Estimating characteristic parameters of hyperbolic systems

Polina Vytnova
joint work with Mark Pollicott and Oliver Jenkinson

University of Warwick and Queen Mary University of London

January 2017

It is a fact of experience that computer simulations of a relatively naive sort are generally fairly reliable indicators of the properties of concrete dynamical systems.

O. E. Lanford III

History

- Periodic points are easy to compute and give a lot of information on hyperbolic systems
- A paper "Calculating Hausdorff dimensions of Julia sets and Kleinian limit sets" by Jenkinson & Pollicott appeared in 2002
- An algorithm for computing Hausdorff dimension of dynamically defined sets based on periodic orbit data was presented
- But back then fast computers were very big and not accessible to general public

the Years Passed . . .

The algorithm has been used to compute various parameters:

- Entropy (Lyapunov Exponents)
- Diffusion Coefficient (Variance)
- Pressure
- Linear Response (Derivatives of Averages of Maps)
- Rate of Mixing (Decay of Correlations)
- Moments
- . . . .
- Hausdorff Dimension of Dynamically Defined Sets (Limit Sets, Julia Sets, . . .)

Open Question: Accuracy

- The algorithm gives a sequence of numbers a_n, each of which depends on periodic points up to period n
- The sequence hopefully converges and
- The limit is the quantity we are interested in

$$\lim_{n\to\infty} a_n = a$$

- We are happy if $\lg |a_n - a_{n-1}| \leq -\alpha n$ for some $\alpha > 0.1$
- We conclude that $|a_{n_{max}} - a| \leq \alpha n_{max}$, where n_{max} is the maximum period our computer can deal with in 24 hours

Main Question

Is the result trustworthy?

Case Study: Lanford map

The Lanford map

$$T(x) :=2x+\frac{1}{2}x(1-x) \mod 1$$

(Visualize a slightly perturbed doubling map)

- Introduced by O. Lanford in 1998 paper "Informal Remarks on the Orbit Structure of Discrete Approximations to Chaotic Maps"
- Brought to my attention by S. Galatolo during his talk "Rigorous estimation of the speed of convergence to equilibrium" in April 2016.

Properties

The Lanford map

1. C^ω expanding map: $|T'| \geq \frac{3}{2}$
2. Admits a unique invariant measure μ equivalent to Lebesgue measure.
3. The abstract dynamical system $(T;\mu)$ is ergodic and isomorphic to a Bernoulli shift.
4. A shadowing theorem which ensures that the computed orbit stays near to some true orbit over arbitrarily large numbers of steps holds
5. A central limit theorem holds

The Variance

Theorem (Central Limit Theorem)

Let g be a real-valued analytic function. Then

$$\mu\left(\left\{x \in [0,1] \colon a \leq \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} g(T^k x) \leq b\right\}\right) \overset{n\to\infty}{\longrightarrow} \frac{1}{\sqrt{2\pi}\sigma} \int_a^b \exp\left(\frac{-t^2}{2\sigma^2}\right) \mathrm{d}t$$

The value

$$\sigma^2 \colon = \lim_{n\to\infty} \frac{1}{n} \int \left(\sum_{k=0}^{n-1} g(T^k x)\right)^2 d\,\mu(x)$$

is called *the variance* of the test function g.

First Numerical Result

W. Bahsoun, S. Galatolo, I. Nisoli, and X. Niu in "Rigorous approximation of diffusion coefficients for expanding maps", using Ulam's method: for $g = x^2 - \int x^2 d\,\mu$ we have that

$$\sigma_\mu^2(g) \in [0.3458, 0.4512].$$

We can do better!

O. Jenkinson, M. Pollicott & P.V.:

$$\sigma_\mu^2(g) \in [0.360109486199160 \pm 10^{-18}]\,.$$

The cost?

1. about $6.7 \cdot 10^8$ periodic points of the period up to 25
2. arbitrary-precision calculations with accuracy of 10^{-200}
3. about 24 computer hours (no special RAM requirements)

Thermodynamic formalism

$$\begin{tikzcd} & \text{Hyperbolic System} \arrow[dl] \arrow[dr] & \\ \text{Pressure Function} \arrow[dr] & & \text{Transfer Operator} \arrow[dl] \\ & \text{Determinant} & \end{tikzcd}$$

Definition

Let $F(x) \stackrel{\text{def}}{=} -\log|T'(x)|$ be a C^ω function. The *pressure function* is $P(F) \stackrel{\text{def}}{=} \sup_{m\in\mathcal{M}}\{h(m)+\int F dm\}$ where $\mathcal{M}$ is the set of f-invariant probability measures $h(m)$ is the entropy. Supremum is achieved at SRB measure μ.

For any $g \in C^\omega$, the pressure $P(F+tg)$ is analytic and

$$\frac{\partial P(F+tg)}{\partial t}\Big|_{t=0} = \int g d\mu$$

Transfer operator

Definition

We let B be the Banach space of complex-valued bounded analytic functions on $U \supset [0,1]$ with supremum norm $\|\cdot\|_\infty$. To a mapping $F \in B$ and a test function $g \in B$ we associate a family of transfer operators $\mathcal{L}_{t,g} : B \to B$:

$$(\mathcal{L}_{t,g}p)(x) = \sum_k e^{(F-tg)(\tau_k x)} p(\tau_k x), \quad t \in \mathbb{R};$$

where $\tau_k : U \to U$ are the local inverses to T, which are C^ω contractions satisfying $\overline{\tau_k(U)} \subset U$.

Determinant

Theorem (Grothendieck–Ruelle)

The transfer operator is nuclear. Its determinant is an entire function in z defined as $d\colon \mathbb{C} \times \mathbb{R} \times C^{\omega}(U) \to \mathbb{C}$

$$d(z,t,g) \stackrel{\mathrm{def}}{=} \det(I - z\mathcal{L}_{t,g}) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{Tr}(\mathcal{L}_{t,g}^n)\right)$$

Lemma (Ruelle)

$$d(z,t,g) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{T^n x = x} \frac{\exp(-t g^n(x))}{|(T^n)'(x)| - 1}\right),$$

where $g^n(x) = \sum_{k=0}^{n-1} g(T^k x)$.

Magic of thermodynamics

Lemma (Ruelle)

For any $z \in \mathbb{C}$, $t \in \mathbb{R}$, and $g \in C^\omega(U)$ we have that:

1. $d(z, t, g)$ *converges to an analytic function for* $|z| < e^{-P(F-tg)}$;
2. $d(z, t, g)$ *has an analytic extension in* $z \in \mathbb{C}$ *to the entire complex plane* $\mathbb{C}$;
3. $z \mapsto d(z, t, g)$ *has a simple zero at* $z(t, g) = e^{-P(F-tg)}$.

Lemma (Grothendieck–Ruelle)

The power series coefficients of the determinant decrease superexponentially and uniformly in $t \in \mathbb{R}$.

Cooking approximations up

1. Write the diffusion coefficient as the 2'nd derivative of pressure

$$\sigma^2(g) = \frac{\partial^2}{\partial t^2} P(-\ln|T'| + tg)\Big|_{t=0}$$

2. Using the Implicit Function Theorem, express the 2'nd derivative of pressure in terms of the Taylor coefficients of the determinant and their derivatives
3. Using Ruelle's Lemma, rewrite the Taylor coefficients and their derivatives in terms of the periodic orbit sums

$$\sum_{T^n(x)=x} \frac{\exp(-tg^n(x))}{|(T^n)'(x)| - 1}$$

Prospective Bounds

Theorem

Given a piecewise real-analytic Markov map $T : X \to X$ with an absolutely continuous invariant probability measure μ, and a real-analytic $g : X \to \mathbb{R}$, there exists a sequence $\{\sigma_n^2\}$ where n'th element depends only on periodic points of period up to n, and the rate of convergence is faster than exponential. Specifically, if $\dim X = 1$, then there exist explicit constants $A = A_{T,\mu,g} > 0$ and $\alpha = \alpha_{T,\mu,g} \in (0,1)$ such that

$$|\sigma_\mu^2(g) - \sigma_n^2| \le A\alpha^{n^2} \quad \textit{for all } n \in \mathbb{N}\,.$$

Aim

Given T and g to estimate A and α.

Invisible Superexponential Convergence

n	σ_n^2	$\lvert\sigma_n^2 - \sigma_{n-1}^2\rvert$
12	0.36 010948 **61**85859588343561990599828878966607	10^{-9}
13	0.360 1094861 **99**222993644688357957828705184562	10^{-11}
14	0.3601 09486199 **160**481163645430040654615882458	10^{-14}
15	0.36010 9486199160 **67**3287014050839470838927840	10^{-16}
16	0.360109 48619916067 **2898**306093693521789682071	10^{-20}
17	0.3601094 86199160672898 **824**643277247080597474	10^{-23}
18	0.36010948 6199160672898824 **186**562820134550885	10^{-26}
19	0.3601094861 9916067289882418 6 **828**679098981571	10^{-29}
20	0.36010948619 9160672898824186828 **5767**23147913	10^{-33}
21	0.360109486199 1606728988241868285767 **4924**6076	10^{-37}
22	0.3601094861991 6067289882418682857674924 **1669**	10^{-41}

Based on numerical data only, one can guess that the convergence is exponential. Our estimates show that $\alpha \approx 0.86\ldots$ and $A \approx \exp(3)$.

Computational Limits

Question

How big n could be?

1. Computational Time
   time $\sim \#$ periodic points $\cdot$ precision
   $\sim \mathrm{const} \cdot \exp(\text{period}) \cdot \#$ digits
   2^{25} points of period 25 with precision $10^{-200} \approx 6$ hours
2. Memory
   space $\sim \#$ periodic points $\cdot$ precision
   2^{20} points of period 20 with precision of $10^{-200} \approx 1$GB

I have heard of a super cluster (available to CUNY) which can do a 10000 hours computation in 50 hours, but even then $n_{max} = 34$ and you have to wait in a queque to get access.

the Space

1. Consider $D \colon = \left\{ z \in \mathbb{C} \mid \left| z - \frac{1}{2} \right| < 1 \right\}$.
2. Choose the space $\mathcal{H}$ to be the Hardy Hilbert space

$$\mathcal{H} \colon = \Big\{ f \colon D \mapsto \mathbb{C} \text{ analytic } \mid \sup_{0<r<1} \int_0^1 |f(z_0 + r\exp(2\pi\theta i)|^2 d\,\theta < +\infty \Big\}.$$

3. The transfer operator $\mathcal{L}$ respects $\mathcal{H}$
4. The norm $\|\mathcal{L}\|$ can be bounded via the Littlewood Subordination Theorem (for composition operators)

Approximation Numbers

Approximation numbers for a compact operator $\mathcal{L}$ on a Hilbert space are

$$s_k(\mathcal{L}) :\ = \inf\{\|\mathcal{L} - \Pi_k\| : \mathrm{rank}(\Pi_k) \leq k - 1\}$$

Lemma

Given a transfer operator $\mathcal{L}_{g,t}$ the n'th Taylor coefficient of the determinant $d(z, t, g)$ is bounded by

$$|c_n(t)| \leq \sum_{j_1 < \ldots < j_n} \prod_{k=1}^{n} s_{j_k}(\mathcal{L}_{g,t})$$

Approximation Bounds

Basis in the Hardy space on the disc $B(c, \rho)$:

$$m_k(z) := \frac{(z-c)^k}{\rho^k}$$

Lemma

The apprximation numbers have *approximation bounds*

$$s_k(\mathcal{L}_{g,t}) \leq \alpha_k(t) := \Big(\sum_{j=k-1}^{\infty} \|\mathcal{L}_{g,t}(m_k)\|^2 \Big)^{1/2}$$

Divide and Rule

1. Euler bound: compute $C_t(g)$ and $\theta(T)$: $\alpha_n(t) \leq C_t \theta^n$;
2. Estimate numerically $||\mathcal{L}_{g,t}(m_k)||$ for $k = 1 \ldots 500$;
3. Compute explicitly $|c_n|$, $|c_n'|$ $|c_n''|$ for $n = 1, \ldots, 25$;
4. Estimate carefully $|c_n|$, $|c_n'|$, $|c_n''|$ for $n = 26, \ldots, 40$;
5. Use Euler bound to estimate the tails.

Other Characteristic Parameters

The same method gives other estimates:

- The entropy (or Lyapunov exponent) of the measure is equal to

$$0.57661780006597677541582411 \pm 10^{-20}$$

- The rate of mixing (i.e., the second eigenvalue of the transfer operator) is equal to

$$0.57807968853712196815306689 \pm 10^{-22}$$

- The linear response $\frac{\partial}{\partial\lambda}\int x^2 d\mu_\lambda|_{\lambda=0.5}$, where μ_λ is the absolutely continuous invariant measure for the map $T_\lambda(x) = 2x + \lambda x(1-x) \mod 1$, is estimated to be

$$0.14082024965148029317326339 \pm 10^{-19}$$

Other Systems

In order for the method to work, we need

1. Markov, analytic, and uniformly expanding map T (or a family of maps, or a group of transformations)
2. Finitely supported invariant measure
3. Banach space of functions
4. Nuclear transfer operator(s)

References

- W. Bahsoun, S. Galatolo, I. Nisoli & X. Niu, Rigorous approximation of diffusion coefficients for expanding maps, *J. Stat. Phys.*, **163** (2016), 1486–1503.
- A. Grothendieck, Produits tensoriels topologiques et espaces nucleaires, *Mem. Amer. Math. Soc.*, 16 (1955), 1–140.
- O. E. Lanford III, Informal remarks on the orbit structure of discrete approximations to chaotic maps, *Exp. Math.*, **7** (1998), 317–324.
- D. Ruelle, Zeta-functions for expanding maps and Anosov flows, *Invent. Math.*, 34 (1976), 231–242.
- J. H. Shapiro, *Composition operators and classical function theory*, Springer-Verlag, 1993.