in \mathbb{Z}_2$ corresponds to the part of the bifurcation diagram that intersects the line $x = 0$ on each interval $[c_{n+1}, c_n]$, $n \in \mathbb{N}$. Indeed, for any n there exists a value $c_{n+1/2} \in [c_{n+1}, c_n]$ such that 0 is a 2^n-periodic point of $f_{c_{n+1/2}}$. Such a cycle of the map $f_{c_{n+1/2}}$ is superattractive.

Lemma 1. *Let d_n be the distance on the (c, x)-plane between the line $x = 0$ and the nonzero point of the superattractive cycle of period 2^n that is closest to it. Then*

$$\lim_{n\to\infty} \left| \frac{d_n}{d_{n+1}} \right| = \alpha = 2.503\dots \tag{4}$$

and hence $\lim_{n\to\infty} d_n = 0$.

Proof. We calculate

$$g_1(0) = \lim_{n\to\infty} (-\alpha)^n f^{2^n\circ}_{c_{n+1+1/2}}(0) = \lim_{n\to\infty} (-\alpha)^n |d_{n+1}|.$$

The claim (4) follows from the last relation, because

$$\left| \frac{d_n}{d_{n+1}} \right| = \alpha \frac{|(-\alpha)^{n-1} d_n|}{|(-\alpha)^n d_{n+1}|} \to \alpha \qquad \text{as} \quad n \to \infty.$$

Hence $d_n \sim \alpha^{-n} \to 0$. $\square$

Lemma 2.

$$\lim_{n\to\infty} \frac{c_{n+3/2} - c_{n+1/2}}{c_{n+1/2} - c_{n-1/2}} = \lim_{n\to\infty} \frac{c_{n+1} - c_n}{c_n - c_{n-1}} = \frac{1}{\delta}. \tag{5}$$

Proof. It is obvious that

$$\varphi_\gamma = \varphi_{\gamma_\infty} + (\gamma - \gamma_\infty), \qquad T\varphi_\gamma = T(\varphi_{\gamma_\infty} + (\gamma - \gamma_\infty)). \tag{6}$$

Consider the linearization of T in the neighborhood of a function $f \in \mathcal{U}_0$:

$$(DT_f[h])(x) := \lim_{\lambda\to 0} \frac{T(f + \lambda h) - Tf}{\lambda}(x) = -\alpha \left[h\left(f\left(\frac{x}{-\alpha} \right) \right) + f'\left(f\left(\frac{x}{-\alpha} \right) \right) h\left(\frac{x}{-\alpha} \right) \right],$$

where $\alpha = \alpha(f)$.

Then equation (6) takes the form

$$\begin{aligned} (T\varphi_\gamma)(x) &= T(\varphi_{\gamma_\infty}) + (\gamma - \gamma_\infty) DT_{\varphi_{\gamma_\infty}}(1) + \bar{o}(\gamma - \gamma_\infty) \\ &= T(\varphi_{\gamma_\infty})(x) - \alpha(\gamma - \gamma_\infty) \left(1 + 2\left(\frac{x^2}{\alpha^2} + \gamma_\infty \right) \right) + \bar{o}(\gamma - \gamma_\infty), \end{aligned}$$

where $\bar{o}(\gamma - \gamma_\infty)$ denotes infinitesimals of higher order. Iterating this relation, we obtain

$$T^{n\circ}(\varphi_\gamma) = T^{n\circ}(\varphi_{\gamma_\infty}) + (\gamma - \gamma_\infty) DT_{T^{(n-1)\circ}\varphi_{\gamma_\infty}} \circ \dots \circ DT_{\varphi_{\gamma_\infty}}(1) + \bar{o}(\gamma - \gamma_\infty). \tag{7}$$

Let $\gamma = \gamma_{n+1/2}$ and $x = 0$. Then

$$\begin{aligned} T^{n\circ}(\varphi_{\gamma_{n+1/2}})(0) &= T^{n\circ}(\varphi_{\gamma_\infty})(0) + (\gamma_{n+1/2} - \gamma_\infty) DT_{T^{(n-1)\circ}\varphi_{\gamma_\infty}} \circ \dots \circ DT_{\varphi_{\gamma_\infty}}(1)(0) \\ &\quad + \bar{o}(\gamma_{n+1/2} - \gamma_\infty)(0). \end{aligned} \tag{8}$$

Take the limit as $n \to \infty$:

$$\lim_{n\to\infty} T^{n\circ}(\varphi_{\gamma_\infty})(0) = \lim_{n\to\infty} \lim_{j\to\infty} T^{n\circ}\varphi_{\gamma_{n+j}}(0) = \lim_{j\to\infty} \lim_{n\to\infty} (-\alpha)^n \varphi^{2^n}_{\gamma_{n+j}}(0) = \lim_{j\to\infty} g_j(0) = g(0) = 1;$$

since a continuous function on a closed interval is uniformly continuous, changing the order of limits is legal. Moreover,

$$\lim_{n\to\infty} DT_{T^{(n-1)\circ} f_{\gamma_\infty}} = DT_g. \tag{9}$$

Fix $N \gg 1$. From equation (9) it follows that whenever $n > N$

$$T^{n\circ}\varphi_{\gamma_{n+1/2}} = T^{n\circ}\varphi_{\gamma_\infty} + (c_{n+1/2} - \gamma_\infty)(DT_g)^{n-N} DT_{T^{N\circ}\varphi_{\gamma_\infty}} \circ \ldots \circ DT_{\varphi_{\gamma_\infty}}(1) + \bar{o}(\gamma - \gamma_\infty).$$

Consider the function $h := DT_{T^{N\circ} f_{\gamma_\infty}} \circ \ldots \circ DT_{\varphi_{\gamma_\infty}}(1)$. It is known [9] that the operator DT_g is not self-adjoint and that it has a unique eigenvector h_0 corresponding to the eigenvalue $\delta > 1$ and the invariant subspace $\widetilde{H}$, on which the norm $\|DT_g\| = \nu < 1$. Then the function h can be represented in the form $h = h_0\varphi_0 + \widetilde{h}$, where $\widetilde{h} \in \widetilde{H}$ and φ_0 is a number.

So

$$(DT_g)^{n-N} h = (DT_g)^{n-N}(h_0\varphi_0 + \widetilde{h}) = \delta^{n-N} h_0\varphi_0 + (DT_g)^{n-N}(\widetilde{h}).$$

Since the space $\widetilde{H}$ is invariant and $\|DT_g\| = \nu < 1$, we have $(DT_g)^{n-N}(\widetilde{h}) \to 0$ as $n \to \infty$. Finally, $\lim_{n\to\infty} T^{n\circ}\varphi_{\gamma_{n+1/2}}(0) = \lim_{n\to\infty}(-\alpha)^n \varphi^{2^n}_{\gamma_{n+1/2}}(0) = 0$. Hence, taking the limit as $n \to \infty$ in (8), we obtain

$$0 = g(0) + \lim_{n\to\infty}(\gamma_{n+1/2} - \gamma_\infty)\delta^{n-N}\varphi_0 h_0(0).$$

Consequently, $\lim_{n\to\infty}(\gamma_{n+1/2} - \gamma_\infty)\delta^n = \mathrm{const}$, from which Lemma 2 follows. $\square$

Theorem 2 follows from Lemma 2 and relation (4). $\square$

5. PROOF OF THE MAIN THEOREM

Proposition 1. *There exists a diffeomorphism conjugating the map f_{c_∞} with a map of the class Φ (see Subsection 2.3).*

Proof. Indeed, f_{c_∞} has a unique unstable periodic orbit of order 2^n. Applying the function $S(x)$ from Subsection 2.1 and using renormalization, we get the desired result. $\square$

For f_{c_∞} one can also construct a system of closed intervals $\{\Delta_i^{(n)}(c_\infty)\}$, $n \ge 1$, $0 \le i < 2^n$, satisfying properties ($1^\star$)–($3^\star$) of Subsection 2.3. The left and the right endpoints of these intervals will be denoted by β_k^n and γ_k^n, respectively. Due to properties ($2^\star$) and ($3^\star$) each interval $\Delta_i^{(n)}$ contains 2^{k-n} points of period 2^k for $k > n$, and by property ($1^\star$) it contains no periodic points of other orders. Each interval of level $\Delta_i^{(n+1)}$ is separated from the other intervals by two repelling points, one of which has period 2^n and the other has the same or smaller period. The numeration of the intervals can be chosen in such a way that $\Delta_0^{(n)}(c_\infty) \ni 0$. On the bifurcation diagram the interval $\Delta_0^{(n)}(c_\infty)$ is contained between the points of the repelling cycle of period 2^n that are the closest to the line $x = 0$.

It follows from Lemmas 1 and 2 that

$$\frac{|\Delta_0^{(n)}(c_\infty)|}{|\Delta_0^{(n+1)}(c_\infty)|} \to \alpha \qquad \text{as} \quad n \to \infty, \tag{10}$$

where $|\Delta|$ is the length of the interval Δ.

In order to prove the existence of the limit $\lim_{c\to c_\infty} X(c,t)$, it suffices to show that the following lemma holds.

Lemma 3. *For any natural k*

$$\lim_{n\to\infty} |\Delta_k^{(n)}| = 0. \tag{11}$$

Proof. It follows from Lemma 1 that $\left|\Delta_0^{(n)}\right| \equiv \frac{\text{const}}{\alpha^n}$. By the construction of the interval system

$$f_{c_\infty}^{2^n-k}\big(\Delta_n^{(k)}\big) \subset \Delta_0^{(n)};$$

so, to prove the convergence of the lengths of intervals, it suffices to show that

$$\forall k\ \exists D_k < \alpha\colon \qquad \alpha^2\big|f_{c_\infty}^{2^n-k}\big(\Delta_k^{(n)}\big)\big|D_k^n \ge \big|\Delta_k^{(n)}\big|. \tag{12_n}$$

For $k=0$ Lemma 3 is proved.

Now we proceed by induction on n. For $n=0$ there is only one interval. In the case $n=k=1$ on the interval $\Delta_1^{(1)}$ the map f_{c_∞} is expanding, and one can take $D_1 = \min\big(f'_{c_\infty}(\gamma_1^1), f'_{c_\infty}(\beta_1^1)\big)^{-1}$. Then

$$\big|f_{c_\infty}\big(\Delta_1^{(1)}\big)\big|D_1 \ge \big|\Delta_1^{(1)}\big|.$$

Let us prove that (12_n) implies (12_{n+1}). We have

$$\big|f_{c_\infty}^{2^{n+1}-k}\big(\Delta_k^{(n+1)}\big)\big| = \big|f_{c_\infty}^{2^n} f_{c_\infty}^{2^n-k}\big(\Delta_k^{(n+1)}\big)\big|.$$

Since $f_{c_\infty}^{2^n-k}$ is a diffeomorphism on the interval $\Delta_k^{(n+1)}$ with negative Schwartzian derivative, according to the choice of D_1 we obtain

$$\big|f_{c_\infty}^{2^n-k}\big(\Delta_k^{(n+1)}\big)\big|D_1^n > \big|\Delta_k^{(n+1)}\big|. \tag{13}$$

Furthermore, $f_{c_\infty}^{2^n-k}\big(\Delta_k^{(n+1)}\big) = \Delta_{2^n}^{(n+1)}$. So now it is time to prove that for any $n\in\mathbb{N}$

$$\big|f^{2^n}\big(\Delta_{2^n}^{2^{n+1}}\big)\big|\frac{\alpha^2}{\alpha-1} > \big|\Delta_{2^n}^{2^{n+1}}\big|.$$

Consider the map f^{2^n} on the interval $\Delta_{2^n}^{2^{n+1}}$. It is a diffeomorphism with negative Schwartzian derivative. Without loss of generality we may suppose that it is increasing. By the definition of partition tower $f^{2^n}\big(\Delta_{2^n}^{2^{n+1}}\big) \subset \Delta_0^{2^{n+1}}$, $\Delta_{2^n}^{2^{n+1}} \supset \Delta_{2^n}^{2^{n+2}} \cup \Delta_{3\cdot 2^n}^{2^{n+2}}$ and $f^{2^n}\big(\Delta_{2^n}^{2^{n+2}}\big) = \Delta_{2^{n+1}}^{2^{n+2}} \subset \Delta_0^{2^{n+1}}$. Moreover, $\Delta_{3\cdot 2^n}^{2^{n+2}} \supset \Delta_{3\cdot 2^n}^{2^{n+3}} \cup \Delta_{7\cdot 2^n}^{2^{n+3}}$ and $f^{2^n}\big(\Delta_{3\cdot 2^n}^{2^{n+3}}\big) = \Delta_{2^{n+2}}^{2^{n+3}}$, $f^{2^n}\big(\Delta_{7\cdot 2^n}^{2^{n+3}}\big) \subset \Delta_0^{2^{n+3}}$. But $\Delta_{2^{n+1}}^{2^{n+2}} < \Delta_0^{2^{n+3}} < \Delta_{2^{n+2}}^{2^{n+3}}$, so $f^{2^n}\big(\Delta_{2^n}^{2^{n+1}}\big) \supset \Delta_0^{2^{n+3}}$. Using relation (10), we get

$$\big|\Delta_{2^n}^{2^{n+1}}\big| < \frac{\big|\Delta_0^{2^n}\big|(\alpha-1)}{\alpha} \qquad \text{and} \qquad \big|\Delta_0^{2^n}\big| > \frac{\big|\Delta_0^{2^{n-3}}\big|}{\alpha^3}.$$

Hence

$$\big|f^{2^n}\big(\Delta_{2^n}^{2^{n+1}}\big)\big|\frac{\alpha^2}{\alpha-1} > \big|\Delta_{2^n}^{2^{n+1}}\big|. \tag{14}$$

Now (12_n) follows from (14) and (13).

Lemma 3 is proved. $\square$

The interval with number $k\ne 0$ appears for the first time on the level $s := [\log_2 k + 1]$. We have to find a number D_k such that $f_{c_\infty}^{2^s-k}(\Delta_k^s)D^s > |\Delta_k^s|$. Take

$$D_k = \max\Big(\min\Big(\big(f_{c_\infty}^{2^s-k}\big)'(\beta_k^s), \big(f_{c_\infty}^{2^s-k}\big)'(\gamma_k^s)\Big), D_{k-2^{s-1}}\Big).$$

Theorem 1 is proved. $\square$

The map f_c acts on the bifurcation diagram:

$$f_c\colon \ \mathrm{BD} \to \mathrm{BD}, \quad (x,c) \mapsto (f_c(x), c),$$

inducing the map $F\colon \mathbb{Z}_2 \to \mathbb{Z}_2$, $t \mapsto t+1$.

Conclusion. The following diagram is commutative:

$$\begin{tikzcd} \mathbb{Z}_2 \arrow[r, "t\mapsto t+1"] \arrow[d, "\kappa(t)"] & \mathbb{Z}_2 \arrow[d, "\kappa(t)"] \\ \mathbb{R} \arrow[r, "f_{c_\infty}"] & \mathbb{R} \end{tikzcd}$$

This is an equivalent form of our main theorem.

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