Exercise Sheet 8 — Unassessed Coursework 2
Smooth manifolds and tangent spaces

Submit solutions to ***three*** problems of your choice from §1 and ***two*** problems of your choice from §2. All sets are considered with the standard topology induced from $\mathbb{R}^n$. Manifold stands for smooth manifold.

Given a manifold $X \subset \mathbb{R}^n$ we say that a subset $Y \subset X$ is a *submanifold of* X if $Y \subset \mathbb{R}^n$ is a manifold.

Easy: check your understanding

Exercise 8.1.1. Find a basis of the tangent space to the cylinder

$$C := \big\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 2\big\}$$

at the point $P = (1, 1, 1)$.

Exercise 8.1.2. Let $M \subset \mathbb{R}^n$ and $N \subset \mathbb{R}^k$ be two smooth manifolds. Show that $M \times N$ and $N \times M$ are diffeomorphic.

Exercise 8.1.3. Show that if $U \subset \mathbb{R}^k$ and $V \subset \mathbb{R}^m$ are open and $\varphi\colon U \to V$ is a diffeomorphism, then $k = m$. *Hint*: Pick a point $p \in U$ and consider the induced map between the tangent spaces T_pU and $T_{f(p)}V$.

Exercise 8.1.4. Let V be a linear subspace of $\mathbb{R}^n$. Show that for any $x \in V$ the tangent space $T_xV = V$.

Exercise 8.1.5. Let $U \subset M$ be an open subset of a smooth manifold. Show that for any $p \in U$ we have $T_pU = T_pM$.

Exercise 8.1.6. It follows from problem 8.3.2 that ρ defines a map from $\mathbb{R}P^2$ to $\mathbb{R}^4$. In particular, we can define a distance between two points on $\mathbb{R}P^2$ by $d([p_1], [p_2]) := d(\rho(p_1), \rho(p_2))$. Show that the open balls with respect to the metric d are open subsets with respect to the quotient topology in $\mathbb{R}P^2$.

Harder: working-level knowledge

Exercise 8.2.1. Let $\mathbb{M}$ be the image of the strip $R := \{(\varphi, t) \in \mathbb{R}^2 \mid -1 < t < 1\}$ under the map

$$F_x(\varphi, t) = \left(4 + t\cos\frac{\varphi}{2}\right)\cos\varphi$$
$$F_y(\varphi, t) = \left(4 + t\cos\frac{\varphi}{2}\right)\sin\varphi$$
$$F_z(\varphi, t) = t\sin\frac{\varphi}{2}.$$

Identify the surface M. Assuming without proof that F is a diffeomorphism on any domain where it is a bijection, explicitly exhibit enough parametrizations to turn $\mathbb{M}$ into a manifold. Find the basis of the tangent space to $\mathbb{M}$ at $(2\sqrt{2}, 2\sqrt{2}, 0)$. *Hint*: Exercise 5.2.2.

Exercise 8.2.2. Let $\mathbb{T} = \tau(\mathbb{R}^2) \subset \mathbb{R}^3$ be a torus, where τ is a map given by

$$\begin{aligned}\tau_1(x_1, x_2) &= (\cos(x_1) + 5)\cos(x_2)\\ \tau_2(x_1, x_2) &= (\cos(x_1) + 5)\sin(x_2)\\ \tau_3(x_1, x_2) &= \sin(x_1)\end{aligned}$$

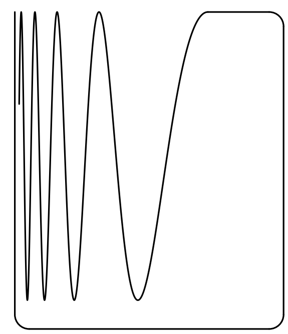
and let $\gamma \subset \mathbb{T}$ be a curve given by

$$\gamma_1(t) = (\cos(t) + 5)\cos(2t), \quad \gamma_2(t) = (\cos(t) + 5)\sin(2t), \quad \gamma_3(t) = \sin(t), \qquad t \in \mathbb{R}.$$

Show that γ is a smooth submanifold. Is the complement $\mathbb{T} \setminus \gamma$ a connected subset of $\mathbb{T}$? Which curve from the classification of 1-dimensional manifolds is γ diffeomorphic to? Justify your answer. *Hint*: Exercise 5.2.1.

Exercise 8.2.3. Prove that the hyperboloid of one sheet in $\mathbb{R}^3$ defined by $x^2 + y^2 - z^2 = a$, is a manifold if $a > 0$. Find the basis of the tangent space at $(0, \sqrt{a}, 0)$.

Exercise 8.2.4. Is the smooth curve pictured on the right a one-dimensional manifold? Justify your answer. (The right part makes infinitely many turns, the left interval doesn't contain the top end point).



Optional: for your curiosity

Exercise 8.3.1. Consider a function $\rho : \mathbb{S}^3 \to \mathbb{R}^4$ defined by

$$\rho(x, y, z) = (yz, xz, xy, x^2 + 2y^2 + 3z^2).$$

Show that $\rho(x_1, y_1, z_1) = \rho(x_2, y_2, z_2)$ if and only if $(x_1, y_1, z_1) = \pm(x_2, y_2, z_2)$. *Hint*: note that on the unit sphere $(x_1 + y_1 + z_1)^2 = 1 + 2(xy + yz + xz)$.

Exercise 8.3.2. With help of a computer draw the image of the unit sphere in $\mathbb{R}^3$ under the map $\rho(x, y, z) = (yz, xz, xy)$. It is another topological *immersion* of $\mathbb{R}P^2$ into $\mathbb{R}^3$.