Kinematic Fast Dynamo Problem

Polina Vytnova

Warwick University

July 2014

The kinematic fast dynamo problem

Under certain simplifying assumptions, the system of magnetohydrodynamics may be reduced to a Navier-Stokes type equation.

The kinematic dynamo equations

$$\begin{cases} \dfrac{\partial B}{\partial t} = (B \cdot \nabla)v - (v \cdot \nabla)B + \varepsilon \Delta B; \\ \nabla \cdot v = \nabla \cdot B = 0. \end{cases}$$

- v is the (known) velocity field of a fluid filling a certain compact domain M;
- B is the (unknown) magnetic field;
- ε is a dimensionless parameter reflecting the magnetic diffusion through the boundary of M.

Problem (Main fast dynamo problem)

Does there exist a divergence-free velocity field v in a compact domain M tangent to the boundary, such that the energy of the magnetic field $B(t)$ grows exponentially in time for some initial field B_0 in the presence of small diffusion ($\varepsilon > 0$)?

This is a Cauchy problem. A case of special interest are stationary velocity fields in three-dimensional domains.

The provisional flow

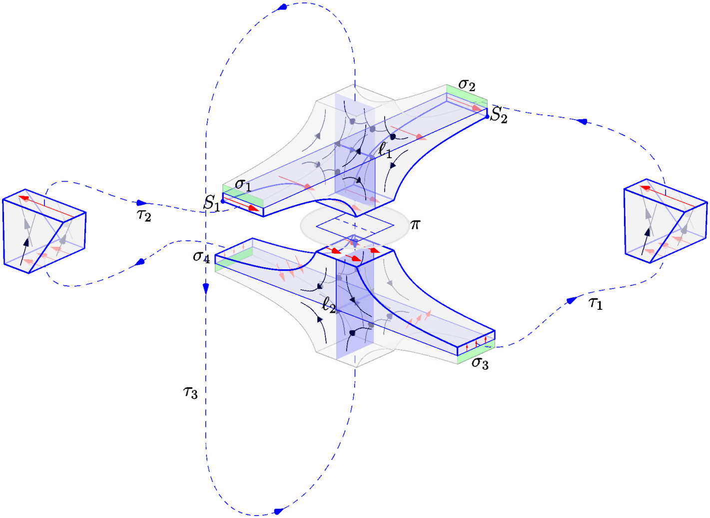
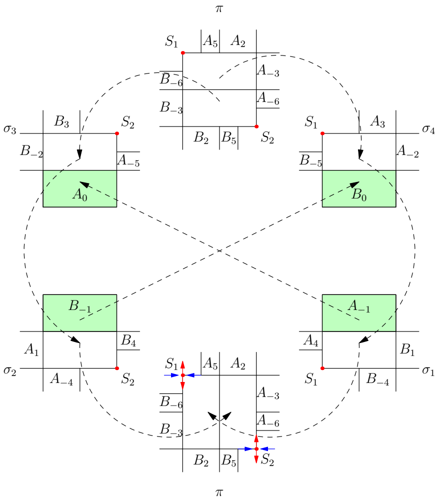


Figure: Dynamo manifold with the fluid flow (blue) and magnetic induction field (red). The labels S_1 and S_2 mark periodic saddle points. $\tau_{1,2,3,4}$ stand for manifolds equivalent to cylinders.

The Poincaré map



The map between the sections σ_1, σ_2, σ_3, σ_4 and π realised by the provisional flow. The points S_1 and S_2 are periodic saddles.

The first return map to the section π is an unfolded Baker's map.

The unfolded Baker's map plays the leading role.

From flows to diffeomorphisms

Lemma

The exponent of the Laplacian is the Weierstrass transform.

$$(\exp(\varepsilon\Delta)B)(z) = (W_\varepsilon B)(z) \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \frac{1}{(\sqrt{2\pi}\varepsilon)^d} \exp\Big(-\frac{|z-t|^2}{2\varepsilon^2}\Big) B(t)\mathrm{d}t$$

The Lemma gives a natural discretization of the dynamo equation, where the action of piecewise diffeomorphisms is used instead of the transport by a flow

$$B \mapsto (W_\varepsilon g_*)B, \qquad g \text{ is a piecewise diffeomorphism.}$$

Theorem (Main)

There exists a piecewise diffeomorphism $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that for some vector field B_0 in $\mathbb{R}^2$

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \frac{1}{n} \log \|(W_\varepsilon F_*)^n B_0\|_{\mathcal{L}_1} > 0.$$

The map F may be realised as the first return map of the provisional flow to the section π.

Noise instead of diffusion

Definition (Small random perturbations)

Given a map $F\colon \mathbb{R}^n \to \mathbb{R}^n$ we define *a natural extension* $\widehat{F}\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by $\widehat{F}(x,y) = F(x) + y$. To any sequence $\xi \in \ell_\infty(\mathbb{R}^n)$ we can associate *a small perturbation* F_ξ^m of the map F by

$$F_\xi^m \stackrel{\text{def}}{=} \widehat{F}_{\xi(m)} \circ \widehat{F}_{\xi(m-1)} \circ \ldots \circ \widehat{F}_{\xi(1)}.$$

(Also known as a skew product representation of a random dynamical system).

Lemma (Noise Lemma)

Let w_ε be the Gaussian kernel in $\mathbb{R}^k$ with isotropic variance ε. In the notations introduced above, for any vector field B in $\mathbb{R}^k$ and for any $m > 0$

$$(W_\varepsilon F_*)^m B(z) = \int_{\mathbb{R}^{k(m-1)}} w_\varepsilon(t_1) w_\varepsilon(t_2) \ldots w_\varepsilon(t_{m-1}) (W_\varepsilon F_{\bar{t}*}^m B)(z) \mathrm{d}\bar{t},$$

where $\bar{t} = (0, t_1, t_2, \ldots, t_{m-1}) \in \mathbb{R}^{km}$.

- The operator $(W_\varepsilon F_*)^n$ was hard to study.
- The operator $W_{\frac{\varepsilon}{2}} F_{\xi*}^m W_{\frac{\varepsilon}{2}}$, where $\xi \in \ell_\infty(\mathbb{R}^2)$, is easier and sufficient.

The operator to study

Main goal

To construct an invariant cone C for the operator $W_{\frac{\varepsilon}{2}} F^m_{\xi *} W_{\frac{\varepsilon}{2}}$ for arbitrary sufficiently large $m \gg 1$, for all $\|\xi\|_\infty \le m2^{-\alpha m}$ and $\varepsilon \le 2^{-\alpha m}$ for some $\alpha < 1$, in the space of essentially bounded vector fields with absolutely integrable components. The cone should satisfy

$$\left\| W_{\frac{\varepsilon}{2}} F^m_{\xi *} W_{\frac{\varepsilon}{2}} \; |_C \right\| \ge 2^m \cdot \mathrm{const}.$$

The bound is justified: $\left\| W_{\frac{\varepsilon}{2}} F^m_{*} W_{\frac{\varepsilon}{2}} \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\chi_\square \right\| \ge 2^{m-1}$ ($\xi = 0$, $\square := [-1;1]^2$).

If the diffeomorphism action causes the field to change direction rapidly; its energy cannot grow exponentially fast in the presence of diffusion.

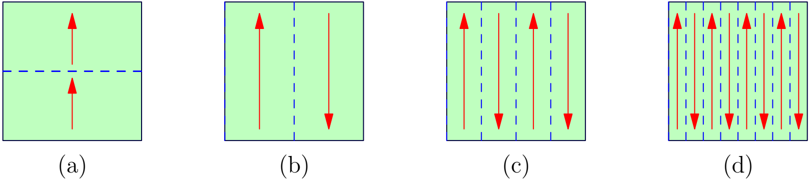


Figure: Evolution of the magnetic field (red) under iterations of the folded Baker's map. (a) initial vector field, (d) vector field after 3 iterations. Blue dashed lines mark discontinuities.

Strategy: key steps

1. Fix a large number $m \gg 1$ and a sequence $\|\xi\|_\infty \le 2^{-m\alpha}$.
2. Choose a norm: maximum of the weighted $\mathcal{L}_1$-norm and weighted supremum norm; the "weights" depend on m and ξ.
3. Introduce a sequence of *canonical partitions*, associated to a sequence of small perturbations ξ, a substitute for a Markov partition for m iterations.
4. Introduce a subspace of piecewise-constant vector fields $\mathfrak{X}_\Omega$, associated to a canonical partition $\Omega(m,\xi)$; and choose a *basis*.
5. Approximate the linear operator $F^{2m}_{\xi*}|_{\mathfrak{X}_{\Omega^1}}$, by a linear operator $\mathcal{A}(m,\xi)\colon \mathfrak{X}_{\Omega^1} \to \mathfrak{X}_{\Omega^2}$ (partitions Ω^1 and Ω^2 depend on ξ and m).
6. Construct a pair cones $C_1 \subset \mathfrak{X}_{\Omega^1}$ and $C_2 \subset \mathfrak{X}_{\Omega^2}$ such that $\mathcal{A}\colon C_1 \to C_2 \ll C_1$. (Both cones depend on ξ and m).
7. Get rid of the dependence on ξ: show that an image of the Weierstrass transform $W_{\frac{\varepsilon}{2}} v$ may be very well approximated by a piecewise-constant vector field, associated to a canonical partition Ω. This is due to $\varepsilon \gg \sup \mathrm{diam}(\Omega_j)$.
8. Construct an invariant cone for the operator $W_{\frac{\varepsilon}{2}} F^{2m}_{\xi*} W_{\frac{\varepsilon}{2}}$ in the space of piecewise-constant vector fields $\mathfrak{X}$.

A sketch of the matrix

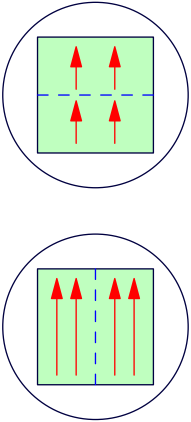
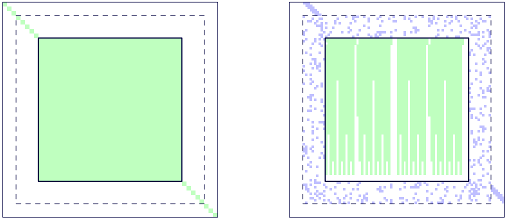


Figure: Baker's map and its action on a constant vector field, which is parallel to the expanding direction.

1. Fix a large number $m \gg 1$ and a sequence $\|\xi\|_\infty \le 2^{-m\alpha}$.
2. In the case $\xi = 0$: the Baker's map itself; take a Markov partition Ω for m iterations;
3. In general take a pair of *canonical partitions* $\Omega^1(\xi)$, $\Omega^2(\xi)$;
4. Define a linear operator $\mathcal{A}_\xi : \mathfrak{X}_{\Omega^1} \to \mathfrak{X}_{\Omega^2}$ by

$$\forall \nu \in \mathfrak{X}_{\Omega^1} : \quad \int_{\Omega^2_{ij}} F^{2m}_{\xi *} \nu = \int_{\Omega^2_{ij}} \mathcal{A}_\xi \nu$$



(a) (b)

Figure: A sketch of the central part of the matrix of $\mathcal{A}_\xi$ restricted to the subspace of vectors, parallel to the expanding direction of the Baker's map. (a) $\xi = 0$ and (b) $\xi \equiv \delta \neq 0$. Green: $a_{ij} = 1$; white: $a_{ij} = 0$.

Sample matrix properties

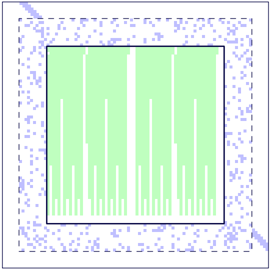


Figure: A central block of the matrix of operator $A(\alpha)$. The size of the internal square is $2^{m+1} \times 2^{m+1}$.

$X = \langle e_i \rangle$; $m \gg 1$; $A(\alpha)\colon X \to X$ linear:

1. $\sup |a_{ij}| \leq 2^{\gamma m}$ for some $0 < \gamma \leq 0.01$;
2. $\# \left\{ -2^m < i, j < 2^m \mid a_{ij} \neq 1 \right\} \leq 2^{\frac{7}{4} m}$;
3. $a_{ij} = 0$ whenever $|i - j| \geq m 2^{(1-\alpha)m}$ and $\max(|i|, |j|) > 2^m (m 2^{-\alpha m} + 1)$.

Norm on X:

$$\left\| \sum x_i e_i \right\| \stackrel{\text{def}}{=} \max \left(2^{-m} \sum |c_i|, 2^{-\frac{m}{4}} \sup |c_i| \right)$$

Cone in X:

$$C(r, X) \stackrel{\text{def}}{=} \left\{ \sum_{i=-2^m}^{2^m} d e_i + x \mid \sum_{i=-2^m}^{2^m} x_i = 0, \ \|x\| \leq r d \right\}$$

Theorem (A prelude to fast dynamo)

Let $\frac{3}{4} < \alpha < 1$. Then $A(\alpha)\colon C(1, X) \to C\left(2^{-\frac{m}{8}}, X\right)$, and $\|A(\alpha)|_{C(1,X)}\| \geq 2^{m-1}$.

Canonical partitions

Fix a large number $m \gg 1$ and a sequence $\|\xi\|_\infty \le 2^{-m\alpha}$.

Definition (Canonical partition)

To a small perturbation F_ξ^{2m} of the map F we associate a partition $\Omega(m,\xi)$ of $\mathbb{R}^2$ that satisfies the following conditions

1. The unit square $\square$ contains at most 2^{2m} and at least $2^{2(m-1)}$ elements of the partition. Interiors of the elements do not intersect the boundary of the square.
2. For any element Ω_{ij} of the partition Ω there exist two rectangles $Rec\left(\frac{1}{m}2^{-m}, \frac{1}{m}2^{-m}\right) \subseteq \Omega_{ij} \subseteq Rec\left(2^{1-m}, 2^{1-m}\right)$.
3. Any rectangle $R \subset \square$ such that $F_\xi^k(R) \subset \square$ for all $0 \le k \le 2m$ is contained in a single element of the partition.

Theorem

Canonical partition does exist for any sequence $\xi \in \ell_\infty(\mathbb{R}^2)$ *with* $\|\xi\|_\infty \le m2^{-\alpha m}$.

Mixed norm

- Keep a large number $m \gg 1$ and a sequence $\|\xi\|_\infty \leq 2^{-m\alpha}$ fixed.
- Given a canonical partition $\Omega(m, \xi)$ of $\mathbb{R}^2$, we define an associated weighted $(\Omega, \mathcal{L}_1)$-norm of a vector field v in $\mathbb{R}^2$ by

$$\|v\|_{\Omega,\mathcal{L}_1} \stackrel{\text{def}}{=} \sum_{ij} \frac{2^{-m}}{|\pi_y(\Omega_{ij})|} \int_{\Omega_{ij}} |v|;$$

where π_y is an orthogonal projection on the expanding direction of the Baker's map.

Definition (Mixed Norm)

We introduce a new norm, associated to the partition Ω, combining weighted $(\Omega, \mathcal{L}_1)$ and supremum norms:

$$\|v\|_\Omega \stackrel{\text{def}}{=} \max\Big(\|v\|_{\Omega,\mathcal{L}_1}, 2^{-m/4} \sup |v|\Big).$$

Main "feature"

We estimate the growth of the $(\Omega, \mathcal{L}_1)$-norm via the supremum norm and vice versa.

Approximating the operator $F_{\xi *}^{2m}$

Keep a large number $m \gg 1$ and a sequence $\|\xi\|_\infty \le 2^{-m\alpha}$ fixed. We may split $X_\Omega = V_\Omega^s \oplus V_\Omega^u$; where V_Ω^s is a span of vectors, parallel to the contracting direction of $F_{\xi *}^{2m}$ and V_Ω^u is a span of vectors parallel to the expanding direction of $F_{\xi *}^{2m}$.

1. Canonical partitions Ω^1 and Ω^2 for ξ and $\sigma^{2m}\xi$, respectively.
2. Linear operator $\mathcal{A}(m,\xi)\colon \mathfrak{X}_{\Omega^1} \to \mathfrak{X}_{\Omega^2}$

$$\forall \nu \in \mathfrak{X}_{\Omega^1} \qquad \int_{\Omega^2_{kl}} F_{\xi *}^{2m}\nu = \int_{\Omega^2_{kl}} \mathcal{A}\nu.$$

3. The operators $W_\delta \mathcal{A}$ and $W_\delta F_{\xi *}^{2m}$ are close on $\mathfrak{X}_{\Omega^1}$. Namely, for $2^{-m} \ll \delta \ll 1$ and $\|\xi\| \le \delta$:

$$\|W_\delta(F_{\xi *}^{2m} - \mathcal{A})\nu\|_{\Omega^2} \le \frac{8}{2^m \delta}\left(\|\mathcal{A}\nu\|_{\Omega^2} + \|F_{\xi *}^{2m}\nu\|_{\Omega^2}\right).$$

4. Decomposition $\mathcal{A} = SS \oplus US \oplus SU \oplus UU$,
   - The operators $SS\colon V^s_{\Omega^1} \to V^s_{\Omega^2}$, $US\colon V^u_{\Omega^1} \to V^s_{\Omega^2}$, and $SU\colon V^s_{\Omega^1} \to V^u_{\Omega^2}$ are small;
   - The operator $UU\colon V^u_{\Omega^1} \to V^u_{\Omega^2}$ is the most important and is responsible for *the exponential growth* of a suitably chosen vector field ν under iterations of $\mathcal{A}$.

Matrix of the operator $UU(\xi, m)$

Definition

We define a basis of the subspace of piecewise constant vector fields $\mathfrak{X}_\Omega$ by

$$\chi^s_{\Omega_{ij}} \stackrel{\text{def}}{=} \frac{1}{|\pi_x(\Omega_{ij})|}\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\chi_{\Omega_{ij}}; \qquad \chi^u_{\Omega_{ij}} \stackrel{\text{def}}{=} \frac{1}{|\pi_x(\Omega_{ij})|}\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\chi_{\Omega_{ij}}.$$

Then the condition

$$\forall \nu \in V^u_{\Omega^1}: \qquad \int_{\Omega^2_{kl}} F^{2m}_{\xi *}\nu = \int_{\Omega^2_{kl}} UU\nu$$

allows us to prove the following estimates.

Theorem (Properties of the matrix $UU(\xi, m)$)

Let $\delta = 2^{-m\alpha}$, $\frac{15}{16} \le \alpha \le 1$. *Consider a sequence* $\xi \in \ell_\infty(\mathbb{R}^2)$ *with* $\|\xi\|_\infty \le \delta$. *Then*

1. *there exists* $0 < \gamma_1 < 0.01$ *such that* $\sup|UU^{kl}_{ij}| \le 2^{\gamma_1 m}$.
2. $UU = UU^B + UU^G$, *where*
   - UU^G *satisfies:* $\#\left\{(ij, kl) \in \square \times \square \mid (UU^G)^{kl}_{ij} \neq 1\right\} \le 2^{\frac{9}{2}}\delta$;
   - UU^B *is small:* $\sum_{\square\times\square} (UU^B)^{kl}_{ij} \le 2^m \cdot 8m\delta$.

A cone in the space $\mathfrak{X}_{\Omega^1}$.

Corollary

The matrix of the operator $UU(\xi, m)$ has a pattern of the "sample matrix" for any m sufficiently large and for any $\|\xi\|_\infty \leq 2^{-m\alpha}$ with $\frac{15}{16} < \alpha < 1$.

Definition (Cone in vector fields)

We define a cone in the space of piecewise-constant vector fields in $\mathbb{R}^2$ centered at the eigenfunction $v_0 = \binom{0}{1}\chi_\square$ of the operator F_*:

$$C(r,\Omega) \stackrel{\text{def}}{=} \left\{\nu = d\binom{0}{1}\chi_\square + \psi,\ \|\psi\|_\Omega \leq rd,\ \sum_\square \psi_y^{ij} = 0\right\}$$

Corollary

Let $m \gg 1$ and a sequence $\|\xi\|_\infty \leq 2^{-m\alpha}$ be fixed. Let Ω^1 and Ω^2 be a pair of canonical partitions associated to the sequence ξ, as above. Define a number $\beta = -\frac{3}{16} + \gamma_1$. Then $\mathcal{A}(\xi, m)\colon C\left(1, \Omega^1\right) \to C\left(2^{-\beta m}, \Omega^2\right)$ and $\|\mathcal{A}\,|_{C(1,\Omega^1)}\,\| \geq 2^{2m-1}$.

Discretization operator

We define the discretization operator $D_\Omega \colon \mathfrak{X} \to \mathfrak{X}_\Omega$ by taking averages by

$$(D_\Omega v)(z) \stackrel{\text{def}}{=} \sum_{ij} \frac{1}{|\Omega_{ij}|} \Big(\int_{\Omega_{ij}} v \Big) \chi_{\Omega_{ij}}(z)$$

Now we can get rid of the dependence of partitions and norm on ξ.

Lemma

There exists $1-\alpha < \gamma_3 < 1-\alpha+\gamma_1$ such that for any $\nu \in \mathfrak{X}$ and for any two canonical partitions Ω^1 and Ω^2

$$\|W_\delta \nu - D_\Omega W_\delta \nu\|_2 \le 2^{-\gamma_3 m} \|\nu\|_1;$$

$$\|W_\delta \chi_\square - D_\Omega W_\delta \chi_\square\|_2 \le 2^{-\frac{m}{4}}.$$

Using this two inequalities, the approximation

$$\|W_\delta (F^{2m}_{\xi *} - \mathcal{A})\nu\|_2 \le \frac{8}{2^m \delta} \left(\|\mathcal{A}\nu\|_2 + \|F^{2m}_{\xi *}\nu\|_2 \right).$$

and Prelude Theorem, we construct an invariant cone C for the operator $W_\delta F^{2m}_{\xi *} W_\delta$, with $\xi \le -2\delta \log \delta$; such that $\left\| W_\delta F^{2m}_{\xi *} W_\delta \mid_C \right\| \ge 2^{2m-2}$. We combine this result with Noise Lemma and get Main Theorem.