Exercise Sheet 10

Submersions. Preimage theorem. Manifolds with a boundary.

All sets are considered with the standard topology induced from $\mathbb{R}^n$. A manifold stands for a smooth manifold.

Easy: check your understanding

Exercise 10.1.1. Show directly by definition that projection $\pi\colon \mathbb{R}^k \to \mathbb{R}^n$ given by

$$\pi(x_1, x_2, \dots, x_k) = (x_1, x_2, \dots, x_n)$$

for $k \geq n$ is a submersion. In fact, it is the prototype of a submersion.

Exercise 10.1.2. Let $f\colon X \to Y$ be a submersion of X into Y, and $U \subset X$ be open. Show that $f(U)$ is open in Y.

Exercise 10.1.3. Show that submersions of compact manifolds into Euclidean spaces do not exist.

Exercise 10.1.4. Show the solid hyperboloid $x^2 + y^2 - z^2 \leq 1$ is a manifold with a boundary.

Exercise 10.1.5. Let U be an open neighbourhood of $0 \in \mathbb{R}^n$ and let V be an open neighbourhood of 0 in $H^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$. Show that there is no diffeomorphism between U and V.

Exercise 10.1.6. Show that the square $[0, 1] \times [0, 1]$ is not a manifold with a boundary. *Hint*: assume that such a suitable diffeomorphism exists and consider its derivative at the corner.

Harder: working-level knowledge

Exercise 10.2.1. Let $X \subset \mathbb{R}^n$ be a compact set and a manifold. Let $Y \subset \mathbb{R}^k$ be a connected set and a manifold. Show that any submersion $X \to Y$ is surjective. *Hint*: first show that the image is open; then show that the image is closed.

Exercise 10.2.2. Show that the map $\rho : \mathbb{S}^3 \to \mathbb{R}^4$ defined by

$$\rho(x, y, z) = (yz, xz, xy, x^2 + 2y^2 + 3z^2).$$

defines an embedding of $\mathbb{R}P^2$ into $\mathbb{R}^4$. *Hint*: see exercise 8.3.1.

Exercise 10.2.3. Show that the map $\rho(x, y, z) = (yz, xz, xy)$ defines an immersion of $\mathbb{R}P^2$ into $\mathbb{R}^3$.

Exercise 10.2.4. Let be a polynomial with complex coefficients, and consider the associated map $z \mapsto p(z)$ of the complex plane $\mathbb{C}$. Prove that this is a submersion except at finitely many points.

Exercise 10.2.5. Show that the product of a manifold without boundary X and a manifold with boundary Y is a manifold with boundary. Furthermore,

(a) $\partial(X \times Y) = X \times \partial Y$;

(b) $\dim(X \times Y) = \dim X + \dim Y$.

Exam-type problems

Exercise 10.3.1. Show that for any open subset $U \subset M$ the inclusion $i \colon U \to M$ is both an immersion and a submersion.

Exercise 10.3.2. (a) Check that 0 is the only critical value of the map $f \colon \mathbb{R}^3 \to \mathbb{R}$ given by $f(x, y, z) = x^2 + y^2 - z^2$.

(b) Prove that if a and b are either both positive or both negative, then $f^{-1}(a)$ and $f^{-1}(b)$ are diffeomorphic. *Hint*: Consider scalar multiplication by $\sqrt{b/a}$.

(c) Find the basis of the tangent space to $f^{-1}(a)$ at $\sqrt{a}, 0, 0)$.

Exercise 10.3.3. Let X and Y be two topological spaces. Show that the projection $\pi \colon X \times Y \to X$ maps open sets to open sets.