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Master thesis

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On nonarchemedean dynamical systems

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1 Introduction.

The main problem that initiated the development of dynamical systems theory from its origins is the problem of turbulence: how a deterministic nature of a dynamical system can be compatible with its apparently chaotic behavior. This problem was studied by the precursors and founding fathers of the dynamical systems theory: A. Poincare, H. Hopf, A. Kolmogorov, V. Arnold, S. Smale... Currently this is one of the principal challenges on the cross-road between mathematics, physics and computer science. Dynamical systems theory heavily uses methods and tools from topology, differential geometry, probability, functional analysis and other branches of mathematics.

Complex dynamics, which was initiated by Fatou and Julia in the late 1910s but which did not draw substantial attention until 1980s, focuses mainly on the iteration of rational functions. In the end of 1990s, number theorists began to study such iterations as well, noting parallels to certain aspects of the theory of elliptic curves. There exists a dictionnary connecting two areas, namely, rational and integral points of varieties are correspondig to the rational and integral points in orbits, and torsion points on abelian varieties is corresponding to periodic and preperiodic points of rational maps. Number-theoretics dynamics has began to emerge as field in its own right, especially concerning the rationality properties of periodic points. This new branch which amalgamates two areas of mathematics, dynamical systems and number theory is calling arithmetic dynamics. Classically, discrete dynamics refers to the study of the iteration of self-maps of the complex plane or real line. Arithmetic dynamics [14] is the study of the number-theoretic properties of integer, rational, p-adic, and/or algebraic points under repeated application of a polynomial or rational function. A fundamental goal is to describe arithmetic properties in terms of underlying geometric structures.

Global arithmetic dynamics refers to the study of analogues of classical Diophantine geometry in the setting of discrete dynamical systems, while local arithmetic dynamics, also called p-adic or non-archimedean dynamics, is an analogue of classical dynamics in which one replaces the complex numbers $\mathbb{C}$ by a p-adic field such as $\mathbb{Q}_p$ or $\mathbb{C}_p$ and studies chaotic behavior and the Fatou and Julia sets.

The first part of this work, dynamics on nonarchimedean spaces, is devoted to dynamics of quadratic maps and contains the results belonging to the classic area. It is shown that under some conditions dynamics over Julia set is conjugated with Bernulli's shift and it's topological entropy is calculated.

In the second part a general concept of dynamical system with non-archimedean time is suggested. It is illustrated by a certain limit on the dynamics on the sets of 2^n-periodic points of real quadratic maps.

This part is devoted to the nonarchimedean *time.* We'll show that at least in one typical example the same dynamical system admits two compatible descriptions: a classical one with discrete time and the one with 2-adic time. Remarkably, the 2-adic version turns out to be considerably simpler.

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2 Dynamics on nonarchimedean spaces

2.1 Introduction

Consider a general discrete dynamical system on a *countable* set (=*phase space*). Formally it is a *deterministic* model of motion (we know *everything* about the orbit of any point) and there seems to be no context for the chaotic considerations.

However, if we are going to study and *describe* the orbits, we need some additional structures on the phase space.

First of all, we need some *language* to specify the points of the phase space. It can be formalized as a *recursive* structure, i.e. the distinguished class of numbering (= bijections with natural numbers) up to recursive renumberings.

For the most dynamical systems the *amount of information* needed to specify a point (it can be formalized in terms of *Kolmogorov complexity*) generically grows along the orbit. In most cases not all this information is valuable for describing the system qualitatively; e.g., if an orbit "goes to infinity" (in some sense) we might be not interested in the details of the positions of the points that are terribly far away.

Thus we impose some *topologies* on the phase space in order to be able to describe the orbits approximately. We emphasize the specific feature of the *nonclassical* discrete dynamics: it is not assumed that the phase space carries some distinguished topology; we rather consider the *set* of natural topologies. The product of the completions of the phase space with respect to all these topologies is provided by a suitable *product topology*; the diagonal embedding of the phase space into this product should induce its true *discrete* topology.

The adelic dynamics provides a perfect framework for this approach, the phase space being global number fields; the topologies are defined by their non-archimedean valuations.

In this part we consider the simplest non-linear model of this kind — the iterations of quadratic maps. Conceptually our main result is the theorem 5, according to which the system demonstrates the chaotic behavior only over the finite number of valuations — precisely over those ones over which the quadratic map is in some sense *averagely expanding* in the fixed points.

The results of the work generalize the earlier results of two of the authors [8] and [3]. The similar results over p-adic fields with $p \neq 2$ were obtained considerably earlier in [10].

This part is organized as follows. Sections 1 and 2 are devoted to certain elementary properties of the quadratic maps over non-archimedean fields. Sections 3 and 4 are technical: under some assumptions the preimages of 0 and of a “large disc” around it are described. In the section 5 the filled Julia sets for all the quadratic maps over all the non-archimedean local fields are described. In the section 6 under the assumptions of the section 3 the isomorphism between the quadratic dynamics on the filled Julia set and some sequence dynamics (Bernoulli shift on the left-infinite sequences) is established. In the section 7 the main results are formulated; the 2-adic case is considered separately. In the section 8 some adelic interpretation of our results is suggested.

2.1.1 Notations

Some of the notations we use are not quite standard.

For a map $T\colon X \mapsto X$ and for $n \in \mathbb{N}$ we denote by $T^{n\circ}$ its nth iterate and by $T^{-n\circ}$ its n inverse iterate (possibly multivalued). By $T^{\mathbb{N}\circ}(x)$ we denote the T- orbit of $x \in X$; finally, for $Y \subseteq X$ denote by $T^{-\mathbb{N}\circ}Y\colon = \bigcup\limits_{n\in\mathbb{N}} T^{-n\circ}Y$ and $T^{-\infty}Y\colon = \bigcap\limits_{n\in\mathbb{N}} T^{-n\circ}Y$.

When X is a metric space denote by $\mathcal{FJ}(T)$ the *filled Julia set*, i.e. the set of elements of X with bounded T- orbits.

For an alphabet (=finite set of characters) A denote by $A^{-\mathbb{N}} = \{\dots a_2 a_1 a_0\}$ (where $a_0, a_1, a_2 \dots \in A$) the set of sequences of elements of A, infinite *to the left.* For a finite sequence ε we denote its length by $|\varepsilon|$.

For a field $\Bbbk$ denote its set of squares by $\Bbbk^{2\cdot} := \{x^2 \mid x \in \Bbbk\}$.

For a field $\Bbbk$ with the norm $\|\cdot\|$ for $a \in \Bbbk$ and $r \in \mathbb{R}_{>0}$ denote the open and closed discs by

$$D(a,r)\colon = \{x \in \Bbbk \mid \|x-a\| < r\}$$
$$D[a,r]\colon = \{x \in \Bbbk \mid \|x-a\| \leq r\}$$

2.2 Canonical forms of quadratic maps

We fix a field $\Bbbk$ with $\operatorname{char}\Bbbk \neq 2$ and consider the general quadratic map

$$q\colon \mathbb{A}(\Bbbk) \mapsto \mathbb{A}(\Bbbk)$$

defined by

$$q(x) = Ax^2 + Bx + C$$

with $A, B, C \in \Bbbk$ and $A \neq 0$.

The dynamical properties of the above q depend only on the similarity class of q; it means that we consider the action of the group of affine transformation of argument

$$x \mapsto L(x) \colon = mx + n \text{ with } m \in \Bbbk^{\bullet},\ n \in \Bbbk$$

on the set of quadratic transformations. This action is defined by

$$L \bullet q = L \circ q \circ L^{-1\circ};$$

q and thus defined $L \bullet q$ are called *similar*. The problem is to find the simplest (and traditional) representatives of similarity classes of the quadratic map.

It's easy to see that any qudaratic map is linear conjugated to

$$x \mapsto x^2 + c$$

This form is universal, and we are going to stick to it in this part. One checks that

$$c \colon = AC - \frac{B^2}{4} + \frac{B}{2} = \frac{1}{4} \prod_{x \in \mathrm{Fix}(q)} q'(x)$$

is always in $\Bbbk$. We'll see that in the case when $\Bbbk$ is equipped with a (usually non-archimedean) metric the dynamical properties of q depend drastically on the norm of c; in particular, q generates the chaotic behavior iff $\|c\| > 1$, i.e., when q is *averagely expanding in the fixed points.* We are not aware of any reasonable generalization of this observation.

2.3 Behavior of norms along the orbits

Every $x \in \Bbbk$ defines a sequence $\|T_c^{n\circ}(x)\|$. In most cases the behavior of the norm is quite simple.

Theorem 1. *According to the values of $\|c\|$ and $\|x\|$ the following statements hold:*

	$\|c\| < 1$	$\|c\| = 1$	$\|c\| > 1$
$\|x\| < 1$	$\lim\limits_{n\to\infty} \|T_c^{n\circ}(x)\| = \|c\|,$	*No general statement*	$\lim\limits_{n\to\infty} \|T_c^{n\circ}(x)\| = \infty$
$\|x\| = 1$	$\|T_c^{n\circ}(x)\| \equiv 1$	*No general statement*	$\lim\limits_{n\to\infty} \|T_c^{n\circ}(x)\| = \infty$
$\|x\| > 1$	$\lim\limits_{n\to\infty} \|T_c^{n\circ}(x)\| = \infty$	$\lim\limits_{n\to\infty} \|T_c^{n\circ}(x)\| = \infty$	$\|T_c^{n\circ}\|$ *is either constant or* $\to \infty$

Proof. All the statements about existing limits and about the norms $\|T_c^{n\circ}\|$ being constant are obvious. In the case $\|c\| = \|x\| = 1$ the $\lim\limits_{n\to\infty} \|T_c^{n\circ}(x)\|$ can exist. E.g., in any field where $\|2\| = 1$, $x = -1$ is a fixed point of $x \mapsto x^2 - 2$. But it is possible as well that $\|c\| = \|x\| = 1$, but $\lim\limits_{n\to\infty} \|T_c^{n\circ}(x)\|$ does not exist. Over any field the map

$$x \mapsto x^2 - 1$$

provides a cycle that gives a sequence of norms $0, 1, 0, 1, \dots$

In the case $\|c\| > 1, \|x\| > 1$ the trajectories generally tend to ∞. E.g., for $\Bbbk = \mathbb{Q}_3$ and $x = c = \frac{1}{3}$ we have the orbit

$$\frac{1}{3} \to \frac{4}{9} \to \frac{43}{81} \to \dots$$

with the sequence of norms $3, 9, 81, \dots$ But in some special cases (which are the most interesting from the viewpoint of the present work) the norms along the orbits are constant. E.g., over $\Bbbk = \mathbb{Q}_5$ the map

$$x \to x^2 - \frac{1}{25}$$

has two fixed points $\frac{1}{2} \pm \frac{\sqrt{21}}{16} \in \mathbb{Q}_5$ of the norm 5. ■

2.4 The preorbit of 0.

We fix the triple $\Bbbk \supset \mathcal{O} \supset \mathcal{M}$ consisting of a local field, its valuation ring and its maximal ideal; let $p = \operatorname{char}(\mathcal{O}/\mathcal{M})$. We fix the non-archimedean norm $\|\cdot\|$ on $\Bbbk$, normalized by the condition $\|p\| = \frac{1}{p}$ and the element $c \in \Bbbk \setminus \mathcal{O}$ (i.e. $\|c\| > 1$; this is the only case we'll need). Our goal is to describe the set $T_c^{-\mathbb{N}\circ}(0)$.

Informally,

$$T_c^{-1\circ}(0) = \{x \mid x^2 + c = 0\} = \pm\sqrt{-c},$$
$$T_c^{-2\circ}(0) = \{x \mid x^2 + c \in T_c^{-1\circ}(0)\} = \{x \mid x^2 = -c \pm \sqrt{-c}\} = \pm\sqrt{-c \pm \sqrt{-c}}$$

and so on. We should is to give the precise sense to the expressions with nested roots

$$\pm\sqrt{\dots \pm \sqrt{-c \pm \sqrt{-c \pm \sqrt{-c}}}}$$

(continued recursively to the *left*).

Note that if the roots do not belong to the corresponding fields our notations would be just the convenient names of the elements of their quadratic extensions; however, we are most interested in the case where these roots belong to $\Bbbk$ and we are going rather to provide for our nested roots certain *analytic* sense.

Proposition. *The following statements are equivalent:*

(i) $-c \in \Bbbk^{2\cdot}$;

(ii) $T_c^{-1\circ}(0)$ *is non-empty*

(iii) For any positive natural n *the set* $T_c^{-n\circ}(0)$ *is non-empty and, moreover,*

$$\#\{T_c^{-n\circ}(0)\} = 2^n.$$

Proof. Implications $(i) \iff (ii) \impliedby (iii)$ are trivial; concentrate on $(i) \implies (iii)$. The assumption (i) implies $c = -a^2$ for some $a \in \mathbb{k}$ with $\|a\| > 1$. In fact, we have *arbitrarily* attributed the signs to $\pm\sqrt{-c}$. Further,

$$\pm\sqrt{-c \pm \sqrt{-c}} = \pm\sqrt{a^2 \pm a} = \pm a(1 \pm \frac{1}{a})^{\frac{1}{2}} :=$$
$$= \pm a\Big[1 + \frac{\frac{1}{2}}{1!}\Big(\pm\frac{1}{a}\Big) + \frac{\frac{1}{2}\big(\frac{1}{2}-1\big)}{2!}\Big(\pm\frac{1}{a}\Big)^2 + \frac{\frac{1}{2}\big(\frac{1}{2}-1\big)\big(\frac{1}{2}-2\big)}{3!}\Big(\pm\frac{1}{a}\Big)^3 + \dots\Big],$$

and this series converges p-adically (we use $p \neq 2$); see lemma 1 below.

The longer expressions with nested roots are also defined by the convergent series; see the next subsection. A similar description in terms of *dichotomic variables* can be found in [10]. ■

Notations of the elements of $T_c^{-\mathbb{N}\circ}(0)$. We assume $c = -a^2$ for all $a \in \mathbb{k}$ and introduce recursively the numbers $b_\epsilon \in \mathbb{k}$ labeled by the strings ϵ of $+$'s and -'s

$$b: = 0,$$
$$b_\pm: = \pm a,$$
$$\dots\dots\dots$$
$$b_{\pm\varepsilon}: = \{ \text{ solution of } x^2 - a^2 = b_\varepsilon \}.$$

In order to choose the signs for $b_{\pm\varepsilon}$ we introduce recursively the following Laurent series $B_\varepsilon \in \mathbb{Q}\big(\big(\frac{1}{A}\big)\big)$:

$$B_\pm: = \pm A,$$

$$B_{\pm\varepsilon}: = \pm\sqrt{A^2 + B_\varepsilon}: = \pm A\Big(1 + \frac{B_\varepsilon}{A^2}\Big)^{\frac{1}{2}} = \pm A\Big[1 + \frac{\frac{1}{2}}{1!}\frac{B_\varepsilon}{A^2} + \frac{\frac{1}{2}\big(\frac{1}{2}-1\big)}{2!}\Big(\frac{B_\varepsilon}{A^2}\Big)^2 + \dots\Big],$$

and it makes sense since one proves inductively that

$$B_\varepsilon \in \pm A + \mathbb{Z}\Big[\frac{1}{2}\Big]\Big[\Big[\frac{1}{A}\Big]\Big]$$

We check that after substituting the free variable A by $a \in \mathbb{k}$ all the B_ε' s converge in $\|\cdot\|$-norm and hence define $b_\varepsilon \in \mathbb{k}$.

2.5 Large disc and the inverse dynamics on it

We keep the same notations, including $c = -a^2$. Besides, for any $S \subset \mathbb{k}$ we denote by $\sqrt{S}$ the set $\{x \in \mathbb{k} \mid x^2 \in S\}$.

Lemma 1 (Effective openness of the set of squares.)**.** *Let $x_0 \in \mathbb{k}^{2\cdot}$. Then $B(x_0, \|x_0\|) \subset \mathbb{k}^{2\cdot}$.*

Proof. Let $y \in \mathbb{k}$ be such that $y^2 = x_0$. By Taylor formula for any x with $\|x\| < \|x_0\|$

$$(y^2 + x)^{1/2} = y\Big(1 + \frac{x}{y^2}\Big)^{1/2} = y\sum_{n=0}^{\infty}\frac{1(-1)(-3)\dots(3-2n)}{2^n n!}\cdot\Big(\frac{x}{y^2}\Big)^n$$

In order to prove the convergence of this series estimate the norm of its general term. Using

$$-\log_p \|n!\|_p = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \ldots \sim \frac{n}{p} \cdot \frac{1}{1-1/p} = \frac{n}{p-1}$$

We see that $\sqrt[n]{\|n!\|_p} \sim p^{-\frac{1}{(p-1)}}$, $\sqrt[n]{\|(2n-1)!!\|_p} = \sqrt[n]{\left\|\frac{(2n)!}{2^n n!}\right\|_p} \sim p^{-\frac{1}{(p-1)}}$. Then nth root of general term satisfies

$$\sqrt[n]{\left\| y\frac{(-1)(-3)\ldots(3-2n)}{2^n n!} \cdot \left(\frac{x}{y^2}\right)^n \right\|} = \sqrt[n]{\left\| y\frac{(2n-3)!!}{2^n n!}\right\|} \cdot \left\|\frac{x}{y^2}\right\| \sim \sqrt[n]{\left\|\frac{(2n-1)!!}{n!}\right\|_p} \left\|\frac{x}{x_0}\right\| < 1$$

■

By definition, for all $\varepsilon \in \bigsqcup_{n=0}^{\infty} \{\pm\}^{\{-n\ldots 0\}}$

$$D_\varepsilon \colon = D\Big[b_\varepsilon; \frac{1}{\|a\|^{|\varepsilon|-1}}\Big].$$

In particular, the one marked by the empty word is

$$D = D[0, \|a\|].$$

Theorem 2. *For any $n \in \mathbb{N}$*

$$T_{-a^2}^{-n\circ}(D) = \bigsqcup_{|\varepsilon|=n} D_\varepsilon.$$

Lemma 2. *Let $a \in \Bbbk$ and $r \in \mathbb{R}_{>0}$ satisfy $\|a\| > 1$ and $D[a^2, r^2] \subset \Bbbk^{2\cdot}$. Then*

$$\sqrt{D[a^2, r^2]} = D\left[a, \frac{r^2}{\|a\|}\right] \sqcup D\left[-a, \frac{r^2}{\|a\|}\right]$$

Proof. First of all note that $\|a\| > r$, since $D[a^2, r^2] \subset \Bbbk^{2\cdot}$.

We are going to show that $\sqrt{D(a^2, r^2)} \supseteq D\big[a, \frac{r^2}{\|a\|}\big] \sqcup D\big[-a, \frac{r^2}{\|a\|}\big]$. Let $x \in D\big[a, \frac{r^2}{\|a\|}\big] \sqcup D\big[-a, \frac{r^2}{\|a\|}\big]$, then $\|x\| = \|a\|$, as $\|x - a\| < \|a\|$ or $\|x + a\| < \|a\|$; since for at least one of the choices of the sign $\|x \mp a\| = \max(\|x\|, \|a\|) = \|a\|$. Then $\|x \pm a\| < \|a\|$, and

$$\|x^2 - a^2\| = \|x \mp a\| \cdot \|x \pm a\| \le \frac{r^2}{\|a\|} \cdot \|a\| \le r^2.$$

Hence $x^2 \in D[a^2, r^2]$.

Now show that $\sqrt{D[a^2, r^2]} \subseteq D\big[a, \frac{r^2}{\|a\|}\big] \sqcup D\big[-a, \frac{r^2}{\|a\|}\big]$. Let $x \in \sqrt{D[a^2, r^2]}$, then (as in the previous case), $\|x\| = \|a\|$. Therefore $\|x \mp a\| = \|a\|$. Hence $\frac{\|a^2\|}{\|a^2 - x^2\|} = \frac{\|a\|}{\|x \pm a\|}$. Therefore $\|a \pm x\| = \frac{\|a^2 - x^2\|}{\|a\|} \le \frac{r^2}{\|a\|}$. So $x \in D[a, \frac{r^2}{\|a\|}] \sqcup D\big[-a, \frac{r^2}{\|a\|}\big]$. ■

Now we prove the theorem 2 by the induction in n. It follows from the effective openness of $\Bbbk^{2\cdot}$ that the disc $D[a^2, \|a\|]$ belongs to $\Bbbk^{2\cdot}$. Therefore by lemma 2

$$T_{-a^2}^{-1\circ} D[\,0, \|a\|\,] = \sqrt{D[a^2, \|a\|\,]} = D[a, 1] \sqcup D[-a, 1] = \bigsqcup_{|\varepsilon|=1} D_\varepsilon.$$

Since $\pm a \in D[\,0, \|a\|\,]$, we have

$$T^{-1\circ}_{-a^2} D[\,0, \|a\|\,] \subset D[\,0, \|a\|\,].$$

So for any n

$$T^{-n\circ}_{-a^2} D[\,0, \|a\|\,] = \bigsqcup_{|\varepsilon|=n} D_\varepsilon \subset D[\,0, \|a\|\,],$$

and the lemma 2 is applicable to every disk it is used for. The theorem 2 is proved.

Corollary 1.

$$T^{-\infty}_{-a^2}(D) = \bigcap_{n=0}^{\infty} \bigsqcup_{|\varepsilon|=n} D_\varepsilon$$

2.6 The filled Julia sets

Keep the notations of the previous section (with the exception of c that now is arbitrary).

Theorem 3. *If $\|c\| \le 1$, then $\mathcal{FJ}(T_c) = \mathcal{O} = D[0,1]$. If $\|c\| > 1$, then*

(a) if $-c \notin \Bbbk^{2\cdot}$, then $\mathcal{FJ}(T_c) = \emptyset$;

(b) if $-c \in \Bbbk^{2\cdot}$, i.e. $c = -a^2$ for some $a \in \Bbbk$, then

$$\mathcal{FJ}(T_{-a^2}) = T^{-\infty} D[0, \|a\|].$$

Proof. The statement in the case $\|c\| \le 1$ follows from the properties of the norm sequence for $T^{n\circ}(x)$, see section 2.

In the case $\|c\| > 1$ we see that if $\|x\| > \sqrt{\|c\|}$, then $\|T^{n\circ}(x)\| = \|x\|^{2^n} \to \infty$ and if $\|x\| < \sqrt{\|c\|}$, then $\|T^{n\circ}(x)\| = \|c\|^{2^{n-1}} \to \infty$. Hence the $\mathcal{FJ}$ lies on the circle defined by $\|x\| = \sqrt{\|c\|}$.

Consider the case *(a)*. The assumption $-c \notin \Bbbk^{2\cdot}$ for any x satisfying $\|x\| = \sqrt{\|c\|}$ implies $\|x^2 + c\| \ge \|c\|$. Indeed, if $\|x^2 + c\| < \|c\|$, then $-c \in D(x^2, \|x^2\|) \subset \Bbbk^{2\cdot}$ by the effective openness of squares. Hence $\|T^{n\circ}(x)\| \ge \|c\|^{2^{n-1}} \to \infty$.

In the case *(b)* we just use our construction of indexed discs:

$$\mathcal{FJ} \subset D = D[0, \|a\|\,].$$

Then $\mathcal{FJ} \subseteq T^{-n\circ}(D) = \bigsqcup_{|\varepsilon|=n} D_\varepsilon$, so $\mathcal{FJ} \subseteq \bigcap_{n=0}^{\infty} T^{-n\circ}(D) = T^{-\infty} D[0, \|a\|\,]$

The opposite inclusion $\mathcal{FJ} \supseteq T^{-\infty} D[0, \|a\|\,]$ is obvious. ∎

2.7 Isomorphism with the sequence dynamics

Keep the notations of the section 4. Consider the space $\{\pm\}^{-\mathbb{N}}\colon = \{\ldots \varepsilon_2, \varepsilon_1, \varepsilon_0 \mid \varepsilon_n \in \{+,-\}\}$ of sequences of pluses and minuses infinite *to the left* endowed with Tikhonov topology. Denote by

$$\sigma\colon \{\pm\}^{-\mathbb{N}} \mapsto \{\pm\}^{-\mathbb{N}}\colon \ \ldots \varepsilon_2 \varepsilon_1 \varepsilon_0 \mapsto \ldots \varepsilon_3 \varepsilon_2 \varepsilon_1$$

the *Bernoulli shift.*

Theorem 4. *For any a satisfying $\|a\| > 1$ there is an isomorphism of dynamical systems (i.e. compacts with continuous endomorphisms)*

$$(\mathscr{J}(T_{-a^2}), T_{-a^2}) \simeq (\{\pm\}^{-\mathbb{N}}, \sigma).$$

Proof. For any $x \in \mathscr{J}(T_{-a^2})$ there exists a unique sequence of embedded discs.

$$D_{\varepsilon_0\varepsilon_1\varepsilon_2} \subset D_{\varepsilon_0\varepsilon_1} \subset D_{\varepsilon_0} \subset D$$

such that $\{x\} = \ldots \cap D_{\varepsilon_0\varepsilon_1} \cap D_{\varepsilon_0} \cap D$ and $\{T(x)\} = \ldots \cap D_{\varepsilon_1} \cap D \cap T(D)$. This construction defines

$$I\colon \mathscr{J}(T_{-a^2}) \mapsto \{\pm\}^{\mathbb{N}}\colon x \mapsto \ldots \varepsilon_2\varepsilon_1\varepsilon_0,$$

and it is easy to check that I is a homeomorphism satisfying $I \circ T_{-a^2} = \sigma \circ I$. ■

2.8 Chaotic properties of quadratic maps

Restore the notations $\Bbbk \supset \mathcal{O} \supset \mathcal{M}$ (a local field, its valuation ring and its maximal ideal); $p := \mathrm{char}(\mathcal{O}/\mathcal{M})$. Extend the polynomial maps we consider from $\mathbb{A}(\Bbbk)$ to the projective line $\mathbf{P}^1(\Bbbk)$, sending infinity to infinity.

Here are the main results of the part.

Theorem 5. *If $p \neq 2$, then the map*

$$T_c\colon\ \mathbf{P}^1(\Bbbk) \to \mathbf{P}^1(\Bbbk)\colon\ x \mapsto x^2 + c$$

has positive topological entropy iff $\|c\| > 1$ and $-c \in \Bbbk^{2\cdot}$.

Proof. Follows from the theorem 4 and the results of [12] and [13]. See details in [8]. ■

Theorem 6. *If $p = 2$, then the map*

$$\mathbf{P}^1(\Bbbk) \to \mathbf{P}^1(\Bbbk)\colon\ x \mapsto x^2 + c$$

has positive topological entropy iff $\|4c\| > 1$ and $(1 - 4c) \in \Bbbk^{2\cdot}$.

Proof. We formulate and outline the proofs of the analogues of our main statements for $p = 2$.

Consider the case $\|c\| \leq \|1/4\|$. Denote the roots of $T_c(x) - x$ by x_1 and x_2. We have $\mathbf{K} := \Bbbk[x_1] = \Bbbk[x_2]$, with $(\mathbf{K} : \Bbbk) \in \{1, 2\}$. Our norm can be extended to the field $\mathbf{K}$. Then $\|2x_1\| \leq 1$, $\|2x_2\| \leq 1$ and moreover $\|x_1 - x_2\| = \|\sqrt{1 - 4c}\| \leq 1$. So $D[x_1, 1] = D[x_2, 1]$.

Now prove the formula $\mathcal{FJ}(T_c) = \Bbbk \cap D_{\mathbf{K}}[x_1, 1]$. For $t\colon = x - x_1$ we obtain $\|T(x) - x_1\| = \|(x_1 + t)^2 + c - x_1\| = \|t(2x_1 + t)\|$. Hence for $\|t\| \leq 1$ we have $\|T_c^{n\circ}(x) - x_1\| \leq 1$ and for $\|t\| > 1$ we have $\|T_c^{n\circ}(x) - x_1\| = \|t\|^{2^n}$.

For any two points $x, y \in \mathcal{FJ}(T_c)$ we have

$$\|T_c(x) - T_c(y)\| = \|(x-y)(x+y)\| \leq \|x-y\|\|2x_1 + (x - x_1) + (y - x_1)\| \leq \|x - y\|.$$

Hence if $\|c\| \leq 1/4$, then the topological entropy of T_c equals zero.

Consider the case $\|c\| > 1/4$. Now we have two distinct disks $D[x_1, 1]$ and $D[x_2, 1]$, with the centres at $\|x_1\| = \|x_2\| = \sqrt{\|c\|}$ and $\|x_1 - x_2\| = \sqrt{\|4c\|}$. We introduce $b_{\pm}\colon = x_{1,2}$, and construct the b_ε's and D_ε as in the subsections 2.4, 2.5 (excluding the empty word). We argue similarly to the case $p \neq 2$, but have to introduce some modifications.

As in the case $p \neq 2$, $\|T_c(x) - x_1\| = \|(x_1 + t)^2 + c - x_1\| = \|t(2x_1 + t)\|$.

For $x_1 \notin \Bbbk$ we have $\|T_c^{n\circ}(x) - x_1\| = \|t\|^{2^n}$ for $\|t\| > \|2x_1\|$ and $\|T_c(x) - x_1\| = \|2x_1\| \cdot \|x - x_1\| > \|x - x_1\|$ for $0 < \|t\| \leq \|2x_1\|$. Hence the filled Julia set is empty and the entropy is equals zero.

But for $x_1 \in \Bbbk$ we have $x_2 = 1 - x_1 \in \Bbbk$ and moreover all the discs D_ε lie within $\Bbbk$ since lemma 1 holds for the disks $D(x_0, \|4x_0\|)$.

Lemma 2 is replaced by the statement $\sqrt{D[a^2, r^2]} = D[a, \frac{r^2}{\|2a\|}] \sqcup D[-a, \frac{r^2}{\|2a\|}]$ for all the discs $D[a^2, r^2]$ with $r^2 < \|4a^2\|$ (in particular, for all the shifted disks in the proof of the theorem 2). Hence for D_ε we obtain the formula $D_\varepsilon = D[b_\varepsilon, \|2a\|^{1-|\varepsilon|}]$.

So we prove that on $\mathcal{FJ}(T_c)$ our dynamical system is equivalent to the Bernoulli shift as in the theorem 4. Its topological entropy is positive. ■

3 Dynamics with nonarchimedean time

3.1 Introduction and main result

3.1.1 General setting.

We treat as the "time" an arbitrary semigroup or group acting on the phase space $\mathbb{T} : X$. Classically $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{N}$ for the continuous and the discrete dynamics respectively. In the case of periodic processes the groups $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $\mathbb{T} = \mathbb{Z}/N\mathbb{Z}$ are more suitable. The product of the latter groups over all $N \in \mathbb{N}$ acts in the obvious way on all the sets of periodic orbits, but the spirit of non-archimedean dynamics rather suggests considering the group of "universal periodic time"

$$\hat{\mathbb{Z}} := \varprojlim \mathbb{Z}/N\mathbb{Z} \cong \prod_{\text{prime } p\text{'s}} \mathbb{Z}_p,$$

acting on the periodic points by means of its finite cyclic factors. However, in the present work we consider only $\mathbb{Z}_2$ acting on the 2^n-periodic points.

3.1.2 The main example.

Consider the family of quadratic maps

$$f_c : x \mapsto x^2 + c$$

The bifurcation values of c, where the 2^n-periodic orbits loose their stability and the 2^{n+1}-stable orbits appear, are well-known. They are $\frac{1}{4} = c_0 > c_1 > \cdots > c_\infty = -1.416\ldots$ and can be defined by the following property: for any $c \in (c_{n+1}, c_n)$ there is exactly one stable 2^n-cycle of f_c.

The stable periodic points constitute the set $StabPer_n(c)$, on which the generator of the group $\mathbb{Z}/2^n\mathbb{Z}$ acts by $x \mapsto x^2 + c$. For every c there exists a distinguished element

$$x_0(c) := \lim_{N\to\infty} f_c^{2^N \circ}(0) \in StabPer_n(c).$$

The orbits of $x_0(c)$'s are pasted together to define the **function of a real and 2-adic variable**

$$X : (c_\infty, c_0] \times \mathbb{Z}_2 \longrightarrow \mathbb{R},$$

where for $c \in (c_{n+1}, c_n]$ we set $X(c,t) := f_c^{[t]_n \circ}(x_0(c))$, denoting by $[t]_n$ the image of the *moment* under the projection $\mathbb{Z}_2 \longrightarrow \frac{\mathbb{Z}}{2^n\mathbb{Z}}$. This function satisfies

$$X(c, t+1) = X(c,t)^2 + c.$$

3.1.3 Statement of the main result.

The 2^n-element sets of connected components

$$\mathcal{X}_n := \pi_0(\{c, X(\mathbb{Z}_2, c) | c \in (c_{n+1}, c_n)\})$$

are related by adjacency maps

$$\mathcal{X}_{n+1} \to \mathcal{X}_n,$$

that map every element ξ of $\mathcal{X}_{n+1}$ to the only element of $\mathcal{X}_n$ representing a component whose closure has a non-empty intersection with the closure of ξ.

The sets $\mathcal{X}_n$ are acted upon by $\mathbb{Z}_2$ in a compatible manner, so that the projective limit

$$\mathcal{X} := \lim_{\leftarrow} \mathcal{X}_n$$

is also acted upon by $\mathbb{Z}_2$.

Denote by $\mathcal{X}_\infty$ the closure of the orbit $f_{c_\infty}^{\mathbb{N}\circ}(0)$. It is well known [11] that $\mathcal{X}_\infty$ is the attractor of f_{c_∞}. It is also true that *the map f_{c_∞} is invertible on $\mathcal{X}_\infty$ – it follows easily from the theorem 1 below.*

Thus $\mathcal{X}_\infty$ is a $\mathbb{Z}$-set. The goal of this part is to prove the following result.

3.1.4 Main theorem.

There exists a $\mathbb{Z} \to \mathbb{Z}_2$-equivariant homeomorphism $\mathcal{X}_\infty \to \mathcal{X}$ sending 0 to the "distinguished element". Under this homeomorphism the action of $\mathbb{Z}_2$ is a continuous extension of the $\mathbb{Z}$-action.

Though the statement seems quite natural, the analysis of the limiting behavior of the 2^n periodic orbits of f_c is rather hard – and the difficulties are purely *archimedean.* We were unable to find easy proofs of our statements and had to use the deep results from the very well-developed one-dimensional dynamics, the survey of which we present below.

3.2 Preliminaries.

3.2.1 Unimodal maps.

Let $I = [-\alpha; \alpha] \subset \mathbb{R}$ be a segment and $f\colon I \to I$ a smooth even map.

Definition 1. A map f is called *unimodal*, if it is monotone on each of the parts of $I \setminus \{0\}$, if it has the unique nondegenerate extremum in 0 and if it has no other critical points.

Definition 2. The Schwartzian of f is defined as

$$Sf\colon = \frac{f'''}{f'} - \frac{3}{2}\Big(\frac{f''}{f'}\Big)^2$$

Theorem (Collet, Eckmann). *[2] Let f_c be a family of unimodal maps of the segment I to itself with negative schwartzians, where c belongs to some segment, such that $c \mapsto f_c$ is a smooth non-trivial map. Then there exists the strictly monotone sequence of values $\{c_n\}_{n=1}^{\infty}$, providing the bifurcation of period-doubling.*

Definition 3. We call a family of unimodal maps of the segment I into itself with negative schwartzian a *Collet-Eckmann family.*

Every function of the classical family $f_c(x) = x^2 + c$, $c \in [-3/2; 1/4]$ is obviously unimodal on the segment $I_c = [-\beta; \beta]$, where β is a positive root of the equation $f_c(x) = x$. The properties of the existence of periodic orbits, as well as the type of their stability are invariant under conjugation by diffeomorphisms. Instead of classical family we consider the truncated family $f_c(x) = x^2 + c$; $c \in [-3/2; -3/4]$. This family is conjugated to the family $\varphi_\gamma(x) = 3/2x^2 + \gamma$, $\gamma \in [-1; -1/2$ by the map $S(x) = 3/2x$, i. e. $\varphi_\gamma = S \circ f_c \circ S^{-1}$. Hence the segment Since the segment $[-1; 1]$ is invariant under the map φ_γ for $\gamma \in [1/2; 1]$ then φ_γ is Collet-Eckmann family as well as f_c for $c \in [-3/2; -3/4]$. Thus families f_c and φ_γ provide the bifurcation sequence of period-doubling values $\{c_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ respectively. These sequences converge with a universal rate (see, e.g., [2]): if c_∞ and γ_∞ are their limits, then

$$c_n - c_\infty \sim \delta^{-n}, \qquad \gamma_n - \gamma_\infty \sim \delta^{-n}$$

where $\delta = 4.669...$ is the famous Feigenbaum constant [5].

Our main functional space is the space $\mathcal{U}$ of the smooth unimodal maps of the segment $[-1;1]$ into itself. Supply it by the topology of uniform convergence.

3.2.2 Doubling transformation.

Define the doubling transformation $T\colon \mathcal{U}_0 \to \mathcal{U}_0$, where

$$\mathcal{U}_0\colon = \{f \in \mathcal{U} \mid f^{2\circ}(0) \neq 0 \text{ and } x = 0 \text{ is the point of maximum}\}$$

Denote

$$\alpha = \alpha(f) = -\frac{f(0)}{f(f(0))}$$

Suppose that $\alpha > 0$, $f(f(\alpha^{-1})) < \alpha^{-1} < f(\alpha^{-1})$, $f(0) > 0$, then

$$h(x) = -\alpha f(f(\alpha^{-1}x))$$

is also the unimodal map of the segment $[-1;1]$ into itself, and $h(0) = f(0)$. Define the doubling map $T\colon \mathcal{U}_0 \to \mathcal{U}_0$ by:

$$(Tf)(x) = -\alpha f(f(\alpha^{-1}x)), \qquad \alpha = \alpha(f) = -\frac{f(0)}{f(f(0))}$$

Consider the functional equation

$$-\alpha g^{\circ 2}\left(\frac{x}{\alpha}\right) \equiv g(x), \quad \alpha = \alpha(g) = -\frac{g(0)}{g(g(0))} \tag{1}$$

in the space of functions $\mathbb{R} \to \mathbb{R}$. This equation is called the *doubling equation*. According to the general theory (see the survey [11]) it has the unique solution, satisfying the following conditions:

1. $g(x) = -\alpha g(g(\alpha^{-1}x)),\ \alpha = 2.503...$
2. $g(0) = 1$, and 0 is the point of the maximum.
3. $g(x) = g(-x)$.

This function was thoroughly studied in the papers of Campino-Epstein [1] and Lanford [6]. It was shown that all the coefficients of the Taylor expansion of g, except the first one, are irrational. The several first terms are

$$g(x) = 1 - 1.527\ldots x^2 + 0.104\ldots x^4 + \ldots$$

Futher in this part we suppose that $\alpha = \alpha(g) = 2,503...$

Consider the sequence of maps $\{g_k\}_{k=1}^{\infty}$

$$g_k(x)\colon = \lim_{n\to\infty} (-\alpha)^n f_{c_{n+k}}^{\circ 2^n}\left(\frac{x}{(-\alpha)^n}\right).$$

In the papers of Collet-Eckmann [2] and Lanford [6] one can find the following results. The sequence $\{g_k\}$ is well-defined (all the limits exist). The functions g_k are unimodal and related by the doubling map:

$$g_{k-1} = Tg_k \tag{2}$$

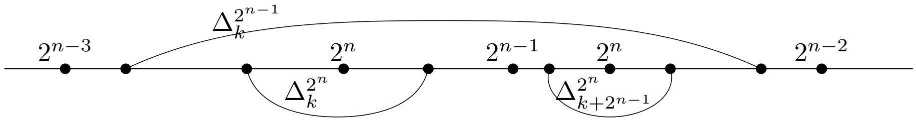

Рис. 1: Segments and periodic points of the period 2^n, 2^{n-1}, 2^{n-2} and 2^{n-3}.

The sequence converges, and because of (2) its limit is the fixed point of the doubling operator:

$$\lim_{k\to\infty} g_k = g$$

3.2.3 The partition tower of the segment.

For the proof of the theorem 7 we use the construction, suggested by Misiurewicz [7]. He considers the class Φ of maps φ of the segment $[-1;1]$ into itself with the properties:

1. $\varphi \in C^1([-1;1])$, $\varphi \in C^2([-1;1] \setminus \{0\})$,
2. $\varphi(-1) = -1$ and $\varphi'(-1) > -1$,
3. $\varphi'(x) \neq 0$ if $x \neq 0$ and $S\varphi(x) < 0$ if $x \neq 0$.
4. For every $n > 0$ the map φ has exactly one periodic orbit of order 2^n and has no other periodic orbits.

For the maps of the class Φ one can construct the system of segments $\{\Delta_i^{(n)}\}$, $n \geq 1$, $0 \leq i < 2^n$ with the following properties:

$1^\star$. $\Delta_i^{(n)} \cap \Delta_j^{(n)} = \varnothing$ for $i \neq j$;

$2^\star$. $f(\Delta_i^{(n)}) = \Delta_{i+1}^{(n)}$ for $0 \leq i < 2^n - 1$; $f(\Delta_{2^i-1}^{(n)}) \subset \Delta_0^{(n)}$ and the endpoints of the segments $f(\Delta_{2^i-1}^{(n)})$ and $\Delta_0^{(n)}$ are different;

$3^\star$. For every n the inclusion $\Delta_i^{(n)} \supset \Delta_i^{(n+1)} \cup \Delta_{i+2^n}^{(n+1)}$ holds and $\Delta_i^{(n)}$ contains no other segments of the level $n+1$.

Convention. We say that the segment $\Delta_k^{(N)}$ has the *number* k and the *level* N. We say that segment $\Delta_k^{(N)}$ is less then $\Delta_m^{(L)}$ if for any points $x \in \Delta_k^{(N)}$ and $y \in \Delta_m^{(L)}$ $x < y$. (See Fig. 1).

3.3 Reformulation of the main theorem

Definition 4. The *bifurcational diagram* (see Fig. 2) is the set of points of the (c,x)-plane defined as follows:

$$BD\colon = \bigsqcup_{n=0}^{\infty} \{(c,x) \mid c \in [c_{n+1}; c_n] \mid f_c^{\circ 2^n}(x) = x, \left| \left(f_c^{\circ 2^n}\right)'(x) \right| \leq 1\}$$

for each $c \in \mathbb{R}$ and $n \in \mathbb{N}$ the equation $f_c^{\circ 2^n}(x) = x$ defines an algebraic curve α_n. The bifurcation diagram consists of pieces of the curves α_n (with various n), defined by the conditions $|(f_c^{\circ 2^n}(x))| < 1$ as well as of the points of neutral cycles where the stability is lost: $\{x\colon f_{c_n}^{\circ 2^n}(x) = x \mid (f^{2^n\circ})'(x) = 1\}$.

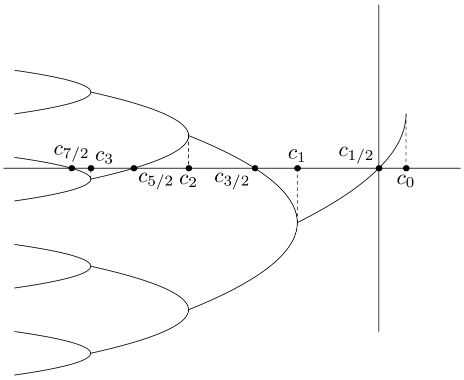

Рис. 2: Bifurcation diagram for the family f_c

Now we define some special numeration of the components of BD over each interval (c_{n+1}, c_n). As in the introduction, we denote these components by $\mathcal{X}_n$ with $\#\mathcal{X}_n = 2^n$. We number the elements of $\mathcal{X}_n$ by the elements of the group $\mathbb{Z}/(2^n\mathbb{Z})$, thus defining the bijection $num : \mathcal{X}_n \to \mathbb{Z}/(2^n\mathbb{Z})$. The only component that intersects line $x = 0$ acquires number 0; the other components are numbered uniquely by the condition

$$num(f_c(\xi)) = num(\xi) + 1.$$

Considering the groups $\mathbb{Z}/(2^n\mathbb{Z})$ as factors of the additive group $\mathbb{Z}_2$ of 2-adic numbers, we paste these numerations to the global map

$$X(c,t)\colon (c_\infty; c_1] \times \mathbb{Z}_2 \to \mathbb{R},$$

it is a real-valued function of a real and 2-adic argument. By definition, it satisfies the equation

$$X(c, t+1) = X(c,t)^2 + c.$$

The main technical result of this part can be formulated as follows:

Theorem 7. *For any “2-adic moment” $t \in \mathbb{Z}_2$ there exists* $\lim\limits_{c\to c_\infty} X(c,t) =: X(c_\infty, t)$

3.4 Important particular case

First we prove the theorem that is a particular case of the theorem 7 (the case of *zero moment*).

Theorem 8.

$$\lim_{c\to c_\infty} X(c, 0) = 0 \tag{3}$$

Proof. The product of the derivatives of f_c, taken over the stable 2^n-periodic orbit, varies from 1 to -1 while c decreases from c_{n+1} to c_n denote by $c_{n+1/2}$ the value when this product vanishes. The number $0 \in \mathbb{Z}_2$ corresponds to the part of the bifurcation diagram, intersecting the line $x = 0$ on each segment $[c_{n+1}, c_n]$, $n \in \mathbb{N}$. Indeed, for any n there exists such a value $c_{n+1/2} \in [c_{n+1}, c_n]$, that 0 is a 2^n-periodic point of $f_{c_{n+1/2}}$. Such cycle of the map $f_{c_{n+1/2}}$ is superattractive.

Lemma 3. *Let d_n be the distances on the (c, x)-plane between the line $x = 0$ and the nonzero point of the superattractive cycle of the period 2^n that is closest to it. Then*

$$\lim_{n\to\infty} \left|\frac{d_n}{d_{n+1}}\right| = \alpha = 2,503\ldots \tag{4}$$

and hence $\lim_{n\to\infty} d_n = 0$

Proof. We calculate:

$$g_1(0) = \lim_{n\to\infty} (-\alpha)^n f^{\circ 2^n}_{c_{n+1+1/2}}(0) = \lim_{n\to\infty} (-\alpha)^n |d_{n+1}|.$$

The claim (4) follows from the last relation, because

$$\left|\frac{d_n}{d_{n+1}}\right| = \alpha \frac{|(-\alpha)^{n-1} d_n|}{|(-\alpha)^n d_{n+1}|} \to \alpha \text{ while } n \to \infty.$$

Hence $d_n \sim \alpha^{-n} \to 0$. ∎

Lemma 4.

$$\lim_{n\to\infty} \frac{c_{n+3/2} - c_{n+1/2}}{c_{n+1/2} - c_{n-1/2}} = \lim_{n\to\infty} \frac{c_{n+1} - c_n}{c_n - c_{n-1}} = \frac{1}{\delta}.$$

Proof. It is obvious that

$$\begin{aligned} \varphi_\gamma &= \varphi_{\gamma_\infty} + (\gamma - \gamma_\infty); \\ T\varphi_\gamma &= T(\varphi_{\gamma_\infty} + (\gamma - \gamma_\infty)). \end{aligned} \tag{5}$$

Consider the linearization of T in the neighborhood of a function $f \in \mathcal{U}_0$:

$$(DT_f[h])(x) \colon = \lim_{\lambda\to 0} \frac{T(f+\lambda h) - Tf}{\lambda}(x) = -\alpha \left[h\left(f\left(\frac{x}{-\alpha}\right)\right) + f'\left(f\left(\frac{x}{-\alpha}\right)\right) h\left(\frac{x}{-\alpha}\right)\right]$$

where $\alpha = \alpha(f)$.

Then the equation (5) takes the form

$$\begin{aligned} (T\varphi_\gamma)(x) &= T(\varphi_{\gamma_\infty}) + (\gamma - \gamma_\infty) DT_{\varphi_{\gamma_\infty}}(1) + \bar{o}(\gamma - \gamma_\infty) = \\ &= T(\varphi_{\gamma_\infty})(x) - \alpha(\gamma - \gamma_\infty)\left(1 + 2\left(\frac{x^2}{\alpha^2} + \gamma_\infty\right)\right) + \bar{o}(\gamma - \gamma_\infty). \end{aligned}$$

where $\bar{o}(\gamma - \gamma_\infty)$ denotes the infinitesimals of higher order. Iterating this relation, we obtain:

$$T^{\circ n}(\varphi_\gamma) = T^{\circ n}(\varphi_{\gamma_\infty}) + (\gamma - \gamma_\infty) DT_{T^{\circ n-1}\varphi_{\gamma_\infty}} \circ \ldots \circ DT_{\varphi_{\gamma_\infty}}(1) + \bar{o}(\gamma - \gamma_\infty). \tag{6}$$

Let $\gamma = \gamma_{n+1/2}$, $x = 0$. Then

$$\begin{aligned} T^{\circ n}(\varphi_{\gamma_{n+1/2}})(0) &= T^{\circ n}(\varphi_{\gamma_\infty})(0) + \\ &+ (\gamma_{n+1/2} - \gamma_\infty) DT_{T^{\circ n-1}\varphi_{\gamma_\infty}} \circ \ldots \circ DT_{\varphi_{\gamma_\infty}}(1)(0) + \bar{o}(\gamma_{n+1/2} - \gamma_\infty)(0) \end{aligned} \tag{7}$$

and take the limit as $n \to \infty$.

$$\lim_{n\to\infty} T^{\circ n}(\varphi_{\gamma_\infty})(0) = \lim_{n\to\infty}\lim_{j\to\infty} T^{\circ n}\varphi_{\gamma_{n+j}}(0) = \lim_{j\to\infty}\lim_{n\to\infty} (-\alpha)^n \varphi^{2^n}_{\gamma_{n+j}}(0) = \lim_{j\to\infty} g_j(0) = g(0) = 1;$$

since the function, continuous on the segment, is uniformly continuous, therefore changing the order of limits is legal. Moreover,

$$\lim_{n\to\infty} DT_{T^{\circ(n-1)}f_{\gamma_\infty}} = DT_g \tag{8}$$

Fix $N \gg 1$. From the equation (8) it follows that whenever $n > N$

$$T^{\circ n}\varphi_{\gamma_{n+1/2}} = T^{\circ n}\varphi_{\gamma_\infty} + (c_{n+1/2} - \gamma_\infty)(DT_g)^{n-N} DT_{T^{\circ N}\varphi_{\gamma_\infty}} \circ \ldots \circ DT_{\varphi_{\gamma_\infty}}(1) + \bar{o}(\gamma - \gamma_\infty).$$

Consider the function $h\colon = DT_{T^{\circ N}f_{\gamma_\infty}} \circ \ldots \circ DT_{\varphi_{\gamma_\infty}}(1)$. It is known, [9] that the operator DT_g is not self-conjugate and that it has the only eigenvector h_0, corresponding to the eigenvalue $\delta > 1$ and the invariant subspace $\tilde{H}$, on which the norm $\|DT_g\| = \nu < 1$. Then the function h can be represented in the form $h = h_0\varphi_0 + \tilde{h}$, where $\tilde{h} \in \tilde{H}$ and φ_0 is a number.

So

$$(DT_g)^{n-N}h = (DT_g)^{n-N}(h_0\varphi_0 + \tilde{h}) = \delta^{n-N}h_0\varphi_0 + (DT_g)^{n-N}(\tilde{h})$$

Since the space $\tilde{H}$ is invariant and $\|DT_g\| = \nu < 1$, we have $(DT_g)^{n-N}(\tilde{h}) \to 0$ when $n \to \infty$. Finally, $\lim\limits_{n\to\infty} T^{\circ n}\varphi_{\gamma_{n+1/2}}(0) = \lim\limits_{n\to\infty}(-\alpha)^n\varphi^{2^n}_{\gamma_{n+1/2}}(0) = 0$. Hence, taking the limit with $n \to \infty$ in (7), we obtain:

$$0 = g(0) + \lim_{n\to\infty}(\gamma_{n+1/2} - \gamma_\infty)\delta^{n-N}\varphi_0 h_0(0).$$

Consequently, $\lim\limits_{n\to\infty}(\gamma_{n+1/2} - \gamma_\infty)\delta^n = const$, from which the lemma 4 follows. ■

The theorem 8 follows from the lemma 4 and the relation (4). ■

3.5 Proof of the main theorem

Proposition. *There exists a diffeomorphism conjugating the map f_{c_∞} with a map of the class Φ. (See subsection 2.3).*

Proof. Indeed, f_{c_∞} has the unique unstable periodic orbit of the order 2^n. Applying function $S(x)$ from the subsection (2.1) and using renormalization we get the desired result.

■

For f_{c_∞} one can also construct the system of segments $\{\Delta_i^{(n)}(c_\infty)\}$, $n \geq 1$, $0 \leq i < 2^n$, satisfying the properties $1^\star - 3^\star$ of the subsection 2.3. The left and the right endpoints of these segments will be denoted β_k^n and γ_k^n respectively. Due to the properties $2^\star, 3^\star$ each segment $\Delta_i^{(n)}$ contains 2^{k-n} points of the period 2^k for $k > n$, and by the property $1^\star$ it contains no periodic points of other orders. Each segment of the level $\Delta_i^{(n+1)}$ is separated from other segments by two repelling points, one of which has the period 2^n, and the other one has the smaller period. The numeration of the segments can be chosen in such a way that $\Delta_0^{(n)}(c_\infty) \ni 0$. On the bifurcational diagram the segment $\Delta_0^{(n)}(c_\infty)$ is contained between the points of the repelling cycle of the period 2^n that are the closest to the line $x = 0$.

It follows from the lemmas 3 and 4 that

$$\frac{|\Delta_0^{(n)}(c_\infty)|}{|\Delta_0^{(n+1)}(c_\infty)|} \to \alpha \text{ with } n \to \infty, \tag{9}$$

where $|\Delta|$ is the length of the segment Δ.

In order to prove the existence of the limit $\lim\limits_{c \to c_\infty} X(c,t)$ it suffices to show that the following lemma 5 holds.

Lemma 5. *For any natural k*

$$\lim_{n \to \infty} |\Delta_k^{(n)}| = 0. \tag{10}$$

Proof. It follows from the lemma 3 that $|\Delta_0^{(n)}| \equiv \frac{\text{const}}{\alpha^n}$. By the construction of the segment system

$$f_{c_\infty}^{2^n-k}(\Delta_n^{(k)}) \subset \Delta_0^{(n)},$$

so for the proof of the convergence of the lengths of segments it suffices to show that

$$\forall k \ \exists D_k < \alpha \quad \alpha^2 |f_{c_\infty}^{2^n-k}(\Delta_k^{(n)})| D_k^n \geq |\Delta_k^{(n)}| \tag{12$_n$}$$

For $k = 0$ the lemma 5 is proved.

Now we induct in n. For $n = 0$ there is only one segment. In the case $n = k = 1$ on the segment $\Delta_1^{(1)}$ the map f_{c_∞} is expanding, and one can take $D_1 = \min(f'_{c_\infty}(\gamma_1^1), f'_{c_\infty}(\beta_1^1))^{-1}$. Then

$$|f_{c_\infty}(\Delta_1^{(1)})| D_1 \geq |\Delta_1^{(1)}|.$$

Let's prove that $(12)_n$ implies $(12)_{n+1}$.

$$|f_{c_\infty}^{2^{n+1}-k}(\Delta_k^{(n+1)})| = |f_{c_\infty}^{2^n} f_{c_\infty}^{2^n-k}(\Delta_k^{n+1})|$$

Since $f_{c_\infty}^{2^n-k}$ is a diffeomorphism on the segment $\Delta_k^{(n+1)}$ and has the negative schwartzian, according to the choice of D_1 we have

$$|f_{c_\infty}^{2^n-k}(\Delta_k^{(n+1)})| D_1^n > |\Delta_k^{(n+1)}| \tag{11}$$

Furthermore, $f_{c_\infty}^{2^n-k}(\Delta_k^{n+1}) = \Delta_{2^n}^{(n+1)}$. So now it's time to prove that for any $n \in \mathbb{N}$

$$|f^{2^n}(\Delta_{2^n}^{2^{n+1}})| \frac{\alpha^2}{\alpha - 1} > |\Delta_{2^n}^{2^{n+1}}|$$

Consider the map f^{2^n} on the segment $\Delta_{2^n}^{2^{n+1}}$. It is a diffeomorphism with negative schwartzian. Without loss of generality we may suppose that it is increasing. By definition of partition tower $f^{2^n}(\Delta_{2^n}^{2^{n+1}}) \subset \Delta_0^{2^{n+1}}$ and $\Delta_{2^n}^{2^{n+1}} \supset \Delta_{2^n}^{2^{n+2}} \cup \Delta_{3 \cdot 2^n}^{2^{n+2}}$ and $f^{2^n}(\Delta_{2^n}^{2^{n+2}}) = \Delta_{2^{n+1}}^{2^{n+2}} \subset \Delta_0^{2^{n+1}}$. Moreover $\Delta_{3 \cdot 2^n}^{2^{n+2}} \supset \Delta_{3 \cdot 2^n}^{2^{n+3}} \cup \Delta_{7 \cdot 2^n}^{2^{n+3}}$ and $f^{2^n}(\Delta_{3 \cdot 2^n}^{2^{n+3}}) = \Delta_{2^{n+2}}^{2^{n+3}}$, $f^{2^n}(\Delta_{7 \cdot 2^n}^{2^{n+3}}) \subset \Delta_0^{2^{n+3}}$. But $\Delta_{2^{n+1}}^{2^{n+2}} < \Delta_0^{2^{n+3}} < \Delta_{2^{n+2}}^{2^{n+3}}$, so $f^{2^n}(\Delta_{2^n}^{2^{n+1}}) \supset \Delta_0^{2^{n+3}}$ using relation (9) we get $|\Delta_{2^n}^{2^{n+1}}| < \frac{|\Delta_0^{2^n}|(\alpha-1)}{\alpha}$ and $|\Delta_0^{2^n}| > \frac{|\Delta_0^{2^{n-3}}|}{\alpha^3}$. Hence

$$|f^{2^n}(\Delta_{2^n}^{2^{n+1}})| \frac{\alpha^2}{\alpha - 1} > |\Delta_{2^n}^{2^{n+1}}| \tag{12}$$

Now $(12)_n$ follows from (12) and (11).

Lemma 3 is proved. ■

The segment number $k \neq 0$ appears for the first time on the level number $s := [\log_2 k + 1]$. We have to find such a number D_k, that $f_{c_\infty}^{2^s-k}(\Delta_k^s)D^s > |\Delta_k^s|$. Take

$$D_k = \max(\min((f_{c_\infty}^{2^s-k})'(\beta_k^s), (f_{c_\infty}^{2^s-k})'(\gamma_k^s),)D_{k-2^{s-1}})$$

■

The theorem 7 is proved.

The map f_c acts on the bifurcation diagram:

$$f_c \colon BD \mapsto BD \colon (x, c) \to (f_c(x), c),$$

inducing the map $F \colon \mathbb{Z}_2 \to \mathbb{Z}_2 \colon t \mapsto t + 1$.

Conclusion. The following diagram is commutative:

$$\begin{array}{ccc} \mathbb{Z}_2 & \xrightarrow{t \mapsto t+1} & \mathbb{Z}_2 \\ \big\downarrow \kappa(t) & & \big\downarrow \kappa(t) \\ \mathbb{R} & \xrightarrow{f_{c_\infty}} & \mathbb{R} \end{array}$$

This is an equivalent form of our main theorem.

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