Apollonian Gasket

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Science is what we understand well enough to explain to a computer. Art is everything else we do.

D. Knuth

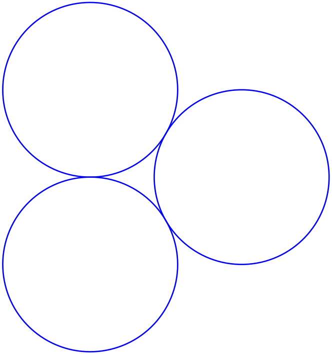
Goal:

- Develop a numerical method for computing the Hausdorff dimension of a parabolic limit set;
- Provide theoretical foundations on the level of computer-assisted proofs;

Model example:

- the Apollonian gasket, the oldest fractal set.

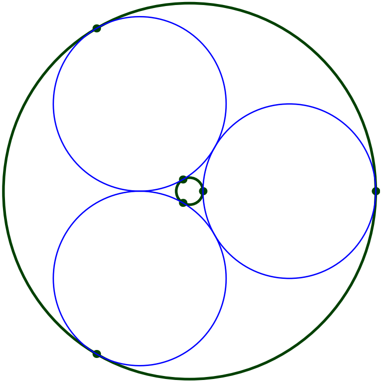
The question of Apollonius



Theorem (Apollonius)

Given three circles tangent to one another, there exist exactly two circles that are tangent to all three.

The question of Apollonius



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Student of Euclid,
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260–190BC;
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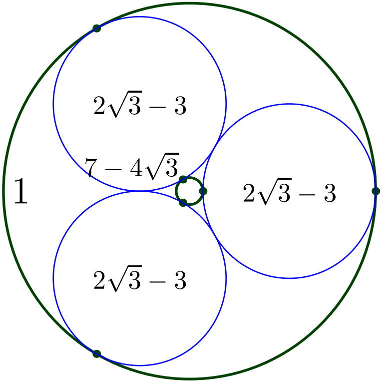

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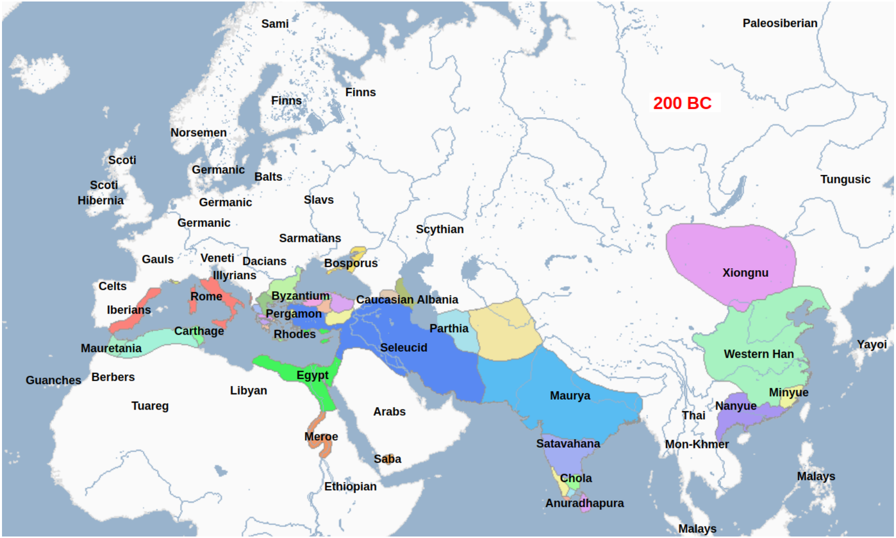
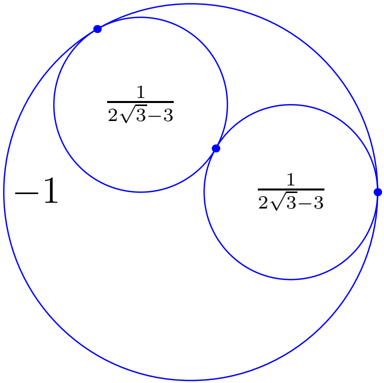
Problem

Identify the two circles (find their radii and centres).

The world during Apollonius' time



Solution(s): from 1643 to 2002

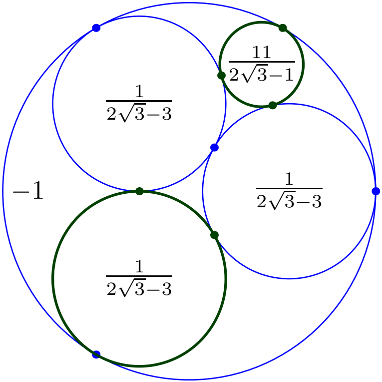


Theorem (Descartes)

The curvatures of the circles satisfy:

$$c_1^2 + c_2^2 + c_3^2 + c_4^2 = \frac{1}{2} \cdot (c_1 + c_2 + c_3 + c_4)^2$$

Solution(s): from 1643 to 2002



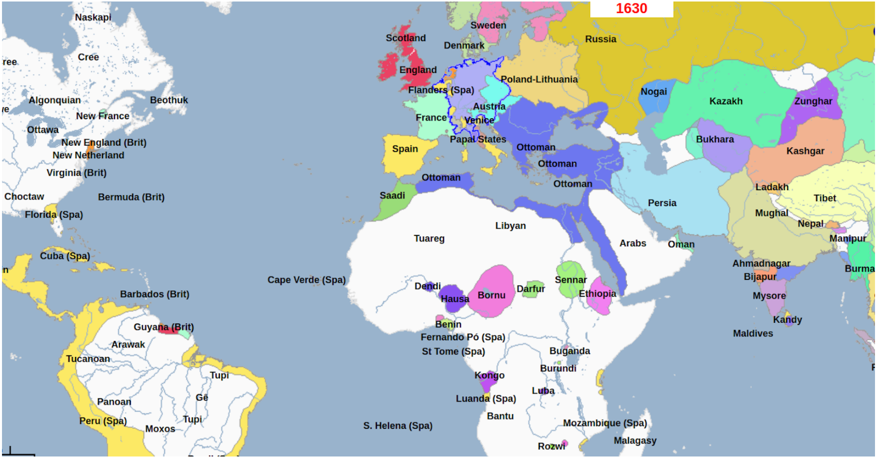
Theorem (Descartes)

The curvatures of the circles satisfy:

$$c_1^2 + c_2^2 + c_3^2 + c_4^2 = \frac{1}{2} \cdot (c_1 + c_2 + c_3 + c_4)^2$$

(In a letter to the Princess Elisabeth of Bohemia.)

The world during Descartes' time



The correspondents

Frans Hals, National Galery of Denmark, Copenhagen; Gerard van Honthorst, Ashdown House, Oxfordshire.

Hyperbolic positioning is used

- to describe the orbits of planets in the solar system (Newton)
- to locate the source of a signal based on the different times the signal is received at three different locations (WW1)
- in modern GPS systems: "gasket anthenna"

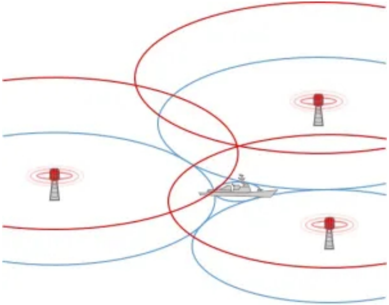
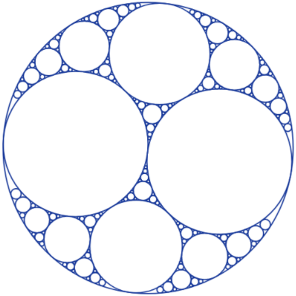
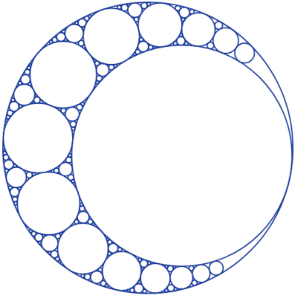


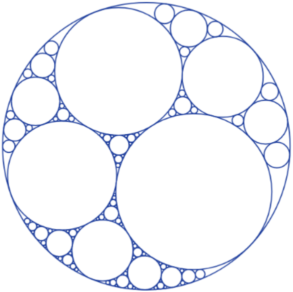
Image credit: Chalkdust, 11/2020

When ingeneers and physicist are done with a topic, mathematicians take it on!

Bounded gaskets: Examples







d-dimensional Hausdorff measure

- Given a d-dimensional set, we want to estimate its volume (think about the area of a piece of paper, $d = 2$) provided we only know how to measure distances.
- Naive idea: we can cover it by sets A_n and consider a $\sum_n \operatorname{diam}(A_n)^d$.
- This is bad: the area is likely to be bigger than $\operatorname{diam}(ball)^2$.
- Take many tiny balls: better bound, but still low.

Hausdorff dimension

Definition

d-dimensional volume of a metric space X *is smaller than* V if for all $\varepsilon > 0$ there exists a countable cover $X = \bigcup_n A_n$ such that (1) $\mathrm{diam}(A_n) \le \varepsilon$ for all n; (2) $\sum_n \mathrm{diam}(A_n)^d < V$.

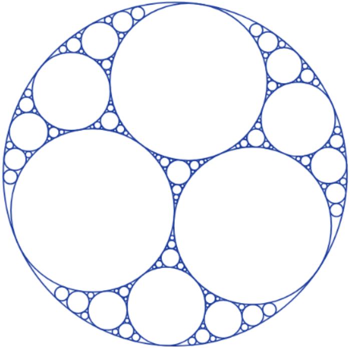
- Vary V continuously from $+\infty$ to 0.
- Spot the moment when the claim "d-dimensional volume smaller than V" fails.

$$H^d(X) := V; \quad d\text{-dimensional Hausdorff measure.}$$

- $H^0(X) = \#X$; Increase d continuously from 0 to $+\infty$.
- Spot the moment when $H^d(X) \neq 0$.

The critical value of d is called the Hausdorff dimension of X.

Hausdorff dimension of the gasket



Let r_n be the sequence of radii, counting multiplicities.
Define *the critical exponent* by

$$\delta = \inf\left\{t \mid \sum_{n=1}^{\infty} r_n^t \text{ is finite}\right\}.$$

Then $\delta = \dim_H(\mathcal{A})$.

This is one of the first known fractals — sets of non-integer Hausdorff dimension

What is in dimension?

Conjecture (Bishop)

Among all sets X whose residual set $R^2 \setminus X$ consists of round disks, the Apollonian gasket has the smallest Hausdorff dimension.

- There are only finitely many circles of radius bigger than a given number $t := NC(t)$.
- Lee–Oh, 2012: There exists $\eta > 0$ such that for any bounded Apollonian packing,

$$NC(t) = \text{const} \cdot t^{-\alpha} + O(t^{\eta-\alpha})$$

where α is the Hausdorff dimension of the packing.

Hausdorff dimension estimates

Consider an Apollonian gasket $\mathcal{A}$ derived from three equal circles; denote $\alpha := \dim_H(\mathcal{A})$.

- David W. Boyd, 1973: $1.300197 < \alpha < 1.314534$.
- Peter B. Thomas and Deepak Dhar, 1994: $\alpha = 1.30568672910\ldots$
- Curtis T. McMullen, 1998: $\alpha = 1.305687\ldots$
- Roberto De Leo, 2014: $\alpha = 1.3056867\ldots$
- Zai-Qiao Bai and Steven Finch, 2018: $\alpha = 1.3056867280\,4987718464\,5986206851\,0\ldots$

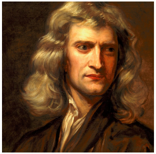
Theorem (V. & Wormell, 2023 (?))

The estimate by Bai and Finch is accurate to all decimal places given.

... more estimates

Theorem (Vytnova & Wormell)

$$\dim_H(\mathcal{A}) = 1.3056867280\,4987718464\,5986206851\,0408911060\,2644149646\,8296446188\,3889969864\,2050296986\,4545216123\,1505387132\,8079246688\,2421869101\,9673056436\,0845303608\,3 \pm 10^{-140}$$



"I am ashamed to tell you to how many figures I carried these computations, having no other business"

— Isaac Newton

(on computing 15 digits of π during the Great Plague in 1666.)

Step 1: Introduce a dynamical system

Idea

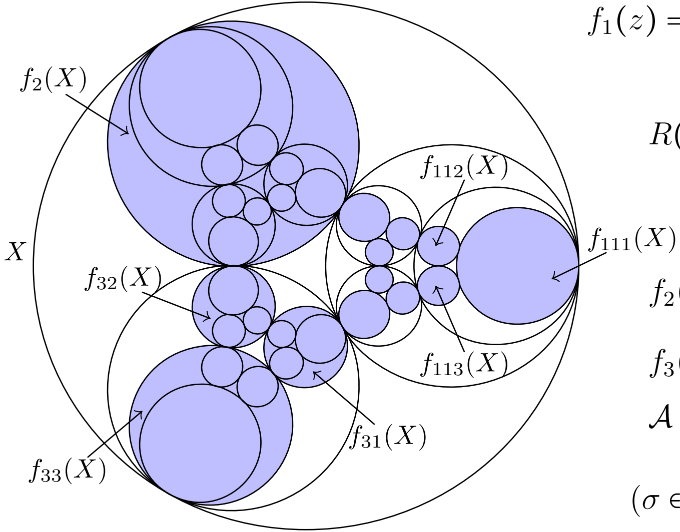
To compute the Hausdorff dimension of a bounded set $X \subset B \subset \mathbb{R}^2$ we want to realise it as a limit set of an iterated function scheme.

More precisely, we want to find a countable family of *uniformly contracting* maps $\mathcal{T} = \{T_j\}_{j \in \mathbb{N}}$ such that $T_j(B) \subset B$ for all $j \in \mathbb{N}$ and X is the limit set for $\mathcal{T}$:
$x \in X \iff$
there exists $y \in B$ and a sequence $\{j_n\} \subset \mathbb{N}$ such that

$$x = \lim_{n\to\infty} T_{j_n} \circ \ldots \circ T_{j_2} T_{j_1}(y)$$

In fact, since all T_j are uniformly contracting, i.e. $|T_j'| < 1 - \varepsilon$ for some $\varepsilon > 0$, the limit is independent on y.

Iterated function scheme for the gasket



$$f_1(z) = \frac{(\sqrt{3}-1)z+1}{-z+\sqrt{3}+1}$$

$$R(z) = \exp\left(\frac{2\pi i}{3}\right) z$$

$$f_2(z) = R(f_1(z))$$

$$f_3(z) = R^2(f_1(z))$$

$$\mathcal{A} = \bigcap_n \bigcup_{|\sigma|=n} f_\sigma(X)$$

$$(\sigma \in \{1,2,3\}^n).$$

f_j are not uniformly contracting!

Indeed, since

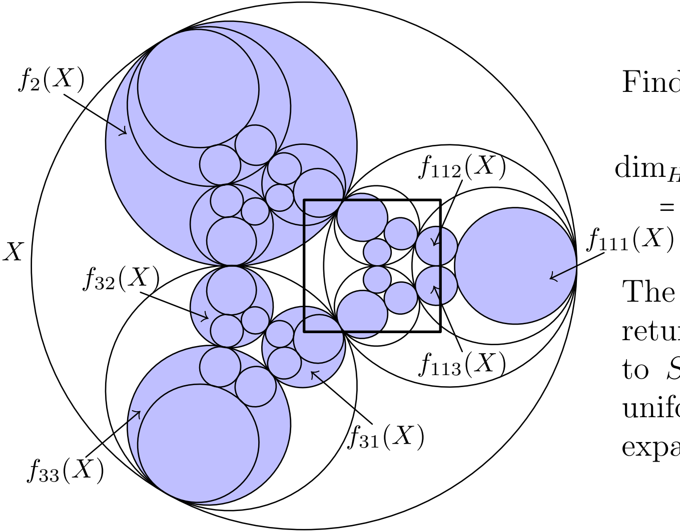
$$f_1(z) = \frac{(\sqrt{3}-1)z+1}{-z+\sqrt{3}+1}$$

we get $f_1'(1) = 1$, $|f_2'(1)| = |f_3'(1)| = 1$.

Idea

We apply inducing to replace the IFS $\{f_1, f_2, f_3\}$ with an IFS $\{T_k, k \in \mathbb{N}\}$ with *countable number of uniformly contracting maps with the limit set of the same Hausdorff dimension.*

Inducing — I

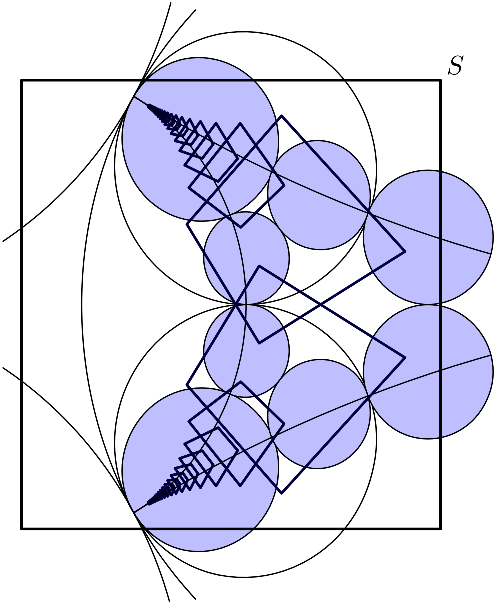


Find $S \subset \mathbb{D}$:

$\dim_H(S \cap \mathcal{A})$
$= \dim_H A$

The first return map to $S \cap \mathcal{A}$ is uniformly expanding.

Inducing — II



For $k \in \mathbb{N}$

$$T_{2k+1} = f_1 \circ R \circ f_1^k$$

$$T_{2k} = f_1 \circ R^2 \circ f_1^k$$

Infinite IFS of uniformly contracting maps on a square S. Dimensions of the limit sets are the same.

Step 2: Introduce the operators

Idea

The estimates on the Hausdorff dimension of the limit set of an iterated function scheme of uniform contractions come from the study of (associated) bounded linear operators.

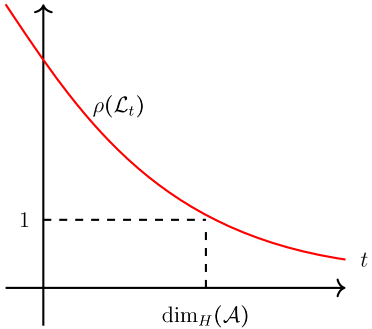
Given the maps $T_j: S \to S$, consider the Banach space of real analytic functions $C^\omega(S)$ and the family of linear operators $\mathcal{L}_t : C^\omega(S) \to C^\omega(S)$:

$$[\mathcal{L}_t w](x) = \sum_{j=1}^{\infty} |T_j(x)'|^t \cdot w(T_j(x))$$

The operator is called the transfer operator for the iterated function scheme.

Spectral radius and dimension

Let $\rho(\mathcal{L}_t)$ denote the spectral radius of $\mathcal{L}_t$.



Lemma (after Bowen and Ruelle, from 1980s)

The map $t \mapsto \rho(\mathcal{L}_t)$ is strictly monotone decreasing and the solution to $\rho(\mathcal{L}_t) = 1$ is $t = \dim_H(\mathcal{A})$.

Approaches to the spectral radius $\rho(\mathcal{L}_t)$

The so-called "periodic points method" or "dynamical zeta functions method" (Jenkinson and Pollicott, 2002) is to consider a real *analytic* function

$$\zeta(z,t) = \det(z\mathcal{L}_t - I)$$

and to compute the largest zero of $\zeta(1,t)$.

Instead, we attempt to compute *an approximation* to the eigenvector of $\mathcal{L}_t$ corresponding to $\rho(\mathcal{L}_t)$.

Useful fact (after Ruelle–Grothendieck):

In the case we consider, the operator $\mathcal{L}_t$ is nuclear and $\rho(t)$ is an isolated eigenvalue.

Step 3: Estimates on $\rho(\mathcal{L}_t)$

Lemma

Let $t_0 < t_1$*.*

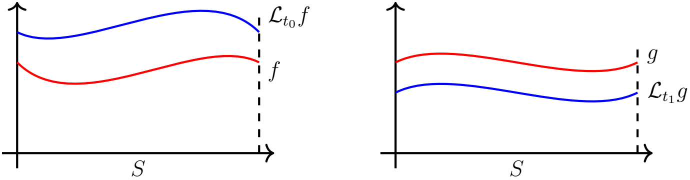
1. *If there exists a (positive) polynomial* $f : S \to \mathbb{R}^+$ *such that*
$$\inf_x \frac{\mathcal{L}_{t_0} f(x)}{f(x)} > 1 \implies \text{then } \rho(\mathcal{L}_{t_0}) > 1.$$
2. *If there exists a (positive) polynomial* $g : S \to \mathbb{R}^+$ *such that*
$$\sup_x \frac{\mathcal{L}_{t_1} g(x)}{g(x)} < 1 \implies \text{then } \rho(\mathcal{L}_{t_1}) < 1.$$

Corollary

If we can find f, g *as above then* $t_0 < \dim_H(\mathcal{A}) < t_1$*.*

Now in pictures...

Given $t_0 < t_1$, to show that $\dim_H(\mathcal{A}) \in [t_0, t_1]$ if suffices to ... guess (or construct) two positive polynomials $f, g : S \to \mathbb{R}^+$ such that



$$\mathcal{L}_{t_0} f \geq f \implies t_0 \leq \dim_H(\mathcal{A}) \qquad \mathcal{L}_{t_1} g \geq g \implies \dim_H(\mathcal{A}) \leq t_1$$

It only remains to construct such functions f and g, which is the final step.

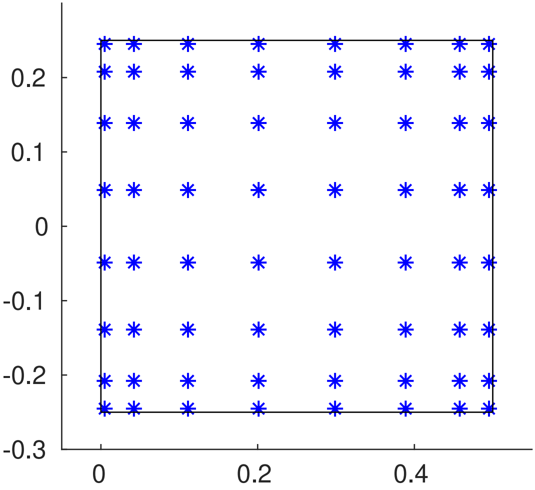
Chebyshev interpolation method

We could just try and guess the functions f and g (and hope we get lucky), but a more systematic approach is to use a bit of interpolation theory.

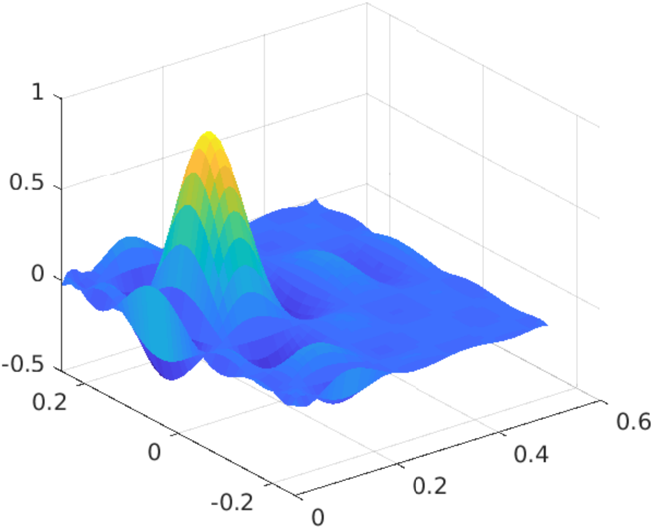
- Fix a natural numbers m (e.g., $m = 12$).
- We can introduce
  1. $p_{j,k}(x) \in C^\omega(S)$ — the Lagrange polynomials $(1 \le j, k \le m\ )$, and
  2. $x_{j,k} \in S$ — the Chebyshev nodes $(1 \le j, k \le m\ )$

  so that $p_{j_1,j_2}(x_{k_1,k_2}) = \delta_{j_1}^{k_1}\delta_{j_2}^{k_2}$, for $1 \le j_{1,2}$, $k_{1,2} \le m$

The nodes



A basis function



Step 4: Cooking up test functions

- Given t consider the $m^2 \times m^2$ matrix $A_t(j,k) = (\mathcal{L}_t p_{j_1,j_2})(x_{k_1,k_2})$ for $1 \le j_{1,2}, k_{1,2} \le m$.
- Let $w_t = (w_t^{1,1}, \cdots, w_t^{m,m})$ be the (left) eigenvector for the largest eigenvalue.
- Finally, set $f_{m,t}(x) = \sum_{j,k=1}^{m} w_t^{j,k} p_{j,k}(x)$.

The matrix A is a finite rank approximation to $\mathcal{L}_t$.

I learned this method from a 1984 paper by K. I. Babenko, Demonstrative Computations In The Problem of Existence of a Solution of the Doubling Equation, Soviet Math. Dokl, Vol. 30.

Step 5: Verification

To apply the “min-max” principle, we need to confirm that

1. $f_{m,t} > 0$; and
2. $\sup_x \frac{\mathcal{L}_t f_{m,t}(x)}{f_{m,t}(x)} < 1$ (or $\inf_x \frac{\mathcal{L}_t g_{m,t}(x)}{g_{m,t}(x)} > 1$)

Fortunately, $f_{m,t}$ is a polynomial, so its derivative can be computed with arbitrary precision, this allows us to verify the first inequality.

To verify the second inequality, we differentiate

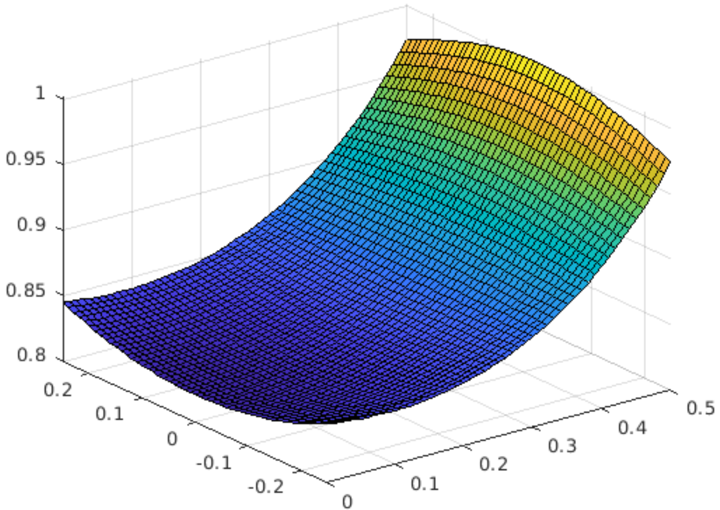
$$\left(\frac{\mathcal{L}_t f_{m,t}}{f_{m,t}}\right)' = \frac{(\mathcal{L}_t f_{m,t})' \cdot f_{m,t} - (f_{m,t})' \cdot \mathcal{L}_t f_{m,t}}{(f_{m,t})^2}$$

It turns out that

$$(\mathcal{L}_t f_{m,t})' \cdot f_{m,t} - (f_{m,t})' \cdot \mathcal{L}_t f_{m,t} \to 0 \text{ as } m \to \infty$$

exponentially fast.

A test function



The hard part

We need to compute the eigenvector of the matrix:

$$A_t(j,k) = [\mathcal{L}_t p_{j_1,j_2}](x_{k_1,k_2}) = \sum_{n=1}^{\infty} |T_n(x)'|^t \cdot p_{j_1,j_2}(T_n(x_{k_1,k_2}))$$

This requires evaluating the infinite sum to a high precision. Recall:

$$T_{2k+1} = f_1 \circ R \circ f_1^k \qquad T_{2k} = f_1 \circ R^2 \circ f_1^k$$

where R is rotation and f_1 is Moebius. Thus the elements of the sum depend on the parameter n analytically!

Contributions

- Exponential convergence of Chebyshev – Lagrange approximation in multiple dimensions, with explicit constants
- Adaptation of min-max idea for faster convergence (in comparison with binary subdivision)
- Effective and efficient estimates for terms in Euler–Maclaurin formula that allows to evaluate the infinite sum.