Exercise Sheet 7

Manifolds: parametrizations and basic constructions

In what follows, a manifold stands for a smooth (real) manifold.

1. Easy: check your understanding.

Exercise 7.1.1. Find an atlas on a circle consisting of two charts only. In other words, find two open sets $U_1 \subset \mathbb{R}$, $U_2 \subset \mathbb{R}$ and diffeomorphisms $f_1 : U_1 \to V_1 \subset \mathbb{S}^2$ and $f_2 : U_2 \to V_2 \subset \mathbb{S}^2$ such that $V_1 \cup V_2 \supset \mathbb{S}^2$.

Exercise 7.1.2. Show that an open subset of a manifold is a manifold.

Exercise 7.1.3. Show that "the shell" of a ball

$$\mathbb{A} := \{x \in \mathbb{R}^n : r < \|x\| < R\}.$$

is a manifold. Find its dimension and write parametrisations down.

Exercise 7.1.4. Show that every k-dimensional vector subspace $V \subset \mathbb{R}^n$ is a manifold diffeomorphic to $\mathbb{R}^k$, and that all linear maps on V are smooth.

Exercise 7.1.5. Prove that the paraboloid in $\mathbb{R}^3$ defined by $x^2 + y^2 - z^2 = a$, is a manifold if $a > 0$. Why doesn't $x^2 + y^2 - z^2 = 0$ define a manifold?

2. Harder: working-level knowledge.

Exercise 7.2.1. Find an atlas on a torus consisting of three charts only. In other words find three open sets $U_j \subset \mathbb{R}^2$, and diffeomorphisms $f_j : U_j \to V_j \subset \mathbb{T}^2$ such that $\cup V_j \supset \mathbb{T}^2$ for $j = 1, 2, 3$.

Exercise 7.2.2. Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^k$ be two manifolds. Show that the projection map $X \times Y \to X$, carrying (x, y) to x, is smooth.

Exercise 7.2.3. Let $X \subset \mathbb{R}^n$. The diagonal $\Delta \subset X \times X$ is the set of points of the form (x, x). Show that Δ is diffeomorphic to X. In particular, if X is a manifold, then Δ is also a manifold.

Exercise 7.2.4. Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^k$ be two manifolds. The graph of a map $f : X \to Y$ is a subset of $X \times Y$ defined by

$$G_f = \{(x, f(x)) \in X \times Y : x \in X,\ y \in Y\}$$

Define $F : X \to G_f$ by $F(x) = (x, f(x))$. Show that if f is smooth, then F is a diffeomorphism; thus if X is a manifold, then G_f is also a manifold.

3. Exam-type questions.

Exercise 7.3.1. Give an example of a smooth bijective map between two manifolds which is not a diffeomorphism.

Exercise 7.3.2. Let $a, b > 0$ and consider

$$X := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \colon (x_1 - y_1)^2 + (x_2 - y_2)^2 + x_3^2 = b^2 \text{ for some } y_1^2 + y_2^2 = a^2\}$$

Identify the space X. For which a, b the set X is a manifold? Justify your answer.

Exercise 7.3.3. Consider the set

$$X := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \colon |x_1| < 1,\ |x_2| < 1,\ |x_3| < 1\} \setminus \{(0, 0, 0)\}.$$

Is X a manifold? Justify your answer.