Exercise Sheet 9

Compactness in topological spaces.
Immersions, Embeddings, and Inverse Function Theorem.

All sets are considered with the standard topology induced from $\mathbb{R}^n$. A manifold stands for a smooth manifold.

Easy: check your understanding

Exercise 9.1.1. Show that any topological space with only finite number of points is compact.

Exercise 9.1.2. Which of the following subspaces is compact in A if (i) $A = \mathbb{R}$, (ii) $A = \mathbb{R}\backslash\{0.5\}$ with respect to the induced topology from $\mathbb{R}$?

$$(\mathbf{a})\ A \cap \big([0,1] \cup [2,3]\big); \qquad (\mathbf{b})\ A \cap [0,\infty); \qquad (\mathbf{c})\ A \cap Q \cap [0,1]$$

Exercise 9.1.3. Which of the following subspaces is compact in A if (i) $A = \mathbb{R}^2$, (ii) $A = \mathbb{R}^2 \setminus \{(0,0)\}$ with respect to the induced topology from $\mathbb{R}$?

$$(\mathbf{d})\ A \cap \big\{x^2 + y^2 = 1\big\}\,; \quad (\mathbf{e})\ A \cap \{|x| + |y| \le 1\}\,; \quad (\mathbf{f})\ A \cap \Big\{x \ge 1,\ 0 \le y \le \frac{1}{x}\Big\};$$

Exercise 9.1.4. Let A be a linear map of $\mathbb{R}^n$. Show that the mapping $x \mapsto Ax + b$ is a diffeomorphism of $\mathbb{R}^n$ if and only if A is nonsingular.

Exercise 9.1.5. Show that if f and g are immersions, then $g \circ f$ and $g \times f$ are also immersions.

Exercise 9.1.6. Check that the map $\varphi\colon \mathbb{R} \to \mathbb{R}^2$ given by $\varphi(t) = \frac{1}{2}\left(e^t + e^{-t}, e^t - e^{-t}\right)$ is an embedding. Prove that its image is one nappe of the hyperbola $x^2 - y^2 = 1$.

Harder: working-level knowledge

We say that a set $X \subset \mathbb{R}^n$ has the property $\mathcal{E}$ if for any compact set $C \subset X$ there exists another compact set K such that $C \subsetneq K \subsetneq X$ and $X \setminus K$ is connected.

Exercise 9.2.1. Show that $\mathbb{R}^n$ has the property $\mathcal{E}$.

Exercise 9.2.2. Show that the Möbius band without boundary has the property $\mathcal{E}$.

Exercise 9.2.3. Show that the cylinder surface doesn't have the property $\mathcal{E}$.

Exercise 9.2.4. Prove that if $A \subset \mathbb{R}^n$ is not compact, then there exists a continuous function $f\colon A \to \mathbb{R}$ which is not bounded on A.

Exercise 9.2.5. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a local diffeomorphism. Prove that the image of f is an open interval and that, in fact, f maps $\mathbb{R}$ diffeomorphically onto this interval.

Exercise 9.2.6. In contrast with the previous problem, construct a local diffeomorphism $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ that is not a diffeomorphism onto its image.

Optional: for your curiosity

Exercise 9.3.1. Show that if $A \subset \mathbb{R}^n$ is compact, then there exists $a < b$ such that A is contained in a cube $[a, b] \times [a, b] \times \ldots \times [a, b]$ (n factors).

Exercise 9.3.2. Show that if $C \subset \mathbb{R}^n$ is closed and $K \subset \mathbb{R}^n$ is compact, then $K \cap C$ is compact.

Exercise 9.3.3. Show that $f \colon \mathbb{R} \to \mathbb{S}^2$, $f(t) = (\cos 2\pi t, \sin 2\pi t)$, is, in fact, a local diffeomorphism. Given a line $\gamma \subset \mathbb{R}^2$, show that the restriction $f \times f \colon \gamma \to \mathbb{S}^2 \times \mathbb{S}^2$ is an immersion.