Exercise Sheet 6

Projective spaces, Klein bottle, and first steps in Manifolds.

1. Easy: check your understanding

Exercise 6.1.1. Show that the circle is a smooth 1-dimensional manifold. More precisely, show that every point has a neighbourhood diffeomorphic to an open interval. First check that the map

$$f_1\colon (-1,1) \to \mathbb{S}^2 \qquad f_1\colon t \mapsto \left(t, \sqrt{1-t^2}\right)$$

is a diffeomorphism between the interval $U_1 := (-1,1)$ and $V_1 := \{(x,y) \in \mathbb{S}^2 \mid y > 0\} \subset \mathbb{S}^2$. Thus every point $p \in V_1$ has a neighbourhood diffeomorphic to an interval.

Then write down explicitly a few more maps to show that *every* point $p \in \mathbb{S}^2$ has a neighbourhood diffeomorphic to an interval.

The pair (U_1, f_1) is called *a parametrisation* of the neighbourhood V_1. The pair (V_1, f_1^{-1}) is called *a chart*. The collection of charts covering a manifold M is called *an atlas* of M.

Exercise 6.1.2. Show that the cylinder is a smooth 2-dimensional manifold by exhibiting explicitly enough parametrisations to cover it.

Exercise 6.1.3. Prove that the union of the two coordinate axes in $\mathbb{R}^2$ is not a manifold.

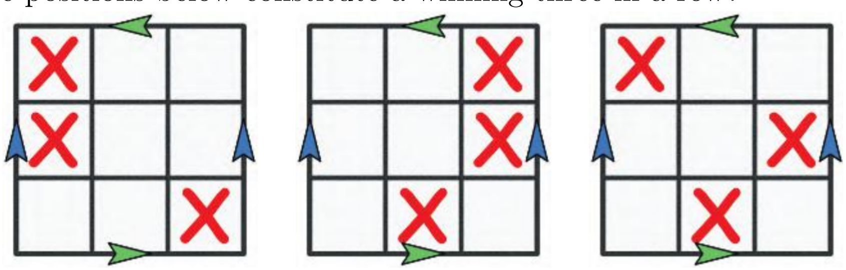
Exercise 6.1.4. Let $\mathbb{B}_r$ be the open ball $\{x \in \mathbb{R}^k\colon \|x\|^2 < r\}$. Show that the map

$$x \mapsto \frac{rx}{\sqrt{r^2 - |x|^2}}$$

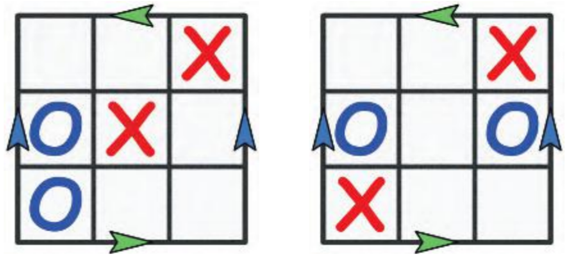
is a diffeomorphism of $\mathbb{B}^r$ onto $\mathbb{R}^k$. *Hint*: Compute its inverse directly.

Deduce that if X is a k-dimensional manifold, then every point in X has a neighborhood diffeomorphic to all of $\mathbb{R}^k$. Thus local parametrizations may always be chosen with all of $\mathbb{R}^k$ for their domains.

Exercise 6.1.5. Consider the Tic-Tac-Toe game on the Klein bottle.
(a) Which of the positions below constitute a winning three-in-a-row?



(b) Find X's best move in each situation shown below.



Exercise 6.1.6. We now have two different models of the projective plane $\mathbb{R}P^2$:

— A sphere with opposite points identified;

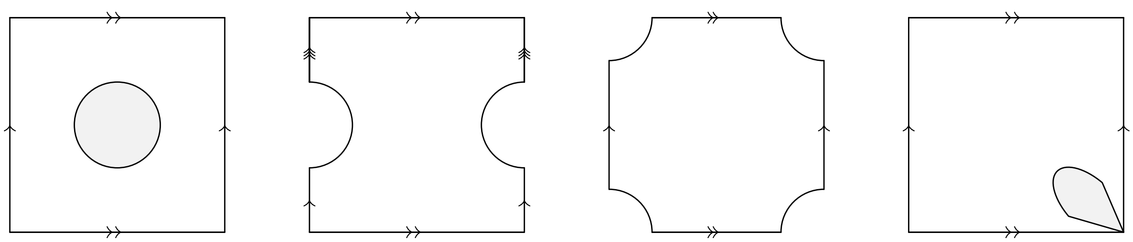
— A set of lines passing through the origin in $\mathbb{R}^3 \setminus \{0\}$;

Describe open sets with respect to the quotient topology.

2. Optional: working-level knowledge

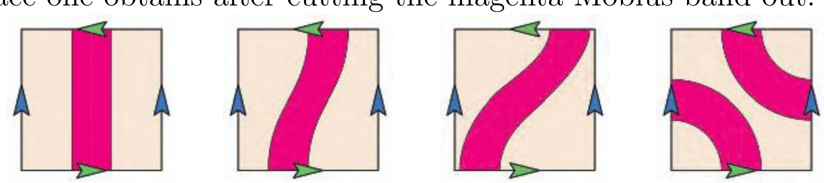
Exercise 6.2.1. Show that the quotient map $\mathbb{S}^2 \to \mathbb{S}^2/_{\{p\}\sim\{-p\}} \approx \mathbb{S}^2$ is continuous with respect to the natural topology on the circle $\mathbb{S}^2$.

Exercise 6.2.2. Show that the four nets below correspond to a torus with a closed disk cut out.



Exercise 6.2.3. Watch the video showing the process of gluing a double torus from the regular octagon. We know that the double torus is a connected sum of two tori. Show how one can to obtain the octagon from the two nets corresponding to a torus with a disk removed and the net for a cylinder.

Exercise 6.2.4. The Klein bottle contains many Moebius bands. For each case shown below identify the surface one obtains after cutting the magenta Möbius band out.



Exercise 6.2.5. Let M_1 and M_2 be two smooth manifolds with atlases (U^1_α, f^1_α) and (U^2_β, f^2_β), for some index set $I \ni \alpha$, $J \ni \beta$. Show that the direct product $M_1 \times M_2$ is a smooth manifold with an atlas $\{(U^1_\alpha \times U^2_\beta, f^1_\alpha \times f^2_\beta)\}_{\alpha\in I,\beta\in J}$.

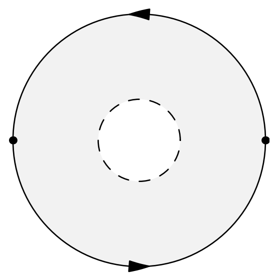
Exercise 6.2.6. Recall that $f\colon X \to Y$ is smooth, if for any $x \in X$ there exists an open subset $U \subset \mathbb{R}^n$ and a smooth function $F\colon U \to Y$ such that $F\big|_{U\cap X} = f$.

Suppose that X is a subset of $\mathbb{R}^N$ and Z is a subset of X. Show that the restriction to Z of any smooth map on X is a smooth map on Z.

Exercise 6.2.7. Let $X \subset \mathbb{R}^N$, $Y \subset \mathbb{R}^M$, $Z \subset \mathbb{R}^L$ be arbitrary subsets, and let $f\colon X \to Y$, $g\colon Y \to Z$ be smooth maps. Then the composition $g \circ f\colon X \to Z$ is smooth. If f and g are diffeomorphisms, so is $g \circ f$.

3. Exam-level questions

Exercise 6.3.1. We know that the real projective plane can be represented as a disk with opposite points identified. Using the net below, show that $\mathbb{R}P^2$ with a disk removed is homeomorphic to the Möbius band.



Exercise 6.3.2. Give an example of an atlas on the torus.

Exercise 6.3.3. Show that one cannot parametrize the k-sphere $\mathbb{S}^k$ by a single parametrization.