Dimension function of the Lagrange and Markov spectra

based on joint works with Matheus, Moreira and Pollicott

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A special feature of our work, which distinguishes it from other traditional mathematical papers, is the fact that we rely heavily on results of computer calculations. Therefore, a full verification of our results with pen and paper only is impossible, and computer calculations are required. It is our opinion that very soon works referring to computer calculations will become much more common in certain fields of mathematics.

K.I. Babenko, On a problem of Gauss (1977)

Markov and Lagrange spectra

The continued fraction of $x \in (0,1)$ is an expression

$$x = [0; \alpha_1, \dots, \alpha_n, \dots] := \cfrac{1}{\alpha_1 + \cfrac{1}{\alpha_2 + \cfrac{1}{\alpha_3 + \ddots}}}, \quad \alpha_n \in \mathbb{N}$$

Consider a set of bi-infinite sequences $(\mathbb{N}^*)^{\mathbb{Z}}$ and Bernoulli shift

$$\sigma\colon (\mathbb{N}^*)^{\mathbb{Z}} \to (\mathbb{N}^*)^{\mathbb{Z}}; \qquad \sigma\left((\alpha_n)_{n\in\mathbb{Z}}\right) = (\alpha_{n+1})_{n\in\mathbb{Z}}.$$

Introduce a map

$$\lambda\colon (\mathbb{N}^*)^{\mathbb{Z}} \to \mathbb{R} \qquad \lambda(\underline{\alpha}) = [\alpha_0; \alpha_1, \alpha_2, \dots] + [0; \alpha_{-1}, \alpha_{-2}, \dots].$$

Definition (Perron, 1921)

The Lagrange value of $\alpha \in (\mathbb{N}^*)^{\mathbb{Z}}$ is $\ell(\alpha) := \limsup_{n\to\infty} \lambda(\sigma^n \alpha)$

The Markov value of $\alpha \in (\mathbb{N}^*)^{\mathbb{Z}}$ is $m(\alpha) := \sup_{n\in\mathbb{Z}} \lambda(\sigma^n \alpha)$.

The collection of Lagrange (Markov) values is called the Lagrange (Markov) spectrum.

$$\mathcal{L} := \left\{\ell(\alpha) \mid \alpha \in (\mathbb{N}^*)^{\mathbb{Z}}\right\} \subsetneq \mathcal{M} := \left\{m(\alpha) \mid \alpha \in (\mathbb{N}^*)^{\mathbb{Z}}\right\}.$$

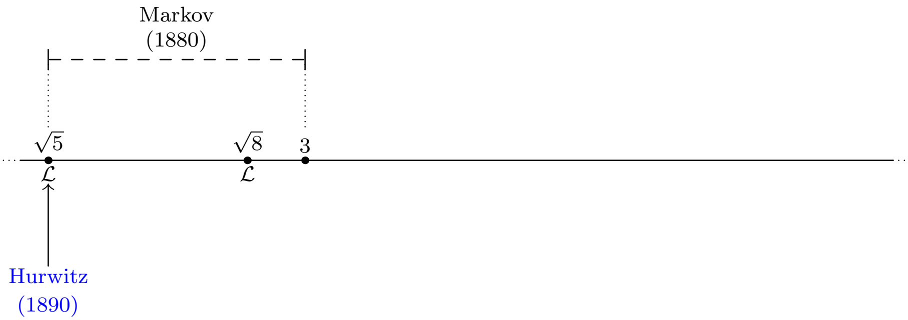
Study of the spectra — I

Markov, 1880

$$\mathcal{L} \cap \left(\sqrt{5}, 3\right) = \mathcal{M} \cap \left(\sqrt{5}, 3\right) = \left\{\sqrt{5} < \sqrt{8} < \sqrt{221}/5 < \dots\right\}$$

is a countable set (the proof uses Markov triples).

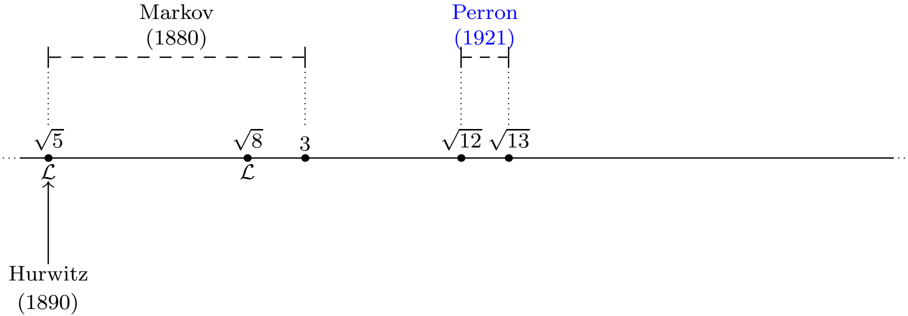
Study of the spectra — II



Hurwitz, 1890

$\min \mathcal{L} = \sqrt{5}$ (the proof uses a more classical definition via best approximants).

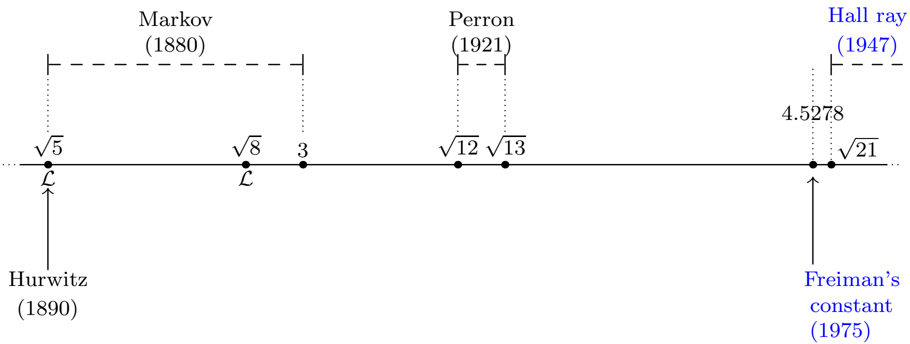
Study of the spectra — III



Perron, 1921

$\left(\sqrt{12}, \sqrt{13}\right) \cap \mathcal{M} = \varnothing$, while $\sqrt{12}, \sqrt{13} \in \mathcal{L}$. Furthermore, $\mathcal{L} \subset \mathcal{M}$ and both sets are closed.

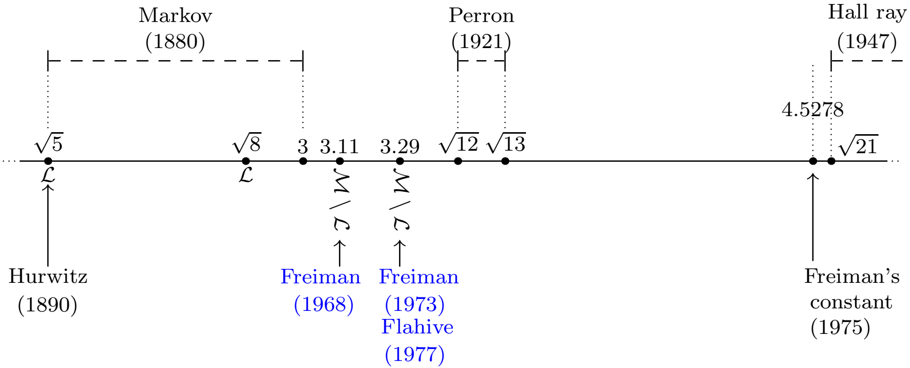
Study of the spectra — IV



Hall, 1947: There exists $c \in \mathbb{R}$ such that $[c, +\infty) \subset \mathcal{L} \subset \mathcal{M}$.
Schecker & Freiman, 1963: One can take $c = \sqrt{21}$ above.
Freiman, 1975: The smallest possible c is

$$c_F = \frac{2221564096 + 283748\sqrt{462}}{491993569}.$$

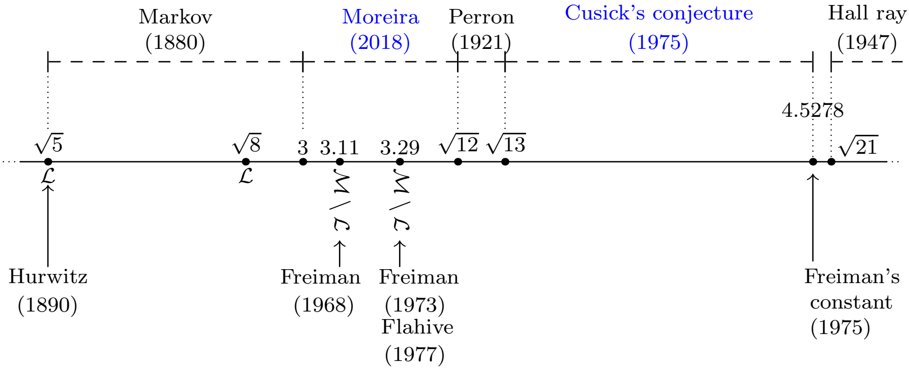
Study of the spectra — V



Freiman 1968, 1973; Flahive 1977

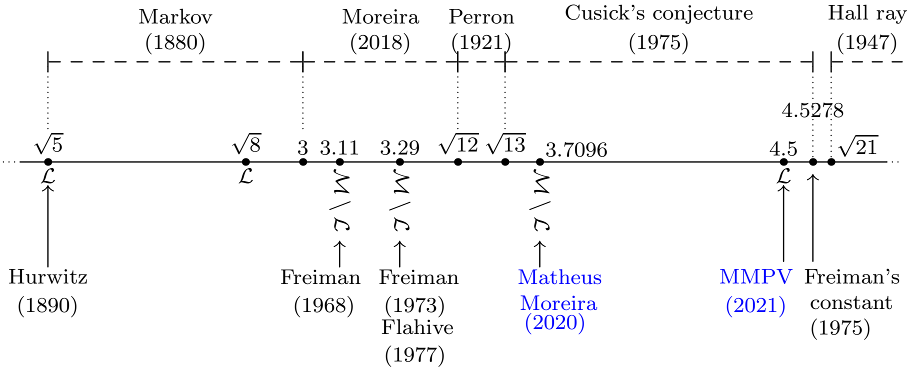
Near 3.11 and 3.29 the set $\mathcal{M} \setminus \mathcal{L}$ contains two countable subsets.

Study of the spectra — VI



Cusick's conjecture, 1975: $(\mathcal{M} \setminus \mathcal{L}) \cap [\sqrt{12}, +\infty) = \varnothing$.
Bernstein's conjecture, 1973: $[4.1, 4.52] \subset \mathcal{L}$.
Moreira, 2018: $\dim(\mathcal{L} \cap (-\infty, t)) = \dim(\mathcal{M} \cap (-\infty, t))$ and the function $f(t) := \dim(\mathcal{M} \cap (-\infty, t))$ is continuous. Moreover, $f(\sqrt{12}) = 1$ and $f(3 + \varepsilon) > 0$ for any $\varepsilon > 0$.

Study of the spectra — VII



~~Cusick's conjecture, 1975: $(\mathcal{M}\setminus\mathcal{L})\cap[\sqrt{12},+\infty)=\varnothing$.~~

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Intermission

$$f(t) := \dim(\mathcal{M} \cap (-\infty, t))$$

END OF PART 1: INTRODUCTION

NEXT

PART 2: BUMBY'S METHOD FOR COMPUTING $f(t)$

Goal: Compute the graph of f

$$f(t) := \dim(\mathcal{M} \cap (-\infty, t))$$

Matheus, Moreira, Pollicott, and P.V., 2021:

The first transition is

$$t_1 := \inf\{t \in \mathbb{R} \mid f(t) = 1\} = 3.334384\ldots$$

So we are mainly interested in $\dim(\mathcal{M} \cap (-\infty, t))$ for $3 \le t \le 3.334385$

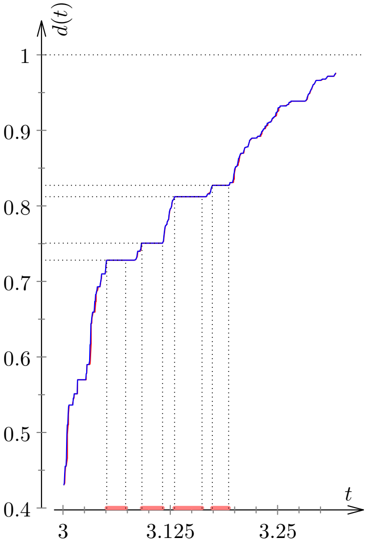
The function f is a Cantor staircase function:

- continuous and monotone increasing
- $f' = 0$ almost everywhere,
- $f(3) = 0$, $f(3+\varepsilon) > 0$ for any $\varepsilon > 0$; and $f(t) = 1$ for all $t \ge t_1$.
- Asymptotic at 3:

$$d(3+\varepsilon) = 2 \cdot \frac{W(c|\log\varepsilon|)}{|\log\varepsilon|} + O\left(\frac{\log|\log\varepsilon|}{|\log\varepsilon|^2}\right),$$

  where W is the Lambert function (the inverse of $g(z) = ze^z$) and $c = \frac{1}{\log(3+\sqrt{5})/2}$.

The graph



The **blue** curve is an upper bound for

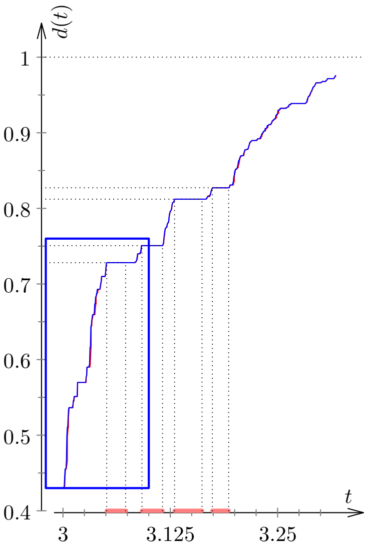
$$\dim(\mathcal{M} \cap (-\infty, t)) = \dim(\mathcal{M} \cap (3, t)).$$

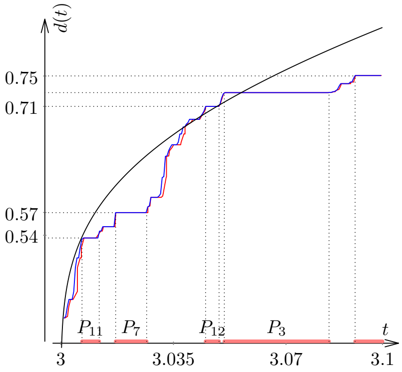
The **red** curve is the lower bound for

$$\dim(\mathcal{M} \cap (-\infty, t)) = \dim(\mathcal{M} \cap (3, t))$$

The **pink** intervals are the *gaps* in the Markov spectrum, identified by Cusick and Flahive in mid 1970s.

Now with Lambert function





In 1982 Bumby gave an heuristic bound $3.33437 < t_1 < 3.33440$, and suggested a technique for approximating $d(t)$, but computers in 1982 were big but not powerful enough; now they are much smaller but more powerful!

We can realise the approach Bumby proposed in practice.

Preliminaries

- Consider a set of continued fractions of 1's and 2's:

$$E_2 := \{a = [0; \alpha_1, \alpha_2, \dots] \mid \alpha_j \in \{1,2\},\ j \ge 1\}$$

  Then $\min E_2 = \frac{1}{2}(\sqrt{3}-1)$, $\max E_2 = \sqrt{3}-1$ and $\dim E_2 > 0.53128$.
- Let $\alpha \in \{1,2\}^{\mathbb{Z}}$. Then $m(\alpha) \ge \sqrt{5}$ and $m(\alpha) = \sqrt{5}$ if and only if $\alpha = \dots, 1, 1, 1, \dots$.
- Let $\alpha \in \{1,2\}^{\mathbb{Z}}$. Then $m(\alpha) \le \sqrt{12}$ and $m(\alpha) = \sqrt{12}$ if and only if $\alpha = \dots, 1, 2, 1, 2, \dots$

Approach to upper bound

Fix $T > 0$ and construct a finite set F of finite strings of 1's and 2's with a property that if $\alpha \in \{1,2\}^{\mathbb{Z}}$ doesn't contain a string from F, then $m(\alpha) < T$.

Claim. Let $K \subset E_2$ be such that for any $x \in K$ its continued fraction expansion doesn't contain a string from F. Then

$$M \cap (\sqrt{5}, T) \subset 2 + K + K.$$

Idea of the proof: $m = \lambda(\alpha) = [2; \alpha_1, \alpha_2, \dots] + [0; \alpha_{-1}, \alpha_{-2} \dots]$.
Since $\dim_H(K + K) \le \dim_H K + \overline{\dim}_B K$, thus

$$\dim(\mathcal{M} \cap (3, T)) \le 2 \dim_H K.$$

Example (Hall, 1971)

If $\alpha \in \{1,2\}^{\mathbb{Z}}$ doesn't contain a substring 121, then $m(\alpha) \le \sqrt{10}$.

$$\dim(\{[0; \alpha_1, \alpha_2, \dots] \mid \alpha_j \in \{1,2\}, (\alpha_j \alpha_{j+1} \alpha_{j+2}) \ne (121), j \ge 1\}) \le 0.45,$$

therefore $\dim(\mathcal{M} \cap (3, \sqrt{10})) \le 0.9$.

Approach to lower bound

Let S be the maximal Markov value of strings which do not contain a substring from a finite set of finite strings F:

$$S = \max m(\alpha), \text{ where } \alpha \in \{1,2\}^{\mathbb{Z}} \text{ doesn't contain a string from } F$$

and let $K \subseteq E_2$ be such that for any $x \in K$ its continued fraction expansion doesn't contain a string from F. Moreira proved

$$\dim_H((\sqrt{5}, S) \cap \mathcal{M}) \geq \min\{2 \cdot \dim_H K, 1\},$$

Example (Perron)

Note that $m(\alpha) \leq \sqrt{12}$ if and only if $\alpha \in \{1,2\}^{\mathbb{Z}}$. Therefore we may choose $F = \varnothing$ and $K = E_2$. Then

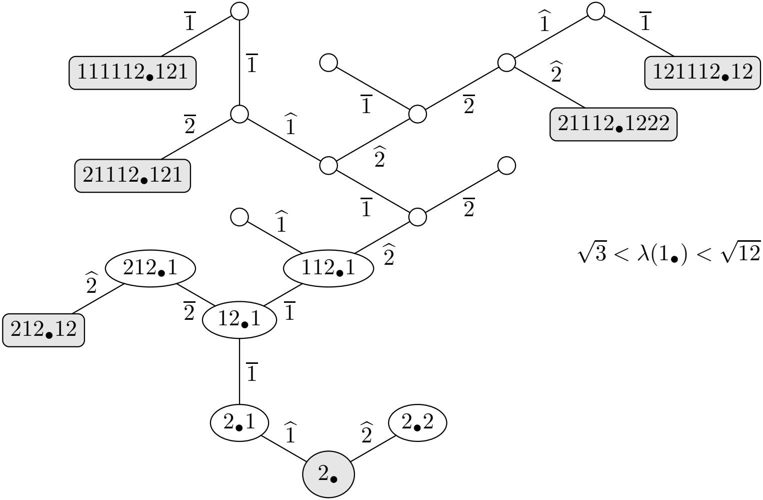
$$\dim_H((\sqrt{5}, \sqrt{12}) \cap \mathcal{M}) \geq \min\{2 \cdot \dim_H E_2, 1\} = \min(2 \cdot 0.54318, 1) = 1,$$

and conclude, in particular, that $t_1 \leq \sqrt{12}$.

Strategy by Bumby $T = 3.333$

Identifying the strings with $\lambda(\alpha) > T$ (because $m(\alpha) \geq \lambda(\alpha)$).

$$\lambda(\alpha) = [\alpha_0; \alpha_1, \alpha_2, \dots] + [0; \alpha_{-1}, \alpha_{-2}, \dots],\ \alpha_j \in \{1, 2\}$$



$\sqrt{3} < \lambda(1_\bullet) < \sqrt{12} - 1 < T$

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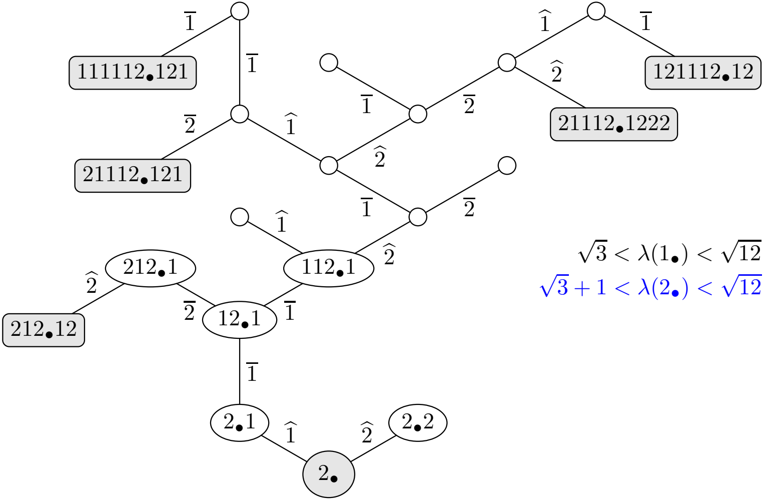

$\sqrt{3} < \lambda(1_\bullet) < \sqrt{12} - 1 < T$

$\sqrt{3} + 1 < \lambda(2_\bullet) < \sqrt{12}$

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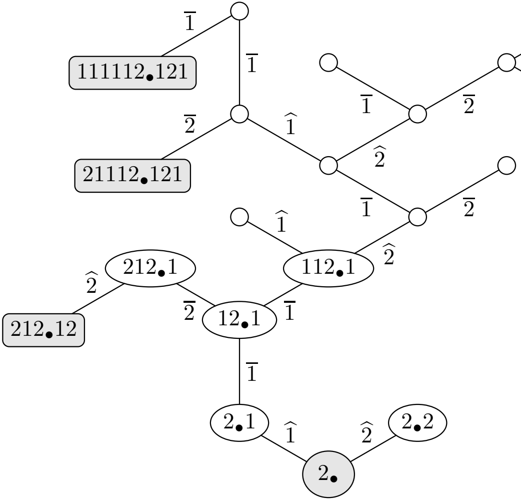

$$\sqrt{3} < \lambda(1_\bullet) < \sqrt{12} - 1 < T$$

$$\sqrt{3} + 1 < \lambda(2_\bullet) < \sqrt{12}$$

$$\sqrt{3} + 1 < \lambda(2_\bullet 2) < 2 + 2/\sqrt{3} < T$$

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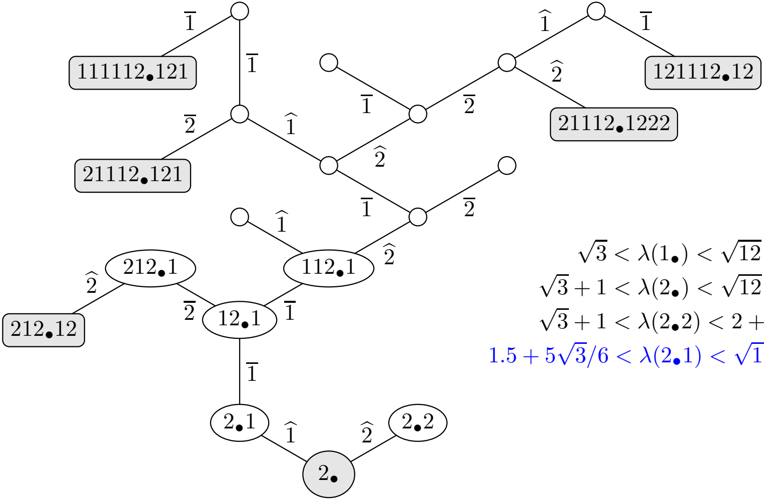

$\sqrt{3} < \lambda(1_\bullet) < \sqrt{12} - 1 < T$

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$\sqrt{3} + 1 < \lambda(2_\bullet 2) < 2 + 2/\sqrt{3} < T$

$1.5 + 5\sqrt{3}/6 < \lambda(2_\bullet 1) < \sqrt{12}$

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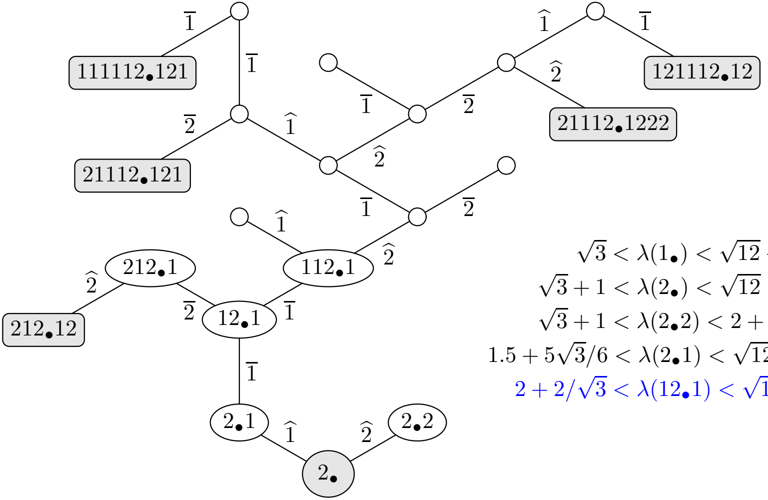

$$\sqrt{3} < \lambda(1_\bullet) < \sqrt{12} - 1 < T$$
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$$1.5 + 5\sqrt{3}/6 < \lambda(2_\bullet 1) < \sqrt{12}$$
$$2 + 2/\sqrt{3} < \lambda(12_\bullet 1) < \sqrt{12}$$

Strategy by Bumby $T = 3.333$

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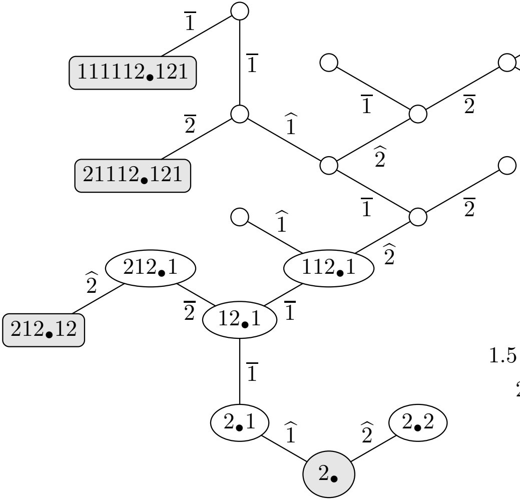

$$\sqrt{3} < \lambda(1_\bullet) < \sqrt{12} - 1 < T$$
$$\sqrt{3} + 1 < \lambda(2_\bullet) < \sqrt{12}$$
$$\sqrt{3} + 1 < \lambda(2_\bullet 2) < 2 + 2/\sqrt{3} < T$$
$$1.5 + 5\sqrt{3}/6 < \lambda(2_\bullet 1) < \sqrt{12}$$
$$2 + 2/\sqrt{3} < \lambda(12_\bullet 1) < \sqrt{12}$$
$$T < 3.4 < \lambda(212_\bullet 12) < \sqrt{12}$$

Strategy by Bumby (continued)

Taking inverses, we obtain the set F of 9 words of length up to 9:

$$F = \{21212, 111112121, 121211111, 12111212, 21211121, \\ 21112121, 12121112, 211121222, 222121112\}.$$

Question

How to estimate the dimension of the set X which we obtain from E_2 after removing all numbers whose continued fraction expansion contains these strings?

Intermission

The usability of the method of computing the first transition point t_1 depends on our ability to estimate the Hausdorff dimension of the Gauss–Cantor set of continued fractions.

$$
\begin{aligned}
X_{\bar r} := \Big\{ [0;a_1,a_2,\ldots] \,\Big|\ & a_n \in \{1,2\},\ \text{with extra restrictions} \\
& a_j a_{j+1} \ldots a_{j+r_1} \neq d_0^{(1)} d_1^{(1)} \ldots d_{r_1}^{(1)},\ d_j^{(1)} \in \{1,2\} \\
& a_j a_{j+1} \ldots a_{j+r_1} \neq d_0^{(2)} d_1^{(2)} \ldots d_{r_2}^{(2)},\ d_j^{(2)} \in \{1,2\} \\
& * \qquad * \qquad * \\
& a_j a_{j+1} \ldots a_{j+r_k} \neq d_0^{(k)} d_1^{(k)} \ldots d_{r_k}^{(k)},\ d_j^{(k)} \in \{1,2\} \Big\} \subsetneq E_2
\end{aligned}
$$

with $k \le 9$ and $r_j \le 9$ for all $1 \le j \le k$.

the end of part 2

approach for computing lower and upper bounds on $f(t)$

next

part 3: computation of dimension of Gauss–Cantor sets

Toy example

$$X = \left\{ [0; a_1, a_2, \ldots,] \mid a_j \in \{1,2\},\ a_j a_{j+1} a_{j+2} \neq 121, 212 \right\}$$

We define a Markov iterated function scheme of 4 maps parametrised by strings $\bar{j} \in \mathcal{A} = \{1,2\}^2$ and a transition matrix M

$$T_{j_1 j_2}(x) = \cfrac{1}{j_1 + \cfrac{1}{j_2 + x}} \qquad M = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

Columns and rows are encoded by $\mathcal{A} = \{11, 12, 21, 22\}$.

$$M_{i_1 i_2, j_1 j_2} = 1 \iff j_1 j_2 i_1 i_2 \text{ doesn't contain 121 or 212.}$$

The limit set of $\{T_j\}_{\bar{j} \in \mathcal{A}}$ with respect to M is

$$\left\{ \lim_{n \to +\infty} T_{\bar{j}_1} \circ \cdots \circ T_{\bar{j}_n}(0) \mid \bar{j}_k \in \mathcal{A}, M_{\bar{j}_k, \bar{j}_{k+1}} = 1, 1 \le k \le n-1 \right\} = X$$

Approximating eigenfunction (after Babenko at al, 1977)

- Fix a small natural number m (e.g., $m = 8$ works).
- We can introduce
  1. $p_k(x) \in C([0,1])$ — the Lagrange polynomials ($1 \le k \le m$), and
  2. $x_k \in [0,1]$ — the Chebyshev nodes ($1 \le k \le m$)

  so that $p_i(x_j) = \delta_{ij}$, for all $1 \le i, j \le m$
- Introducing $d = |\mathcal{A}|$ small $m \times m$ matrices

$$B^{j,t}(i,l) := |T_j'(x_i)|^t \cdot p_l(T_j(x_i))$$

  we get a $dm \times dm$ matrix A^t given by

$$A^t = \begin{pmatrix} M_{1,1} \cdot B^{1,t} & M_{2,1} \cdot B^{2,t} & \dots & M_{d,1} \cdot B^{d,t} \\ M_{1,2} \cdot B^{1,t} & M_{2,2} \cdot B^{2,t} & \dots & M_{d,2} \cdot B^{d,t} \\ \vdots & \vdots & \ddots & \vdots \\ M_{1,d} \cdot B^{1,t} & M_{2,d} \cdot B^{2,t} & \dots & M_{d,d} \cdot B^{d,t} \end{pmatrix}.$$

- Let $w_t = (w_t^1, \cdots, w_t^{dm})$ be the (left) eigenvector for the largest eigenvalue.
- Finally, set $f_j(x) = \sum_{k=1}^{m} w_t^{(j-1)m+k} p_k(x)$.

Realisation challenges

1. The construction of matrix M which gives Markov condition requires analysing of 2^{2n} words of length $2n$ looking for forbidden substrings
2. For $n = 17$ the matrix M would take $2GB$ (and we need $n = 24$)
3. The matrix A^t is even larger: for $n = 17$ and $m = 6$ it would take $1512GB$ to store (and we need its eigenvector!)
4. The best method for the computation of the eigenvector is the power method, it has complexity a bit more than $O(n^{2.5})$

Lemma (Matheus, Moreira, Pollicott, & V. 2021)

Assume that the columns j_1 and j_2 of the Markov matrix M are identical, i.e. for all $1 \le k \le md$ we have that $M_{k,j_1} \equiv M_{k,j_2}$. Then any eigenvector $\bar{f}$ of A^t lies in the subspace of $C^2(S)$ for which $f_{j_1} = f_{j_2}$.

This is a huge help: In the case of the set X the Markov matrix has 3940388 columns of which only 429 are pairwise distinct. A reduction procedure allows to replace the matrix A^t of size $\approx 31 \cdot 10^8$ with a matrix of size $429 \cdot 8 = 3432$ only!

Key Contributions

Advances on the study of Markov and Lagrange spectra:

1. We show that the first transition point

$$t_1 := \inf\{t \in \mathbb{R} \mid \dim(\mathcal{M} \cap (-\infty, t)) = 1\} = 3.334384\ldots$$

2. We identify several non-affine Cantor sets in $\mathcal{M} \setminus \mathcal{L}$ and demonstrate that $\mathcal{M} \setminus \mathcal{L}$ has a rich structure.

3. We give an effective and efficient method for computing the Hausdorff dimension of fairly complicated Gauss–Cantor sets and apply it to approximate the devil staircase of the dimension function.